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Isometric Reflectionless Eigenfunction Transforms for Higher-order AΔOs

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Abstract

In a previous paper (*Regular and Chaotic Dynamics* **7** (2002), 351–391, Ref. [1]), we obtained various results concerning reflectionless Hilbert space transforms arising from a general Cauchy system. Here we extend these results, proving in particular an isometry property conjectured in Ref. [1]. Crucial input for the proof comes from previous work on a special class of relativistic Calogero-Moser systems. Specifically, we exploit results on action-angle maps for the pertinent systems and their relation to the 2D Toda soliton tau-functions. The reflectionless transforms may be viewed as eigenfunction transforms for an algebra of higher-order analytic difference operators.

1 Introduction

In previous work we studied reflectionless second-order AΔOs (analytic difference operators) of Shabat type [1, 2, 3] and of Toda type [4, 5, 6, 7, 8] (cf. also [9, 10]). Their eigenfunctions are of the form

$$\mathcal{W}(a,b,\mu;x,p) = e^{ixp}\left(1 - \sum_{k=1}^{N} \frac{R_k(a,b,\mu;x)}{e^p - b_k}\right) \tag{1.1}$$

and satisfy

$$\mathcal{W}(x,p) \sim \exp(ixp), \qquad \mathrm{Re}\, x \to \infty, \tag{1.2}$$

$$\mathcal{W}(x,p) \sim a(p)\exp(ixp), \qquad \mathrm{Re}\, x \to -\infty, \tag{1.3}$$

where

$$a(p) = \prod_{k=1}^{N} \frac{e^p - a_k}{e^p - b_k}. \tag{1.4}$$

Here we have $a, b \in \mathbb{C}^N$, whereas the generalized norming 'constants', $\mu_1(x), \ldots, \mu_N(x)$, are i-periodic functions in the space

$$\mathcal{M}^* \equiv \{f \text{ meromorphic, not identically zero}\}, \tag{1.5}$$

obeying further restrictions, cf. Section 2. The 'residue functions', $R_1(x), \ldots, R_N(x) \in \mathcal{M}^*$, solve a linear Cauchy system we also specify in Section 2.

As was shown in [1], a great many features of the wave functions $\mathcal{W}(x, p)$ corresponding to these two classes of AΔOs hold true for an extensive set of parameters, a, b, and μ, that encompasses both the Shabat and Toda parameters. Moreover, for constant $\mu \in \mathbb{C}^{*N}$ and under suitable reality restrictions, the same is true for the eigenfunction transforms (generalized Fourier transforms)

$$\begin{aligned} \mathcal{F} \,:\, \mathcal{H}_p \equiv L^2(\mathbb{R}, dp) \quad &\to \quad \mathcal{H}_x \equiv L^2(\mathbb{R}, dx) \\ \phi(p) \quad &\mapsto \quad (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp \mathcal{W}(a, b, \mu; x, p)\phi(p). \end{aligned} \tag{1.6}$$

For the general case, however, we could not prove the isometry of $\mathcal{F}$, though we stated a conjecture (cf. [1] (4.41)). A principal aim of this paper is to prove this isometry conjecture.

For the Shabat and Toda specializations we could show the critical isometry property of $\mathcal{F}$ via the time-dependent scattering theory associated to the relevant second-order AΔOs. For the far larger parameter set at issue here, we are still able to show that the transforms $\mathcal{F}$ diagonalize an algebra of higher-order AΔOs. It appears out of the question, however, to use these AΔOs in the same way as before to arrive at an isometry proof.

Accordingly, the proof presented below proceeds along quite different lines, not involving AΔOs at all. As a starting point, we summarize in Section 2 various findings from [1] regarding the general situation that play a role in the proof. Of particular relevance is Theorem 4.2 in [1] (cf. Theorem 2.5 below), inasmuch as this amounts to an isometry result for $\mathcal{F}^*$ when the integer N_+ vanishes.

