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Uniqueness Issues on Permanent Progressive Water-Waves

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Abstract

We consider two-dimensional water-waves of permanent shape with a constant propagation speed. Two theorems concerning the uniqueness of certain solutions are reported. Uniqueness of Crapper's pure capillary waves is proved under a positivity assumption. The proof is based on the theory of positive operators. Also proved is the uniqueness of the positive gravity waves of infinite depth with moderately large amplitude. This is accomplished by a combination of new inequalities and a numerical verification algorithm. Possibilities and impossibilities of other uniqueness theorems are discussed.

1 Introduction

We consider progressive waves of permanent shape on a 2D irrotational flow of incompressible inviscid fluid. We show that, under a positivity assumption, the pure capillary waves of Crapper are unique. Also, the positive gravity waves of infinite depth are shown to be unique if their amplitudes are not large.

The shape of a water-wave is determined by solving a free boundary problem for the Laplace equation with a nonlinear boundary condition, which is derived from Bernoulli's theorem. This is a mathematically difficult problem in its primitive form, but, thanks to a happy idea of Stokes, Nekrasov, and Levi-Civita, the set of governing equations is actually transformed into a problem of finding a certain analytic function in a fixed domain (= the unit disk). In the present paper, we use the following concise form:

$$e^{2H\theta}\frac{\mathrm{d}H\theta}{\mathrm{d}\sigma} - pe^{-H\theta}\sin\theta + q\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(e^{H\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right) = 0 \quad (-\pi \le \sigma \le \pi). \tag{1.1}$$

Here H is the Hilbert transform (or can be called the conjugate operator; its concrete form is given in [11]), and p, q are dimensionless parameters defined by

$$p = \frac{gL}{2\pi c^2}, \qquad q = \frac{2\pi T}{mc^2 L},$$

where g is the gravity acceleration, c is the propagation speed of the wave, L is the wave length, m is the mass density, and T is a surface tension. The unknown θ represents the angle between the tangent at the free boundary and the horizontal line. σ is a Lagrangian variable along the free boundary. See [11] for more detail. In this way we get to the problem to seek a 2π-periodic θ which satisfies (1.1) and

$$\int_{-\pi}^{+\pi} \theta(\sigma)\, d\sigma = 0. \tag{1.2}$$

Once θ is obtained, we can compute from it the wave profile and the velocity vector field ([11]).

For $p > 0, q > 0$, a lot of numerical solutions have been found (see [11] and the references therein). What we are going to consider in the present paper is the cases where either p or q vanishes. In this case, numerical experiments by various authors suggest that the set of the solutions is rather simply structured ([11]). In fact, as we will soon show, some uniqueness theorems holds true.

From now on, only symmetric waves are considered: this means that we assume that

$$\theta(-\sigma) = -\theta(\sigma). \tag{1.3}$$

We first consider pure capillary waves. Namely, we neglect the gravity ($p = 0$). We then obtain, after integrating (1.1) once,

$$q\frac{d\theta}{d\sigma} = -\sinh(H\theta) \qquad (-\pi \le \sigma \le \pi). \tag{1.4}$$

(See [11].)

In 1957, G.D. Crapper [5] found a family of solutions of (1.4), which are written, in our context, as follows:

$$\begin{aligned} q &= \frac{1+A^2}{1-A^2}, \\ \theta(\sigma) &= -2\arctan\left(\frac{2A\sin\sigma}{1-A^2}\right) \\ &= -4\left(A\sin\sigma + \frac{A^3}{3}\sin 3\sigma + \frac{A^5}{5}\sin 5\sigma + \cdots\right), \end{aligned}$$

where A is a parameter satisfying $-1 < A < 1$. One easily notices that $(q/n, \theta(n\sigma))$ are also solutions for $n = 2, 3, \cdots$. These solutions are called solutions of mode n.

A natural question would be: Does the differential equation (1.4) has a solution other than Crapper's waves?

