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Wavelet Transforms in Quantum Calculus

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Abstract

This paper aims to study the q-wavelets and the q-wavelet transforms, using only the q-Jackson integrals and the q-cosine Fourier transform, for a fix $q \in]0,1[$. For this purpose, we shall attempt to extend the classical theory by giving their q-analogues.

1 Introduction

Wavelets were introduced by J. Morlet in 1982 as tool to study the analysis of seismic data. Taking account of the success of this method, this author joint with A. Gossmann (see [8]) gave mathematical assizes for the so-called wavelet transform. In 1985 Y. Meyer recognized this theory and contributed for showing that it can be used to explain easily many mathematical tools. This motivated many authors who published intensively in this way.
The continuous wavelet transform on $\mathbb{R}$ was presented, in particular, by T. H. Koornwinder in [15]. Since we are concerned, in the present paper, by showing the q-analogue of this, we summarize some of their results in the even case as follows:

Let $\mathcal{F}_0$ be the even Fourier transform

$$\mathcal{F}_0(f)(\lambda) = \int_0^\infty f(x)\cos(\lambda x)dx, \quad \lambda \in \mathbb{C}. \tag{1.1}$$

We have

$$\mathcal{F}_0 o \sigma_x(\lambda) = \cos(\lambda x)\mathcal{F}_0(\lambda), \quad \mathcal{F}_0 o H_a = H_a^{-1} o \mathcal{F}_0, \tag{1.2}$$

where σ_x, $x \geq 0$, is the even translation operator defined by

$$\sigma_x(f)(y) = \frac{1}{2}[f(x+y) + f(x-y)], \quad y \in [0, +\infty[\tag{1.3}$$

and H_a and H_a^{-1}, $a > 0$, are the dilatation operators defined by

$$\begin{cases} H_a(f)(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a}); \\ H_a^{-1}(f)(x) = \sqrt{a} f(ax). \end{cases} \tag{1.4}$$

A wavelet is an even function g defined on $\mathbb{R}$, square integrable (with respect the Lebesgue measure), such that for $\lambda > 0$,

$$C_g = \int_0^\infty \mid \mathcal{F}_0(g)(a\lambda) \mid^2 \frac{da}{a}, \tag{1.5}$$

is finite, positive and independent of λ.
We construct a family of wavelets by putting for all $a > 0$ and $b \geq 0$,

$$g_{a,b}(x) = \sigma_b o H_a(g)(x), \quad x \geq 0, \tag{1.6}$$

where σ_b and H_a are defined respectively in (1.3) and (1.4).
The continuous wavelet transform of an even square integrable function f on $\mathbb{R}$ is given by

$$\Phi_g(f)(a,b) = \int_0^\infty f(x)\overline{g_{a,b}}(x)dx. \tag{1.7}$$

This transform Φ_g satisfies the following Plancherel formula

$$\int_0^\infty \mid f(x) \mid^2 dx = \frac{1}{C_g} \int_0^\infty \mid \Phi_g(f)(a,b) \mid^2 \frac{dadb}{a^2} \tag{1.8}$$

and can be inverted as follows

$$f(x) = \frac{1}{C_g} \int_0^\infty \int_0^\infty \Phi_g(f)(a,b) g_{a,b}(x) \frac{dadb}{a^2}. \tag{1.9}$$

In the present paper, we give and study the q-analogue of the continuous wavelet transform (1.7) by the use of the q-Jackson integral and some elements of q-harmonic analysis. We also give its inversion formula which is a q-analogue of (1.9). Furthermore, a q-analogue of Plancherel formula is proved and a q-analogue of the Parseval formula is established.

This paper is organized as follows: in Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we summarize some results stated in [4] and [2] and we establish some properties of the dilatation operators and the q-Fourier cosine transform. In Section 4, we define the q-wavelet and the q-wavelet transform, and discuss their properties. A special attention is devoted to the q-analogue of the Plancherel formula and the Parseval formula, and an inversion formula is proved. In Section 5, we characterize the image set of the q-wavelet transform.

2 Notations and preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [7] and [12]. Throughout this paper, we will fix $q \in]0,1[$.

2.1 Basic symbols.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1, 2, \ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also denote

$$(a_1, a_2, \ldots, a_p; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_p;q)_n, \quad n = 0, 1, 2, 3, \ldots \infty, \tag{2.2}$$

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \tag{2.3}$$

2.2 Operators and elementary functions.

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if} \quad x \neq 0, \tag{2.4}$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If f is differentiable, then $(D_q f)(x)$ tends to $f'(x)$ as q tends to 1.
For a function f, we note

$$D_q^1 f = D_q f, \quad D_q^n f = D_q(D_q^{n-1} f), \quad n \in \mathbb{N}^*, \tag{2.5}$$

$$\Lambda_q(f)(x) = f(qx), \quad \Lambda_q^{-1}(f)(x) = f(q^{-1}x) \tag{2.6}$$

and

$$\Delta_q f = \Lambda_q^{-1} D_q^2 f. \tag{2.7}$$

We remark that if f is two times continuously differentiable, we have

$$\lim_{q \to 1^-} \Delta_q(f) = \frac{d^2 f}{dx^2}. \tag{2.8}$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [10])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.9}$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{2.10}$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a, b]$ is given by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.11}$$