In Section 3 we exploit the latter isometry feature and a duality argument to obtain isometry of $\mathcal{F}$ for a special class of parameters. For the latter we can express the reflectionless wave function kernel $\mathcal{W}(x, p)$ of $\mathcal{F}$ in terms of the positions and momenta of a class of relativistic Calogero-Moser systems. The pertinent class $\mathrm{II}_{\mathrm{rel}}(c, N_+, N_-)$, $c \in (0, \pi/2]$, was studied in great detail in [11]. We have occasion to use some of the findings of [11], especially as concerns the spectral characteristics of the Lax and dual Lax matrices. They enable us to show that $\mathcal{W}(x, p)$ admits an alternative ('dual') representation such that isometry of $\mathcal{F}$ can be deduced from the $N_+ = 0$ isometry result for $\mathcal{F}^*$ already obtained in [1].

Section 4 is concerned with isometry for general parameters. The main idea is to exploit the isometry for the specializations obtained in Section 3; it enables us to deduce isometry for a dense set in the full parameter space from an analytic continuation reasoning. This strategy involves some formalism involving an 'adjoint wave function' and various constructions from our paper [12]. More specifically, the notion of 'fusion' for 2D Toda soliton tau-functions plays a pivotal role, just as it did in [12]. Using additional analytic continuation arguments we can then obtain isometry for the parameters already specified in [1] to complete the proof.

We should add that the previous paragraph attempts to give a very concise summary of our general isometry proof. The technical details are quite involved, and we have relegated long proofs of two key lemmas to Appendix A. Moreover, we have tried to follow a line of exposition that explains the main ideas before we embark on the technicalities and somewhat elaborate notation we were unable to avoid. (Unfortunately, we simply did not find an easy answer to the main question we address: How do you recognize an isometric operator when you see one?)

As we mentioned above, AΔOs play no role in the isometry proof. Even so, all of the isometric transforms may be viewed as eigenfunction transforms for AΔOs whose order is in general larger than two. More precisely, when the 'pole number' N is greater than one, then $\mathcal{W}(x,p)$ can only be an eigenfunction for a second-order AΔO for nongeneric parameters, whereas there are plenty of AΔOs whose order is greater than N and for which $\mathcal{W}(x,p)$ is an eigenfunction. Due to our isometry results we can very easily address self-adjointness questions, which would appear quite elusive otherwise. We devote Section 5 to a study of these matters.

Section 6 concludes the paper with a number of remarks. The first two remarks are mostly concerned with data (a,b,μ) for which we do not know whether $\mathcal{F}$ is an isometry. This issue is intimately connected to the eventual occurrence of tau-function zeros for real x, which is the subject of remark (i). In remark (ii) we collect some open problems that center around the character of the data subset where $\mathcal{F}$ might fail to be isometric.

Remark (iii) concerns eventual relations to hierarchies of nonlocal soliton evolution equations. Though this is a quite interesting issue, we have very little to offer here. Indeed, just as in our previous work on the special Shabat and Toda cases, the 'soliton perspective' can be studied independently of isometry questions, but to date we simply have not done this.

As already mentioned, we have occasion to use various results from our previous papers [11] and [12]. On the other hand, the connection between the $\mathrm{II}_{\mathrm{rel}}(c,N_+,N_-)$ systems studied in [11] and the fusion construction from [12] must be made in a different way in the present setting. To bring this out more clearly we compare the two constructions in remark (iv).

2 Some results from [1]

In this section we collect definitions and results from [1] we have occasion to use. We begin by specifying the range of variation for the data a,b and $\mu(x)$. Let

$$\mathbb{C}_- \equiv \mathbb{C} \setminus [0,\infty). \tag{2.1}$$

Then (a,b) varies over

$$\Pi_-(N) \equiv \{(a,b) \in \mathbb{C}^{2N} \mid a_j, b_j, a_j/b_j \in \mathbb{C}_-,\ \ j=1,\ldots,N,\ \ a_1,\ldots,a_N,b_1,\ldots,b_N \text{ distinct}\}. \tag{2.2}$$