We have suspected that the answer would be No. The reason we believe so is stated in [11] and we do not reiterate. The first author recently found a uniqueness theorem ([12]), which guarantees that Crapper's solutions are only possible solutions among positive solutions.

2 Result of [12]

The theorem in [12] is stated here:

Theorem 1. *Suppose that a solution of* (1.2)(1.3)(1.4) *satisfies either the following* **A1** *or* **A2**.

A1 $0 \le \theta(\sigma) \le \pi$ *everywhere in* $0 \le \sigma \le \pi$*;*

A2 $\frac{d\tau}{d\sigma}(\sigma) \ge 0$ *everywhere in* $0 \le \sigma \le \pi$*, where* $\tau = H\theta$.

Then it is one of Crapper's solutions of mode one.

The proof in [12] uses the Perron-Frobenius theory for positive operators. By applying the theory it is shown in [12] that θ satisfies

$$\frac{d\theta}{d\sigma} = \gamma \sin(H\theta) \qquad (-\pi \le \sigma \le \pi), \tag{2.1}$$

where γ is a constant. This was called by Toland [17] the Peierls-Nabarro equation, all solutions of which were concretely written down in [17]. By his theory we can write down the solution θ in terms of elementary functions, which leads to our conclusion.

In writing the solution of (2.1), [17] used an unexpected relation between (2.1) and the stationary Benjamin-Ono equation. Thus our result revealed a relation between Crapper's waves and the Benjamin-Ono equation. More specifically, Toland discovered the following fact. He considered

$$H\frac{df}{d\sigma} = \sin f \tag{2.2}$$

(the Peierls-Nabarro equation), and showed that there existed two solutions, say g_1 and g_2, of

$$H\frac{dg}{d\sigma} = -g + g^2 \tag{2.3}$$

such that $\frac{df}{d\sigma} = g_1 - g_2$. But (2.3) is actually the stationary Benjamin-Ono equation. All of its solutions are concretely written in [2, 3].

This is fine, but one may feel that our proof is unnecessarily indirect. Note that, by putting $f = H\varphi$, the equation (2.2) can be written as

$$\frac{d\varphi}{d\sigma} = -\sin\left(H\varphi\right).$$

The equation for Crapper's waves, (1.4), differs from this equation only in sin and sinh. We may therefore expect that there exists a direct relation between the solutions of (1.4) and (2.3). We tried to find this relation but we did not succeed.

Also, we note that the relation between (1.4) and (2.1) was established only for positive solutions. Since we relied on a theory of positive operators, no conclusion was derived for solutions of mixed signs (solutions of mode ≥ 2). Although we could not prove, we believe that the uniqueness holds true without assuming a positivity assumption for Crapper's waves.

3 A uniqueness theorem on the gravity waves

We now move on to a uniqueness theorem on the gravity waves, which has been recently obtained by the second author. Now the assumption is that the surface tension is neglected and only the gravity acts. Putting $q = 0$ in (1.1) and integrating, we obtain what is called Nekrasov's equation:

$$\theta(\sigma) = \frac{1}{3\pi} \int_0^\pi \log \left| \frac{\sin \frac{\sigma+s}{2}}{\sin \frac{\sigma-s}{2}} \right| \frac{\mu \sin \theta(s)}{1 + \mu \int_0^s \sin(\theta(u)) \mathrm{d}u} \mathrm{d}s. \tag{3.1}$$

Here μ is a new parameter, which is related to p in a nontrivial manner.

The equation (3.1) has a rather long history but the structure of the solutions had long been unclear except for those solutions of small amplitude. See [11]. The first satisfactory answer about the existence was given by [8] as in the following form:

Theorem 2 (Keady & Norbury, '78). *For all $3 < \mu < \infty$, there exists at least one non-trivial solution satisfying $0 \le \theta \le \pi/2$.*

As μ tends to 3, the solution tends to the trivial solution $\theta \equiv 0$. Namely, the solution branches off the trivial solution at $\mu = 3$. As μ tends to ∞, however, the solution tends to what is called Stokes' extreme wave. See, for instance, [11].