The improper integral is defined in the following way (see [14])

$$\int_0^{\frac{\infty}{A}} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}. \tag{2.12}$$

In particular, for $n \in \mathbb{Z}$, we have

$$\int_0^{\frac{\infty}{q^n}} f(x)d_qx = \int_0^{\infty} f(x)d_qx. \tag{2.13}$$

The q-trigonometric functions q-cosine and q-sine are given by (see [4] and [16])

$$\cos(x;q^2) ={}_1\varphi_1\Big(0,q;q^2,(1-q)^2x^2\Big) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)}\frac{x^{2n}}{[2n]_q!} \tag{2.14}$$

and

$$\sin(x;q^2) = x\ {}_1\varphi_1\Big(0,q^3;q^2,(1-q)^2x^2\Big) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)}\frac{x^{2n+1}}{[2n+1]_q!}. \tag{2.15}$$

Here ${}_1\varphi_1$ is a basic hypergeometric function (see [7]).
The functions q-cosine and q-sine are majorized by $\dfrac{1}{(q;q)^2_\infty}$ and we have for $\lambda \in \mathbb{C}$: $\cos(\lambda x;q^2)$ is the unique solution of

$$\begin{cases} \Delta_q u(x) = -\lambda^2 u(x), \\ u(0) = 1, u'(0) = 0. \end{cases} \tag{2.16}$$

2.3 Sets and spaces.

We denote by

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}, \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}. \tag{2.17}$$

• $\mathcal{E}_{*q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even infinitely q-differentiable functions on $\mathbb{R}$, equipped with the induced topology of uniform convergence on all compact, for all functions and its q-derivatives.
• $\mathcal{D}_{*q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even infinitely q-differentiable functions on $\mathbb{R}$ with compact supports, equipped with the induced topology of uniform convergence, for all functions and its q-derivatives.
• $\mathcal{C}_{*q,0}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even smooth functions, continued in 0 and vanishing at ∞, equipped with the induced topology of uniform convergence.
• $\mathcal{S}_{*q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of infinitely q-differentiable, even and fast decreasing functions and all its q-derivatives **i.e.**

$$\forall n,m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R};0\le k\le n} \mid (1+x^2)^m D_q^k f(x) \mid < +\infty.$$

$\mathcal{S}_{*q}(\mathbb{R}_q)$ is equipped with the induced topology defined by the semi-norms $P_{n,m,q}$.
• $L^p_q(\mathbb{R}_{q,+})$, $p > 0$, the set of all functions defined on $\mathbb{R}_{q,+}$ such that

$$\|f\|_{q,p} = \left\{\int_0^{\infty} \mid f(x) \mid^p d_qx\right\}^{\frac{1}{p}} < \infty. \tag{2.18}$$

3 The q-even translation and the q-cosine Fourier transform

The q-even translation operator $T_{q,x}$, $x \in \mathbb{R}_{q,+}$ was defined (see [4]) on $\mathcal{D}_{*q}(\mathbb{R}_q)$ by

$$T_{q,x}(f)(y) = \sum_{s=-\infty}^{\infty} \mathcal{D}(x,y,q^s) f(q^s y), y \in \mathbb{R}_{q,+}, \tag{3.1}$$

where

$$\mathcal{D}(x,y,q^s) f(q^s) = q^s (\frac{x}{y})^{2s} \frac{(q\frac{x^2}{y^2};q)_\infty}{(q;q)_\infty} \quad {}_1\phi_1(0; q\frac{x^2}{y^2}; q, q^{2s+1}). \tag{3.2}$$

It verifies the following properties (see [4]), for $f, g \in \mathcal{D}_{*q}(\mathbb{R}_q)$,

$$T_{q,x}(f)(y) = T_{q,y}(f)(x), \quad x, y \in \mathbb{R}_{q,+}, \tag{3.3}$$

$$\int_0^\infty T_{q,x}(f)(y) d_q y = \int_0^\infty f(y) d_q y, \quad x \in \mathbb{R}_{q,+}, \tag{3.4}$$

$$\int_0^\infty T_{q,x}(f)(y) g(y) d_q y = \int_0^\infty f(y) T_{q,x}(g)(y) d_q y, \quad x \in \mathbb{R}_{q,+}, \tag{3.5}$$

$$T_{q,x} \cos(ty; q^2) = \cos(tx; q^2) \cos(ty; q^2), \quad x, y, t \in \mathbb{R}_{q,+} \tag{3.6}$$

and $T_{q,x}$ tends to σ_x, defined by (1.3), when q tends to 1^-.
It was shown in [2], that if we note

$$\widetilde{S} = \{q \in [0,1[: \ {}_1\phi_1(0; q^{1+2r}; q, q^{1+2s}) \geq 0, \forall r, s \in \mathbb{N}\}, \tag{3.7}$$

then, $[0, \frac{3-\sqrt{5}}{2}[\subset \widetilde{S}$ and for all $q \in \widetilde{S}$ and $x \in \mathbb{R}_{q,+}$, the operator $T_{q,x}$ is positive.
In the sequel, we suppose that $q \in \widetilde{S}$.