Moreover, setting

$$\mathcal{R}(c) \equiv \{\nu(x) = F(\exp(2\pi x)) \mid F(z) \text{ rational}, F(0)=c, F(\infty)\neq 0\}, \quad c\in\mathbb{C}^*, \tag{2.3}$$

we choose

$$\mu_n(x) \in \mathcal{R}(c_n), \qquad n=1,\ldots,N. \tag{2.4}$$

Next we introduce the Cauchy matrix

$$C(a,b)_{jk} \equiv \frac{1}{a_j - b_k} \tag{2.5}$$

and the diagonal matrix

$$D(a,b,\mu;x) \equiv \mathrm{diag}(d(a_1,b_1,\mu_1;x),\ldots,d(a_N,b_N,\mu_N;x)) \tag{2.6}$$

where d is defined by

$$d(\alpha,\beta,\nu;x) \equiv \nu(x)\exp(-ix\mathrm{Ln}(\alpha/\beta)), \quad \nu \in \mathcal{R}(c), \quad \alpha/\beta \in \mathbb{C}_- \tag{2.7}$$

The branch of the logarithm occurring here is fixed by

$$\mathrm{Ln}(w) \equiv \ln(|w|) + i\mathrm{Arg}(w), \quad \mathrm{Arg}(w) \in (0,2\pi), \quad w \in \mathbb{C}_-, \tag{2.8}$$

and

$$\ln r \in \mathbb{R}, \quad r \in (0,\infty). \tag{2.9}$$

Now the vector function $R = (R_1,\ldots,R_N)$ is the solution to the Cauchy system

$$[D(a,b,\mu;x) + C(a,b)]R = \zeta, \quad \zeta \equiv (1,\ldots,1)^t \in \mathbb{R}^N. \tag{2.10}$$

Besides the wave function $\mathcal{W}(x,p)$ (1.1) and the (reciprocal) transmission coefficient $a(p)$ (1.4) we need the tau-function

$$\tau(x) \equiv |\mathbf{1}_N + D(x)^{-1}C|. \tag{2.11}$$

We proceed by collecting crucial properties of the above functions. We begin with the solution R to the system (2.10). By Cramer's rule and (2.11), it can be written as

$$R_n(x) = E_n(x)/\tau(x), \quad n = 1,\ldots,N, \tag{2.12}$$

where the functions $E_n(x)$ are entire whenever all of $\mu_1(x),\ldots,\mu_N(x)$ are constant. Less obvious features now follow, cf. Lemma 2.1 in [1].

Lemma 2.1. (**Properties of** $R(x)$) *We have* $R(x) \in \mathcal{M}^{*N}$. *The limits of* $R(x)$ *for* $|\mathrm{Re}\,x| \to \infty$ *exist and are given by*

$$\lim_{\mathrm{Re}\,x\to\infty} R(x) = 0, \tag{2.13}$$

$$\lim_{\mathrm{Re}\,x\to-\infty} R(x) = C^{-1}\zeta. \tag{2.14}$$

More precisely we have

$$R_n(x) = O(\exp[-\mathrm{Arg}(a_n/b_n)\mathrm{Re}\,x]), \quad \mathrm{Re}\,x \to \infty, \quad n = 1,\ldots,N, \tag{2.15}$$

$$R_n(x) - (C^{-1}\zeta)_n = O(\exp(\rho\mathrm{Re}\,x)), \quad \mathrm{Re}\,x \to -\infty, \quad n = 1,\ldots,N, \tag{2.16}$$

where

$$\rho \equiv \min_{n\in\{1,\ldots,N\}} \mathrm{Arg}(a_n/b_n). \tag{2.17}$$

The bounds are uniform for $\mathrm{Im}\,x$ *in compact subsets of* $\mathbb{R}$. *Moreover, for* $\mu(x)$ *constant the bounds are uniform for* (a,b,μ) *in compact subsets of* $\Pi_-(N) \times \mathbb{C}^{*N}$, *and* $R(a,b,\mu;x)$ *is meromorphic in* a,b *and* μ *for* $(a,b,\mu) \in \Pi_-(N) \times \mathbb{C}^{*N}$.