The Levi-Civita equation (1.1) (with $q = 0$) possesses solutions which change signs in $0 \le \sigma \le \pi$. It is known (numerically by [4] and mathematically [7]) that secondary bifurcations exist along the branch of such solutions. However, based on numerical computation by [4] and others, there persisted a speculation that no secondary bifurcation exists along the branch of positive solutions. This was partly proved rigorously by [7], which seems to be a nice step toward the uniqueness of the positive solutions. The second author proved in [10] the following

Theorem 3. *For all $3 < \mu \le 170.0$, there exists at most one non-trivial solution satisfying $0 \le \theta \le \pi/2$.*

The proof in [10] uses the validated numerics or "interval analysis", which gives us exact (i.e., including round-off errors) bound for numerical computations.

The idea of the proof is easily stated, although its real implementation is far from trivial. To explain the idea, we use the following setting;

$$E = \{ f \in C[0,\pi]\ ;\ f(0) = f(\pi) = 0\ \},$$

which we regard as a Banach space with the usual norm $\|f\| = \max_{0 \le \sigma \le \pi} |f(\sigma)|$. A closed subset of E is defined as follows:

$$K_\alpha = \{ f \in E\ ;\ \alpha \sin\sigma \le f(\sigma) \le \pi/2\ \ (0 \le \sigma \le \pi\ )\ \},$$

where α is a sufficiently small positive number.

Nekrasov's equation can be written as an abstract form $\theta = F(\theta)$. Then our goal is realized if we have shown that any positive solution is contained in K_α and that $F : K_\alpha \to K_\alpha$ has a unique fixed point. Uniqueness would be proved if F are shown to be a contraction mapping. However, it is very unlikely that F is a contraction in the whole K_α. But, based upon numerical experiments in [11] and others, we may expect that F is a contraction in a small neighborhood of the solution. So our task is to show the following three propositions:

- There exists a small positive number α such that any positive solution of Nekrasov's equation is contained in K_α;
- There exist an integer n and a closed subset $\tilde{K}$ of K_α such that if $\theta \in K_\alpha$ then $F^n(\theta) \in \tilde{K}$, where F^n denotes the n iterates of F;
- $F : \tilde{K} \to \tilde{K}$ is a contraction mapping.

The second author proved these propositions with the aid of validated numerical computations. $\tilde{K}$ is in our case the continuous functions bounded from and below by two functions shown in Fig. 1. Actually, he defined a sequence of functions such that

$$0 < \theta_0^-(\sigma) \le \theta_1^-(\sigma) \le \cdots \le \theta(\sigma) \le \cdots \le \theta_1^+(\sigma) \le \theta_0^+(\sigma) \qquad (0 < \sigma < \pi).$$

Such upper bounds $\{\theta_n^+\}$ and lower bounds $\{\theta_n^-\}$ are constructed numerically by a certain iteration scheme. After 30–700 times of iterations depending on μ, these upper and lower bounds become nearly stationary as is shown in Fig. 1. With such θ_n^+ and θ_n^-, we define $\tilde{K}$ as

$$\tilde{K} = \{ f \in K \; ; \; \theta_n^-(\sigma) \le f(\sigma) \le \theta_n^+(\sigma) \quad (0 \le \sigma \le \pi) \ \}.$$

The numerical solutions in the figure was obtained by approximating the function by piecewise quadratic functions. See [10].