The q-cosine Fourier transform and the q-convolution product are defined (see [4]), by:

$$\mathcal{F}_q(f)(\lambda) = c_q \int_0^\infty f(x) \cos(\lambda x; q^2) d_q x, \quad f \in \mathcal{D}_{*q}(\mathbb{R}_q), \tag{3.8}$$

$$f *_q g(x) = c_q \int_0^\infty T_{q,x} f(y) g(y) d_q y, \quad f, g \in \mathcal{D}_{*q}(\mathbb{R}_q), \tag{3.9}$$

where

$$c_q = \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}. \tag{3.10}$$

In [4], the authors proved that $\mathcal{F}_q$ can be extended to $L^1_q(\mathbb{R}_q)$ and we have:

Theorem 1. *For* $f \in L^1_q(\mathbb{R}_q)$,

$$\mathcal{F}_q(f) \in \mathcal{C}_{*q,0}(\mathbb{R}_q) \tag{3.11}$$

and

$$\|\mathcal{F}_q(f)\|_{\mathcal{C}_{*q,0}(\mathbb{R}_q)} \leq \frac{1}{(q(1-q))^{\frac{1}{2}} (q;q)_\infty} \|f\|_{q,1}. \tag{3.12}$$

Theorem 2. *For $f, g \in \mathcal{D}_{*q}(\mathbb{R}_q)$, we have*

$$\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g), \tag{3.13}$$

$$\mathcal{F}_q(T_{q,x}f)(\lambda) = \cos(\lambda x; q^2)\mathcal{F}_q(f)(\lambda), \quad x \in \widetilde{\mathbb{R}}_{q,+}, \quad \lambda \in \mathbb{R}_{q,+} \tag{3.14}$$

and

$$\mathcal{F}_q(\Delta_q f)(\lambda) = -\lambda^2 \mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.15}$$

In [2], the authors proved the following result.

Theorem 3. *.*
1) If f and $\mathcal{F}_q(f)$ are in $L^1_q(\mathbb{R}_{q,+})$, then for all $x \in \mathbb{R}_{q,+}$, we have

$$f(x) = c_q \int_0^\infty \mathcal{F}_q(f)(y)\cos(xy; q^2) d_q y, \tag{3.16}$$

where c_q is given by (3.10).
2) $\mathcal{F}_q(f)$ is an isomorphism of $\mathcal{S}_{*q}(\mathbb{R}_q)$ and $\mathcal{F}_q^2 = Id$.

They also proved that $\mathcal{F}_q$ can be extended to $L^2_q(\mathbb{R}_{q,+})$ and we have

Theorem 4. *$\mathcal{F}_q$ is an isomorphism of $L^2_q(\mathbb{R}_{q,+})$, $\mathcal{F}_q^{-1} = \mathcal{F}_q$ and for $f \in L^2_q(\mathbb{R}_{q,+})$, we have*

$$\|\mathcal{F}_q(f)\|_{q,2} = \|f\|_{q,2}. \tag{3.17}$$

Remak 1.
Using the previous theorem and the relation (3.14), one can see that, for $f \in L^2_q(\mathbb{R}_{q,+})$, we have for all $x \in \widetilde{\mathbb{R}}_{q,+}$, $T_{q,x}f \in L^2_q(\mathbb{R}_{q,+})$ and

$$\|T_{q,x}f\|_{q,2} \leq \frac{1}{(q;q)^2_\infty}\|f\|_{q,2}. \tag{3.18}$$

Theorem 5. *Let $p, p', r \in]1,2]$, such that $\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r}$. If $f \in L^p_q(\mathbb{R}_{q,+})$ and $g \in L^{p'}_q(\mathbb{R}_{q,+})$, then $f *_q g \in L^r_q(\mathbb{R}_{q,+})$*

$$\|f *_q g\|_{q,r} \leq B_{q,p}B_{q,p'}B_{q,r'}\|f\|_{q,p}\|g\|_{q,p'}, \tag{3.19}$$

where $B_{q,p} = \left(\frac{1}{(q(1-q))^{\frac{1}{2}}(q;q)_\infty}\right)^{1-\frac{p}{2}}$ and r' is given by $\frac{1}{r} + \frac{1}{r'} = 1$.

To achieve this section, we state the following propositions, useful for the sequel.

Proposition 1. *.*
i) The dilatation operators satisfy

$$H_1 = id; \tag{3.20}$$

$$H_a o H_b = H_{ab}, \quad a, b \in \mathbb{R}_{q,+}; \tag{3.21}$$

$$H_a^{-1} = H_{a^{-1}}, \quad a \in \mathbb{R}_{q,+}. \tag{3.22}$$

ii) For all $a \in \mathbb{R}_{q,+}$, the operator H_a is linear and isometric from $L^2_q(\mathbb{R}_{q,+})$ into itself.
iii) For all $a \in \mathbb{R}_{q,+}$, the operator H_a is a topological automorphism of $\mathcal{S}_{*q}(\mathbb{R}_q)$ (resp $\mathcal{C}_{*q}(\mathbb{R}_q)$).