We continue with properties of the functions (1.4) and (2.11), cf. [1] Lemma 2.2. For $m(x) \in \mathcal{M}$ we define the conjugate meromorphic function by

$$m^*(x) \equiv \overline{m(\overline{x})}, \quad x \in \mathbb{C}. \tag{2.18}$$

Lemma 2.2. (**Properties of** $a(p)$ and $\tau(x)$) *We have the identity*

$$a(p) = 1 - \sum_{k=1}^{N} \frac{(C^{-1}\zeta)_k}{e^p - b_k}. \tag{2.19}$$

Now suppose that a, b *and* $\mu(x)$ *are restricted by*

$$a_j = \overline{b}_j, \quad j = 1, \dots, N, \tag{2.20}$$

$$\mu_j^*(x) = -\mu_j(x) b_j / a_j, \quad j = 1, \dots, N. \tag{2.21}$$

Then we have

$$\tau^*(x) = \tau(x - i). \tag{2.22}$$

Finally we focus on $\mathcal{W}(x,p)$, cf. [1] Lemma 2.3.

Lemma 2.3. (**Properties of** $\mathcal{W}(x,p)$) *We have the limits*

$$\lim_{\mathrm{Re}\, x \to \infty} e^{-ixp} \mathcal{W}(x,p) = 1, \qquad \lim_{\mathrm{Re}\, x \to -\infty} e^{-ixp} \mathcal{W}(x,p) = a(p), \tag{2.23}$$

and the identity

$$\mathcal{W}(x,p) = e^{ixp} | \mathbf{1}_N + \mathrm{diag}((e^p - a_1)/(e^p - b_1), \dots, (e^p - a_N)/(e^p - b_N)).D(x)^{-1} C | / \tau(x) \tag{2.24}$$

We now turn to features of the transform $\mathcal{F}$ (1.6) already established in [1]. For the case of general data (a, b, μ) given by (2.2)–(2.4) it need not be true that $R(x)$ has no poles on the real axis. Generically, however, this is the case. Clearly, for $\mathcal{F}$ to be a bounded operator we should insist on the absence of real poles. If we do so, $\mathcal{F}$ is indeed bounded. This is one of the results for general data listed in the following theorem (cf. Theorem 4.1 in [1]).

Theorem 2.4. *Suppose that the data* (a, b, μ) *satisfy (2.2)–(2.4) and* $R(x)$ *has no real poles. Then* $\mathcal{W}(x,p), p \in \mathbb{C}$*, satisfies*

$$p + 2\pi i k \neq \mathrm{Ln}(a_1), \dots, \mathrm{Ln}(a_N), \quad \forall k \in \mathbb{Z} \Rightarrow \mathcal{W}(x,p) \notin \mathcal{H}_x, \tag{2.25}$$

and

$$\mathcal{W}(x, \mathrm{Ln}(a_n)) \in \mathcal{H}_x \Leftrightarrow \mathrm{Arg}(a_n) < \mathrm{Arg}(b_n), \quad n = 1, \dots, N. \tag{2.26}$$

The transform $\mathcal{F}$ *(1.6) is a bounded operator with adjoint given by*

$$(\mathcal{F}^* \psi)(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx \overline{\mathcal{W}}(a, b, \mu; x, p) \psi(x), \quad \psi \in \mathcal{H}_x. \tag{2.27}$$

When $\mu(x)$ is nonconstant, it seems quite unlikely that $\mathcal{F}$ can be isometric. Accordingly we assume $\mu \in \mathbb{C}^{*N}$ from now on (so that $\mu_n = c_n$, cf. (2.4)). For the same reason, we assume that a and b are related by (2.20). Therefore b_j either belongs to the (open) upper half plane (UHP) or to the lower half plane (LHP). By permutation invariance we may choose

$$b_1, \ldots, b_{N_+} \in \mathrm{LHP}, \quad b_{N-N_-+1}, \ldots, b_N \in \mathrm{UHP}, \quad N_+ + N_- = N. \tag{2.28}$$

Thus Theorem 2.4 entails

$$\psi_n(x) \equiv \mathcal{W}(x, \mathrm{Ln}(\overline{b}_n)) \in \mathcal{H}_x, \quad n = 1, \ldots, N_+, \tag{2.29}$$

whereas $\mathcal{W}(x,p), x \in \mathbb{R}$, is not square-integrable for other values of p (mod $2\pi i$, of course). By virtue of the asymptotic behavior

$$\psi_n(x) \sim c_n(C^{-1}\zeta)_n \exp(x[2\pi + i\mathrm{Ln}(b_n)]), \quad \mathrm{Re}\, x \to -\infty \tag{2.30}$$