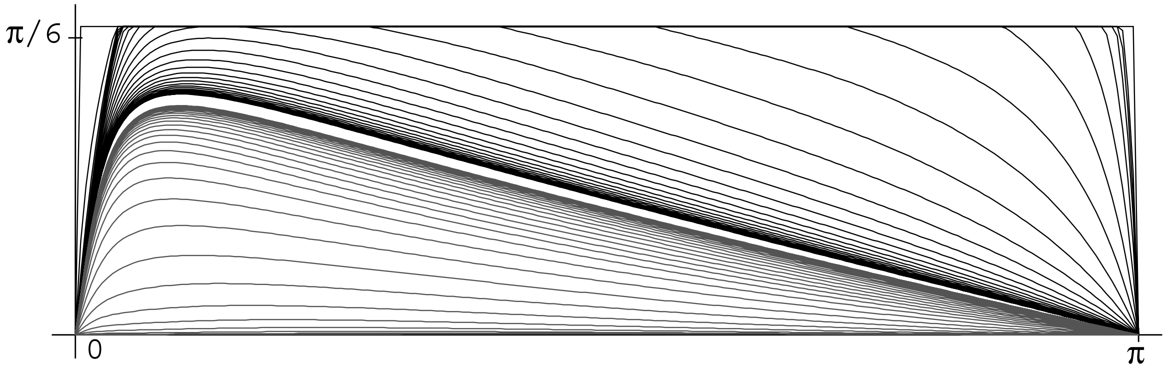

Figure 1. Convergence of the upper and lower bounds. 31 iterations are plotted. $38.96 \le \mu \le 40.0$

The restriction that $\mu \le 170.0$ is a technical one. Uniqueness actually seems to hold for much larger μ. We do not know, however, whether or not the uniqueness holds for all $\mu \in (3, \infty)$. The wave profile at $\mu = 170.0$ is plotted in Figure 2

The technique in [10] can be applied to other problems. For instance, the case of $\mu = \infty$ (Stokes' extreme wave) is treated in a similar fashion. In this case, the governing equation becomes:

$$\theta(\sigma) = \frac{1}{3\pi} \int_0^\pi \log \left| \frac{\sin \frac{\sigma+s}{2}}{\sin \frac{\sigma-s}{2}} \right| \frac{\sin \theta(s)}{\int_0^s \sin(\theta(u)) \mathrm{d}u} \mathrm{d}s. \tag{3.2}$$

The second author recently applied his method to (3.2), and he is confident that the following theorem holds true:

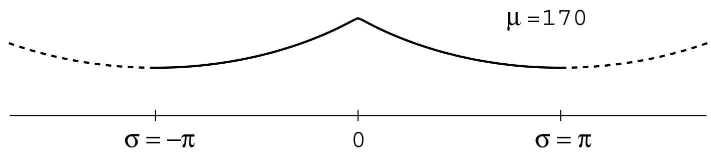

Figure 2. Wave profile at $\mu = 170.0$. This is a wave of infinite depth; the horizontal line is just a reference.

Theorem 4. *Among solutions satisfying* (i) $0 \leq \theta \leq \pi/2$ *in* $0 \leq \sigma \leq \pi$ *and* (ii) θ *is monotone decreasing in* $0 \leq \sigma \leq \pi$*, there exists at most one non-trivial solution.*

The solution of (3.2) is called Stokes' extreme wave and a numerical solution looks like the one in Fig. 3. (The reader may notice that the solution in Fig. 2 is not very far from the extreme wave.) The solution is continuous but is no longer smooth at $\sigma = 0$.

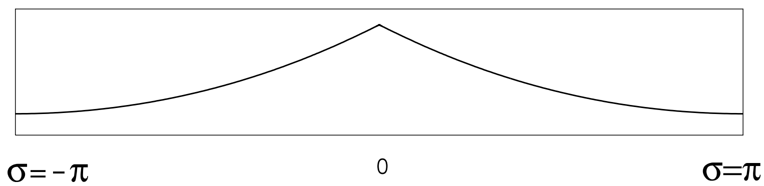

Figure 3. Stokes' extreme wave. This is again a wave of infinite depth.