Proposition 2. *For $a \in \mathbb{R}_{q,+}$, we have*

$$\mathcal{F}_q o H_a = H_a^{-1} o \mathcal{F}_q. \tag{3.23}$$

Proof. The change of variables rule (see [12]) gives for a suitable function f

$$\begin{aligned}\mathcal{F}_q o H_a(f)(x) &= \frac{c_q}{\sqrt{a}} \int_0^\infty f(\frac{t}{a}) \cos(tx; q^2) d_q t \\ &= \sqrt{a} c_q \int_0^\infty f(u) \cos(axu; q^2) d_q u \\ &= H_a^{-1}(\mathcal{F}_q(f))(x).\end{aligned}$$

■

Proposition 3. *Let f and g be in $L_q^2(\mathbb{R}_{q,+})$. Then*
1) $f *_q g \in L_q^2(\mathbb{R}_{q,+})$ iff $\mathcal{F}_q(f)\mathcal{F}_q(g) \in L_q^2(\mathbb{R}_{q,+})$,
2)

$$\int_0^\infty \mid f *_q g(x) \mid^2 d_q x = \int_0^\infty \mid \mathcal{F}_q(f)(x) \mid^2 \mid \mathcal{F}_q(g)(x) \mid^2 d_q x, \tag{3.24}$$

where both sides are finite or infinite.

Proof. The proof is a direct consequence of Theorem 4 and the fact that $\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(f)$. ■

4 q-Wavelet transforms

Definition 1. A q-wavelet is an even function g defined on $\mathbb{R}_q$ and square q-integrable such that

$$0 < C_g = \int_0^\infty \mid \mathcal{F}_q(g)(a) \mid^2 \frac{d_q a}{a} < \infty. \tag{4.1}$$

Example

Put $e_{q^2}^x = \dfrac{1}{((1-q^2)x; q^2)_\infty}$ the q-analogue of the exponential function (see [7], and [12]).

Let $G(x, t; q^2) = A(t; q^2) e_{q^2}^{-\frac{x^2}{qt(1+q^2)}}$, where $A(t; q^2) = (q^{-1} - 1) \dfrac{\left(-\frac{1+q}{1-q} q^2 t, -\frac{1-q}{(1+q)t}; q^2\right)_\infty}{\left(-\frac{1-q}{(1+q)qt}, -\frac{1+q}{1-q} q^3 t; q^2\right)_\infty}$.

We have (see [4]) for all $t \in \mathbb{R}_{q^2,+}$, $x \mapsto G(x, t; q^2)$ is in $\mathcal{S}_{*q}(\mathbb{R}_q)$ and

$$\mathcal{F}_q(G(., t; q^2))(x) = e_{q^2}^{-tx^2}, \quad x \in \mathbb{R}_{q,+}.$$

Then, $g = \Delta_q G(., t; q^2)$ is in $\mathcal{S}_{*q}(\mathbb{R}_q)$ and we have

$$\mathcal{F}_q(g)(x) = -x^2 \mathcal{F}_q(G(., t; q^2))(x) = -x^2 e_{q^2}^{-tx^2}, \quad x \in \mathbb{R}_{q,+}.$$

Thus,

$$\forall a \in \mathbb{R}_{q,+}, \quad 0 <\mid \mathcal{F}_q(g) \mid^2 (a) \leq a^4 e_{q^2}^{-ta^2}$$

and

$$
\begin{aligned}
0 < \int_0^\infty \mid \mathcal{F}_q(g) \mid^2 (a) \frac{d_q a}{a} &\leq \int_0^\infty a^3 e_{q^2}^{-ta^2} d_q a \\
&= \frac{1}{(1+q)t^2} \frac{(-q^4, -q^{-2}; q^2)_\infty}{(-q^2, -1; q^2)_\infty} \\
&= \frac{1}{q^2(1+q)t^2}.
\end{aligned}
$$

So, g is a q-wavelet and it constitutes a q-analogue of the so-called Mexican hat wavelet.

Remark 2.