(cf. [1] (4.21)) and the distinctness of $b_1, \ldots, b_{N_+}$, the vectors $\psi_1, \ldots, \psi_{N_+} \in \mathcal{H}_x$ are linearly independent. Setting

$$\mathcal{H}_{bs} \equiv \mathrm{Span}(\psi_1, \ldots, \psi_{N_+}), \tag{2.31}$$

we deduce

$$\dim(\mathcal{H}_{bs}) = N_+. \tag{2.32}$$

For the next result we need a further restriction on μ: It should be of the form

$$\mu_n \equiv (ib_n\nu_n)^{-1}, \quad \nu_n \in \mathbb{R}^*, \quad n = 1, \ldots, N. \tag{2.33}$$

(Observe this implies (2.21).) Then the following theorem amounts to [1] Theorem 4.2.

Theorem 2.5. *Suppose that the data a, b and μ satisfy (2.28), (2.20) and (2.33), and assume $R(x)$ has no real poles. Then we have*

$$(\mathcal{F}^* f_1, \mathcal{F}^* f_2) = (f_1, f_2) - \sum_{n=1}^{N_+} \nu_n (f_1, \psi_n)(\psi_n, f_2), \quad \forall f_1, f_2 \in C_0^\infty(\mathbb{R}), \tag{2.34}$$

with ψ_n given by (2.29).

The next result is of a very general nature, cf. Theorem 4.3 in [1].

Theorem 2.6. *Assume that $\mathcal{F}$ is an isometry satisfying (2.34), with $\psi_1, \ldots, \psi_{N_+}$ linearly independent and $\nu_1, \ldots, \nu_{N_+} \in \mathbb{C}^*$. Then we have*

$$\nu_1, \ldots, \nu_{N_+} \in (0, \infty) \tag{2.35}$$

and

$$(\psi_m, \psi_n) = \nu_n^{-1} \delta_{nm}, \quad n, m = 1, \ldots, N_+. \tag{2.36}$$

On account of this theorem (2.35) is necessary for $\mathcal{F}$ to be isometric. More generally we believe, but cannot prove, that $\nu_{N_++1},\ldots,\nu_N$ must also be positive for $\mathcal{F}$ to be an isometry. Therefore we require henceforth that

$$\nu_1,\ldots,\nu_N \in (0,\infty), \tag{2.37}$$

together with our previous assumptions, namely,

$$a_n = \overline{b_n}, \quad \mu_n(x) = (i\nu_n b_n)^{-1}, \quad n = 1,\ldots,N, \tag{2.38}$$

$$b_1,\ldots,b_{N_+} \in \mathrm{LHP}, \quad b_{N_++1},\ldots,b_N \in \mathrm{UHP}, \quad b_1,\ldots,b_N,\overline{b_1},\ldots,\overline{b_N} \text{ distinct}. \tag{2.39}$$

From now on we denote the set of parameters (a,b,μ) obeying (2.37)–(2.39) by $\mathcal{P}(N)$.

To prove isometry of $\mathcal{F}$ we need one more restriction on (a,b,μ): We require that

$$\tau(a,b,\mu;x) \neq 0, \quad \forall x \in \mathcal{S}, \tag{2.40}$$

where $\mathcal{S}$ is the strip

$$\mathcal{S} \equiv \{x \in \mathbb{C} \mid \operatorname{Im} x \in [-1,0]\}. \tag{2.41}$$

In view of (2.12) and the entireness of $E_n(x)$ for constant μ, this last assumption implies in particular that $R(x)$ has no real poles. In Section 6 we show by example that (2.40) need not hold for all $(a,b,\mu) \in \mathcal{P}(N)$. Moreover, for the examples in hand isometry of $\mathcal{F}$ indeed breaks down. (It seems plausible that this happens whenever (2.40) is violated.)