The first proof of the existence of a solution of (3.2) was given by Toland [16]. The details of the solution were studied later (see the references in [11]) but its uniqueness does not seem to be established. Recently, Plotnikov and Toland [14] proved the existence of a continuous decreasing solution of (3.2). An ideal uniqueness theorem would be such a one that guarantees the uniqueness of a continuous positive solution (without assuming the monotonicity). But this seems to be difficult; what the second author can claim is the uniqueness in a restricted class of solutions.

4 Comments

As for the solutions of pure capillary waves, we may consider the case of finite depth. In this case, Kinnersley's explicit solutions, which are written by the elliptic functions, are known. But, the method in [12] cannot be applied to them. This is because we use a certain eigenvalue problem in the proof, which is not available for Kinnersley's solutions.

There is another important issue on the uniqueness; the uniqueness of the solitary gravity waves. Solitary gravity waves are constructed by the solution θ of the following equation:

$$e^{2\tau}\frac{d\tau}{d\sigma} = \frac{p}{\cos(\sigma/2)}e^{-\tau}\sin\theta \qquad (-\pi \leq \sigma \leq \pi), \tag{4.1}$$

where $\tau = H\theta + \tau_0$ with the constant τ_0 being so determined that $\tau(\pm\pi) = 0$. Amick and Toland [1] proved that for $p_0 < p < 1/\pi$ there exists at least one non-trivial solution satisfying $0 \le \theta \le \pi/2$. Here $p_0 = 0.1925\cdots$. As $p \to p_0$, the solitary wave tends to the extreme wave, which has a 120° angle at the crest. Thus, the existence and some geometric properties are already established. Plotnikov [13] claims that there exists a secondary bifurcation along the branch of positive solutions to (4.1). Nonuniqueness is therefore claimed affirmatively by the paper. This shows a striking contrast to the periodic water-waves, for which no secondary bifurcation along the branch of positive solutions is expected (see [11, 6]). Tanaka [15] showed numerically that the solitary waves became unstable as the amplitude became large enough. But he also concluded that the situation was the same as in the case of periodic gravity waves. Namely, he concluded that no secondary bifurcation was likely to occur at the onset of instability. The results of [13] and [15] are not necessarily contradictory to each other: Tanaka's numerical computation applies only to the first instability, whence the secondary bifurcation may occur at solutions of higher amplitude.

In view of Plotnikov's result, we must be able to numerically compute the solutions on the secondary branch. This seems to be a future challenge.

The periodic gravity waves of finite depth are governed ([11]) by Yamada's equation:

$$e^{2\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\sigma} - \frac{pe^{-\tau}\sin\theta}{\sqrt{1-k^2\sin^2(\sigma/2)}} = 0 \qquad (-\pi \le \sigma \le \pi). \tag{4.2}$$

where τ is the same as in (4.1) and $k \in [0,1]$ is a parameter depending on the depth. If $k = 0$, this becomes the periodic gravity waves of infinite depth, and (4.2) with $k = 0$ is equivalent to (3.1). As $k \to 1$, the flow becomes shallower. If $k = 1$, the equations describe the solitary waves (4.1). Since secondary bifurcations are predicted by Plotnikov, and the numerical computation is easier in $k < 1$ than in $k = 1$, it may be worth trying to numerically solve (4.2) with $k < 1$.

We can also consider the pure capillary solitary waves, which are governed by

$$q\frac{d\theta}{d\sigma} = -\frac{\sinh(\tau)}{\cos(\sigma/2)} \qquad (-\pi \le \sigma \le \pi). \tag{4.3}$$

Numerical solutions can be obtained ([11]), but we do not know the structure of them. The equation (4.3) is generalized as

$$q\frac{d\theta}{d\sigma} = -\frac{\sinh(\tau)}{\sqrt{1-k^2\sin^2(\sigma/2)}} \qquad (-\pi \le \sigma \le \pi), \tag{4.4}$$

where k is the same as in (4.2). Kinnersley's solutions are solutions of this equation. We cannot deny the possibility that the pure capillary waves of mode one lose their uniqueness as k increases.

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