1) For all $\lambda \in \mathbb{R}_{q,+}$, we have

$$C_g = \int_0^\infty \mid \mathcal{F}_q(g)(a\lambda) \mid^2 \frac{d_q a}{a}.$$

2) Let f be a nonzero function in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (resp. $\mathcal{C}_{*q}(\mathbb{R}_q)$). Then $g = \Delta_q f$ is a q-wavelet, in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (resp. $\mathcal{C}_{*q}(\mathbb{R}_q)$) and we have

$$C_g = \int_0^\infty a^3 \mid \mathcal{F}_q(f)(a) \mid^2 d_q a.$$

Proposition 4. *Let $g \neq 0$ be a function in $L_q^2(\mathbb{R}_{q,+})$ satisfying:*

1. *$\mathcal{F}_q(g)$ is continuous at 0.*

2. *$\exists \alpha > 0$ such that $\mathcal{F}_q(g)(x) - \mathcal{F}_q(g)(0) = O(x^\alpha)$, as $x \to 0$.*

Then, (4.1) is equivalent to

$$\mathcal{F}_q(g)(0) = 0. \tag{4.2}$$

Proof. - Assume that (4.1) is satisfied.
If $\mathcal{F}_q(g)(0) \neq 0$, then there exist $p_0 \in \mathbb{N}$ and $M > 0$, such that

$$\forall n \geq p_0, \quad \mid \mathcal{F}_q(g)(q^n) \mid \geq M.$$

Then, the integral in (4.1) would be equal to ∞.
- Conversely, assume that $\mathcal{F}_q(g)(0) = 0$.
Since $g \neq 0$, we deduce from Theorem 4, that the first inequality in (4.1) holds.
On the other hand, from the relation (2), there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$, such that for all $n \geq n_0$, we have

$$\mid \mathcal{F}_q(g)(q^n) \mid \leq \epsilon q^{n\alpha}.$$

Then, using the definition of the q-integral and Theorem 4, we obtain

$$\begin{aligned}\int_0^{\infty} \mid \mathcal{F}_q(g)(a) \mid^2 \frac{d_q a}{a} &= (1-q)\sum_{n=-\infty}^{\infty} \mid \mathcal{F}_q(g)(q^n) \mid^2 \\ &= (1-q)\sum_{n=-\infty}^{n_0} \mid \mathcal{F}_q(g)(q^n) \mid^2 + (1-q)\sum_{n=n_0+1}^{\infty} \mid \mathcal{F}_q(g)(q^n) \mid^2 \\ &\leq \frac{(1-q)}{q^{n_0}}\sum_{n=-\infty}^{\infty} q^n \mid \mathcal{F}_q(g)(q^n) \mid^2 + (1-q)\epsilon\sum_{n=0}^{\infty} q^{2n\alpha} \\ &\leq \frac{\|\mathcal{F}_q(g)\|_{q,2}^2}{q^{n_0}} + \frac{1-q}{1-q^{2\alpha}}\epsilon \\ &= \frac{\|g\|_{q,2}^2}{q^{n_0}} + \frac{1-q}{1-q^{2\alpha}}\epsilon.\end{aligned}$$

Which completes the proof. ■

Remark 3.
Owing to (3.11), the continuity assumption in the previous proposition will certainly hold if g is moreover in $L_q^1(\mathbb{R}_{q,+})$. Then (4.2) can be equivalently written as

$$\int_0^{\infty} g(x) d_q x = 0.$$

Using a q-wavelet, the operator H_a, $a \in \mathbb{R}_{q,+}$ and the q-even translation operator $T_{q,b}$, $b \in \widetilde{\mathbb{R}}_{q,+}$, we are able to construct a family of q-wavelets, by

$$g_{a,b}(x) = T_{q,b}(H_a(g))(x), \quad \text{forall} \quad x \in \mathbb{R}_{q,+}, \tag{4.3}$$

where $T_{q,b}$ and H_a are defined respectively by (3.1) and (1.4).

Proposition 5. *Let g be a q-wavelet in $L_q^2(\mathbb{R}_{q,+})$. Then for all $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, $g_{a,b}$ is a q-wavelet in $L_q^2(\mathbb{R}_{q,+})$ and we have*

$$C_{g_{a,b}} = a\int_0^{\infty} \cos^2\left(\frac{xb}{a}; q^2\right) \mid \mathcal{F}_q(g)(x) \mid^2 \frac{d_q x}{x}. \tag{4.4}$$

Proof. According to Remark 1 and to the properties of H_a, we can easily see that $g_{a,b}$ is an even function in $L_q^2(\mathbb{R}_{q,+})$.
Now, using Proposition 2 and the properties of the q-even translation, we obtain for $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$,

$$\begin{aligned}\mathcal{F}_q(g_{a,b})(x) &= c_q\int_0^{\infty} T_{q,b}(H_a(g))(t)\cos(xt; q^2) d_q t \\ &= c_q\int_0^{\infty} H_a(g)(t) T_{q,b}\cos(xt; q^2) d_q t \\ &= \cos(xb; q^2) c_q\int_0^{\infty} H_a(g)(t)\cos(xt; q^2) d_q t \\ &= \cos(xb; q^2)\mathcal{F}_q(H_a(g))(x) = \cos(xb; q^2) H_a^{-1}\mathcal{F}_q(g)(x) \\ &= \sqrt{a}\cos(xb; q^2)\mathcal{F}_q(g)(ax).\end{aligned}$$

So,

$$\begin{aligned} C_{g_{a,b}} &= a\int_0^{\infty} \cos(xb;q^2)^2 \mid \mathcal{F}_q(g)(ax) \mid^2 \frac{d_qx}{x} \\ &= a\int_0^{\infty} \cos^2(\frac{xb}{a};q^2) \mid \mathcal{F}_q(g)(x) \mid^2 \frac{d_qx}{x}, \end{aligned}$$

by the change of variable $u = ax$.
Thus,

$$0 < C_{g_{a,b}} \leq \frac{a}{(q;q)_\infty^4} \int_0^{\infty} \mid \mathcal{F}_q(g)(x) \mid^2 \frac{d_qx}{x} = \frac{aC_g}{(q;q)_\infty^4},$$

which proves the result. ■

Proposition 6. *Let g be a q-wavelet in $L^2_q(\mathbb{R}_{q,+})$. Then the mapping*

$$F : (a,b) \mapsto g_{a,b}$$

is continuous from $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ into $L^2_q(\mathbb{R}_{q,+})$.

Proof. It is clear that F is a mapping from $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ into $L^2_q(\mathbb{R}_{q,+})$ and it is continuous at all $(a,b) \in \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$.
Now, let $a \in \mathbb{R}_{q,+}$. For $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\begin{aligned} \| F(a,b) - F(a,0) \|^2_{q,2} &= \| T_{q,b}(H_a(g)) - H_a(g) \|^2_{q,2} \\ &= \| \mathcal{F}_q\left(T_{q,b}(H_a(g)) - H_a(g)\right) \|^2_{q,2} \\ &= \int_0^{\infty} \mid 1 - \cos(xb;q^2) \mid^2 \mid \mathcal{F}_q(H_a(g)) \mid^2 (x) d_qx. \end{aligned}$$

However, for all $x \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\mid 1 - \cos(xb;q^2) \mid^2 \mid \mathcal{F}_q(H_a(g)) \mid^2 (x) \leq (1 + \frac{1}{(q;q)_\infty^2})^2 \mid \mathcal{F}_q(H_a(g)) \mid^2 (x)$$

and $\mathcal{F}_q(H_a(g)) \in L^2_q(\mathbb{R}_{q,+})$. So, the Lebesgue theorem leads to

$$\lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_{q,+}}} \| F(a,b) - F(a,0) \|_{q,2} = 0.$$

Then, for all open neighborhood V of $F(a,0)$ in $L^2_q(\mathbb{R}_{q,+})$, there exists an open neighborhood U of 0 in $\widetilde{\mathbb{R}}_{q,+}$ such that, $\forall b \in U$, we have

$$F(a,b) \in V.$$

Thus, $\{a\} \times U$ is an open neighborhood of $(a,0)$ in $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ and $F(\{a\} \times U) \subset V$. The continuity of F at $(a,0)$ holds. ■

Definition 2. Let g be a q-wavelet in $\mathcal{D}_{*q}(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the operator Δ_q by

$$\Phi_{q,g}(f)(a,b) = c_q \int_0^\infty f(x)\overline{g_{a,b}}(x)d_qx, \quad a \in \mathbb{R}_{q,+}, \quad b \in \widetilde{\mathbb{R}}_{q,+} \quad \text{and} \quad f \in \mathcal{D}_{*q}(\mathbb{R}_q), \tag{4.5}$$

which is equivalent to

$$\begin{aligned}
\Phi_{q,g}(f)(a,b) &= f *_q \overline{H_a(g)}(b) \\
&= \mathcal{F}_q(\mathcal{F}_q(f *_q \overline{H_a(g)}))(b) \\
&= \mathcal{F}_q\left[\mathcal{F}_q(f).\mathcal{F}_q(H_a(\overline{g}))\right](b) \\
&= \sqrt{a}\ c_q \int_0^\infty \mathcal{F}_q(f)(x).\mathcal{F}_q(\overline{g})(ax)\cos(bx;q^2)d_qx,
\end{aligned}$$

where c_q is given by (3.10).

We establish some properties of $\Phi_{q,g}$ in the two following propositions.

Proposition 7. *Let g be a q-wavelet in $L_q^2(\mathbb{R}_{q,+})$ and $f \in L_q^2(\mathbb{R}_{q,+})$. Then*
i) for all $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\mid \Phi_{q,g}(f)(a,b) \mid \leq \frac{c_q}{(q;q)_\infty^2}\|f\|_{q,2}\|g\|_{q,2}; \tag{4.6}$$

ii) for all $a \in \mathbb{R}_{q,+}$, the mapping $b \mapsto \Phi_{q,g}(f)(a,b)$ is continuous on $\widetilde{\mathbb{R}}_{q,+}$ and we have

$$\lim_{b\to\infty} \Phi_{q,g}(f)(a,b) = 0. \tag{4.7}$$

Proof. i) From the relation (3.18), we have for $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$,

$$\begin{aligned}
\mid \Phi_{q,g}(f)(a,b) \mid &= c_q \mid \int_0^\infty f(x)\overline{g_{a,b}}(x)d_qx \mid \\
&\leq c_q \int_0^\infty \mid T_{q,b}f(x) \mid \frac{1}{\sqrt{a}} \mid g(\frac{x}{a}) \mid d_qx \\
&\leq \frac{c_q}{(q;q)_\infty^2}\|f\|_{q,2}\|g\|_{q,2}.
\end{aligned}$$

ii) It is sufficient to prove the continuity at 0. For $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\Phi_{q,g}(f)(a,b) = \mathcal{F}_q\left[\mathcal{F}_q(f).\mathcal{F}_q(H_a(\overline{g}))\right](b)$$

and

$$\forall x \in \mathbb{R}_{q,+}, \quad \mid \cos(bx;q^2) \mid \leq \frac{1}{(q;q)_\infty^2}.$$

Since $f, g \in L_q^2(\mathbb{R}_{q,+})$, then from Theorem 4, we have $\mathcal{F}_q(f)$ and $\mathcal{F}_q(H_a(\overline{g}))$ are in $L_q^2(\mathbb{R}_{q,+})$. and the product $\mathcal{F}_q(f).\mathcal{F}_q(H_a(\overline{g}))$ is in $L_q^1(\mathbb{R}_{q,+})$. Thus, the Lebesgue theorem, gives

$$\begin{aligned}
\lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_{q,+}}} \Phi_{q,g}(f)(a,b) &= \lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_{q,+}}} c_q \int_0^\infty \mathcal{F}_q(f)(x).\mathcal{F}_q(H_a(\overline{g}))(x)\cos(bx;q^2)d_qx \\
&= \Phi_{q,g}(f)(a,0).
\end{aligned}$$

Which proves the continuity of $\Phi_{q,g}(f)(a,.)$ at 0.
Finally, (3.11) implies that

$$\Phi_{q,g}(a,b) = \sqrt{a}\mathcal{F}_q[\mathcal{F}_q(f).\mathcal{F}_q(H_a(\overline{g}))](b)$$

tends to 0 when b tends to ∞. ■

Proposition 8. *i) For all $a \in \mathbb{R}_{q,+}$ and $f \in L^p_q(\mathbb{R}_{q,+})$, $p \in]1,2]$, the mapping $b \mapsto \Phi_{q,g}(f)(a,b)$ is in $L^r_q(\mathbb{R}_{q,+})$, with $r \in [1,\infty]$ such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and we have*

$$\|\Phi_{q,g}(f)(a,.)\|_{q,r} \leq B_{q,p}B_{q,2}B_{q,r'}\|f\|_{q,p}\|g\|_{q,2}, \tag{4.8}$$

where $B_{q,p} = (\frac{1}{(q(1-q))^{\frac{1}{2}}(q;q)_\infty})^{1-\frac{p}{2}}$ and r' verifies $\frac{1}{r} + \frac{1}{r'} = 1$.

ii) If g is in $\mathcal{S}_{*q}(\mathbb{R}_q)$, then for all f in $\mathcal{S}_{*q}(\mathbb{R}_q)$, the mapping $b \mapsto \Phi_{q,g}(f)(a,b)$ is in $\mathcal{S}_{*q}(\mathbb{R}_q)$.

Proof. The proof is easily deduced from the relation

$$\Phi_{q,g}(f)(a,b) = f *_q \overline{H_a(g)}(b)$$

and the properties of the q-convolution product. ■

Theorem 6. *Let $g \in L^2_q(\mathbb{R}_{q,+})$ be a q-wavelet.*
i) **Plancheral formula for $\Phi_{q,g}$**
For $f \in L^2_q(\mathbb{R}_{q,+})$, we have

$$\frac{1}{C_g}\int_0^\infty\int_0^\infty \mid \Phi_{q,g}(f)(a,b) \mid^2 \frac{d_qad_qb}{a^2} = \|f\|^2_{q,2}. \tag{4.9}$$

ii) **Parseval formula for $\Phi_{q,g}$**
For $f_1, f_2 \in L^2_q(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty f_1(x)\overline{f}_2(x)d_qx = \frac{1}{C_g}\int_0^\infty\int_0^\infty \Phi_{q,g}(f_1)(a,b)\overline{\Phi_{q,g}}(f_2)(a,b)\frac{d_qad_qb}{a^2}. \tag{4.10}$$

Proof. The use of Fubini's theorem, Theorem 4, Proposition 2 and the relation (3.24) gives

$$\begin{aligned}
\int_0^\infty\int_0^\infty \mid \Phi_{q,g}(f)(a,b) \mid^2 \frac{d_qd_q}{a^2} &= \int_0^\infty\left(\int_0^\infty \mid f *_q H_a o\overline{g} \mid^2 (b)d_qb\right)\frac{d_qa}{a^2}\\
&= \int_0^\infty\left(\int_0^\infty \mid \mathcal{F}_q(f)(x) \mid^2\mid \mathcal{F}_q(H_a o\overline{g}).\mid^2 (x)d_qx\right)\frac{d_qa}{a^2}\\
&= \int_0^\infty \mid \mathcal{F}_q(f)(x) \mid^2 \left(\int_0^\infty \mid \mathcal{F}_q(g)(ax) \mid^2 \frac{d_qa}{a}\right)d_qx\\
&= C_g\int_0^\infty \mid \mathcal{F}_q(f)(x) \mid^2 d_qx = C_g\|f\|^2_{q,2}.
\end{aligned}$$

The relation (4.9) is then proved. ■

ii) The result follows from (4.9).

Remark 4.

If $g \in Ł_q^2(\mathbb{R}_{q,+})$ is a q-wavelet, then for all $f \in Ł_q^2(\mathbb{R}_{q,+})$, we have $\Phi_{q,g}(f) \in Ł_q^2(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; \dfrac{d_q a d_q b}{a^2})$ and

$$\|\Phi_{q,g}(f)\|^2_{Ł_q^2\left(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; \frac{d_q a d_q b}{a^2}\right)} = C_g \|f\|^2_{q,2}.$$

Theorem 7. *Let g be a q-wavelet in $Ł_q^2(\mathbb{R}_{q,+})$, then for $f \in Ł_q^2(\mathbb{R}_{q,+})$, we have*

$$f(x) = \frac{c_q}{C_g} \int_0^\infty \int_0^\infty \Phi_{q,g}(f)(a,b) g_{a,b}(x) \frac{d_q a d_q b}{a^2}, \quad x \in \mathbb{R}_{q,+}. \tag{4.11}$$

Proof. According to the relation (4.10) of the previous theorem and the definition of $\Phi_{q,g}$, we have for all $h \in Ł_q^2(\mathbb{R}_{q,+})$,

$$\begin{aligned} \int_0^\infty f(t)\overline{h}(t) d_q t &= \frac{c_q}{C_g} \int_0^\infty \int_0^\infty \Phi_{q,g}(f)(a,b) \left(\int_0^\infty \overline{h}(t) g_{a,b}(t) d_q t \right) \frac{d_q a d_q b}{a^2} \\ &= \int_0^\infty \left(\frac{c_q}{C_g} \int_0^\infty \int_0^\infty \Phi_{q,g}(f)(a,b) g_{a,b}(t) \frac{d_q a d_q b}{a^2} \right) \overline{h}(t) d_q t. \end{aligned}$$

Now, let $x \in \mathbb{R}_{q,+}$ and $h = \delta_x$. We have $h \in Ł_q^2(\mathbb{R}_{q,+})$ and the previous equality is equivalent to

$$f(x) = \frac{c_q}{C_g} \int_0^\infty \int_0^\infty \Phi_{q,g}(f)(a,b) g_{a,b}(x) \frac{d_q a d_q b}{a^2}.$$

■

5 Coherent states

Theorem 6 shows that the continuous q-wavelet transform $\Phi_{q,g}$ is an isometry from the Hilbert space $Ł_q^2(\mathbb{R}_{q,+})$ into the Hilbert space $Ł_q^2(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; \frac{d_q a d_q b}{a^2 C_g})$ (the space of square integrable functions on $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ with respect to the measure $\frac{d_q a d_q b}{a^2 C_g}$). For the characterization of the image of $\Phi_{q,g}$, we think of the vectors $g_{a,b}$, $(a,b) \in \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$, as a set of coherent states in the Hilbert space $Ł_q^2(\mathbb{R}_{q,+})$ (see [15]).

Definition 3. A set of coherent states in a Hilbert space $\mathcal{H}$ is a subset $\{g_l\}_{l \in \mathcal{L}}$ of $\mathcal{H}$ such that

i) $\mathcal{L}$ is a locally compact topological space and the mapping $l \mapsto g_l$ is continuous from $\mathcal{L}$ into $\mathcal{H}$.

ii) There is a positive Borel measure dl on $\mathcal{L}$ such that, for $f \in \mathcal{H}$,

$$\| f \|^2 = \int_{\mathcal{L}} | (f, g_l) |^2 \, dl,$$

where $(.,.)$ and $\| . \|$ are respectively the scalar product and the norm of $\mathcal{H}$.

Let now $\mathcal{H} = Ł_q^2(\mathbb{R}_{q,+})$, $\mathcal{L} = \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ equipped with the induced topology of $\mathbb{R}^2$. Choose a nonzero function $g \in Ł_q^2(\mathbb{R}_{q,+})$ and put $g_l = g_{a,b}$ given by the relation (4.3) with $l = (a, b) \in \mathcal{L}$. Then we have a set of coherent states. Indeed, i) of Definition 3 is satisfied, because of Proposition 6, and ii) of Definition 3 is also satisfied, for the measure $\dfrac{d_q a d_q b}{a^2 C_g}$ (see Theorem 6). By adaptation of the approach introduced by T. H. Koornwinder in [15], we obtain the following result:

Theorem 8. *Let F be in $Ł_q^2(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; \frac{d_q a d_q b}{a^2 C_g})$. Then F belongs to $Im\Phi_{q,g}$ if and only if*

$$F(a,b) = \frac{1}{C_g} \int_0^\infty \int_0^\infty F(a', b') \left(\int_0^\infty g_{a',b'}(x) \overline{g_{a,b}}(x) d_q x \right) \frac{d_q a' d_q b'}{(a')^2}. \tag{5.1}$$

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