At face value our final restriction (2.40) may seem elusive (especially as compared to the restrictions (2.37)–(2.39)). In fact, however, it can be shown to hold true when either N_+ or N_- vanishes. By contrast, for $N_+N_- > 0$ it is indeed far less accessible. On the other hand, when the numbers $b_1,\ldots,b_{N_+}$ lie in the fourth quadrant and the numbers $b_{N_++1},\ldots,b_N$ in the second one, (2.40) is valid. These facts are summarized in the last theorem of this section (Theorem 4.4 in [1]).

Theorem 2.7. *Assume that the parameters (a,b,μ) belong to $\mathcal{P}(N)$. Then (2.40) holds for $N_- = 0$ and for $N_+ = 0$; for $N_+N_- > 0$, (2.40) is satisfied when*

$$\operatorname{Re}(b_j) \geq 0, \quad j = 1,\ldots,N_+, \quad \operatorname{Re}(b_{N_++l}) \leq 0, \quad l = 1,\ldots,N_-. \tag{2.42}$$

3 Isometry for the $\mathrm{II}_{\mathrm{rel}}(c,N_+,N_-)$ parameters

In this section we show that $\mathcal{F}$ is an isometry for the special parameters

$$b_j = \begin{cases} \rho_j e^{-ic}, & j = 1,\ldots,N_+, \\ -\rho_j e^{-ic}, & j = N_+ + 1,\ldots,N, \end{cases} \quad a_j = \overline{b}_j, \ \ \mu_j = (ib_j\nu_j)^{-1}, \ \ j = 1,\ldots,N, \tag{3.1}$$

where

$$c \in (0,\pi/2], \tag{3.2}$$

$$0 < \rho_{N_+} < \cdots < \rho_1, \quad 0 < \rho_N < \cdots < \rho_{N_++1}, \quad \nu_1,\ldots,\nu_N \in (0,\infty), \tag{3.3}$$

$$\rho_j \neq \rho_k, \quad j \neq k \quad (c = \pi/2). \tag{3.4}$$

Consequently the assumptions (2.37)–(2.39) and (2.42) of Theorem 2.7 hold true. (Observe that the extra restriction (3.4) for $c = \pi/2$ is needed to ensure distinctness of $b_1, \ldots, b_N, \overline{b}_1, \ldots, \overline{b}_N$.) In particular, this entails that $R(x)$ has no real poles, so that the assumptions of Theorem 2.5 are met as well.

We proceed to tie in the above parameter choices with the relativistic integrable N-particle systems denoted $\mathrm{II}_{\mathrm{rel}}(c, N_+, N_-)$ in [12], which we previously studied in [11]. To this end we introduce particle and antiparticle positions by setting

$$\rho_j = \exp(x_j^+), \quad j = 1, \ldots, N_+, \quad \rho_{N_+ + l} = \exp(x_l^-), \quad l = 1, \ldots, N_-, \tag{3.5}$$

and define the positive pair potentials

$$f(c; x) \equiv [1 + \sin^2(c)/\sinh^2(x/2)]^{1/2}, \quad c \in (0, \pi/2], \quad x \in \mathbb{R}^*, \tag{3.6}$$

$$\tilde{f}(c; x) \equiv [1 - \sin^2(c)/\cosh^2(x/2)]^{1/2}, \quad c \in (0, \pi/2], \quad x \in \mathbb{R}, \tag{3.7}$$

and the multiparticle potentials

$$V_j^+ \equiv \prod_{1 \le k \le N_+, k \ne j} f(x_j^+ - x_k^+) \prod_{1 \le l \le N_-} \tilde{f}(x_j^+ - x_l^-), \tag{3.8}$$

$$V_l^- \equiv \prod_{1 \le m \le N_-, m \ne l} f(x_l^- - x_m^-) \prod_{1 \le j \le N_+} \tilde{f}(x_l^- - x_j^+). \tag{3.9}$$

(Here and in the remainder of this section the indices j, k take the values $1, \ldots, N_+$, whereas the indices l, m take the values $1, \ldots, N_-$.) Next we introduce particle and antiparticle momenta by reparametrizing ν as

$$\nu_j = 2\sin(c)\exp(p_j^+)V_j^+, \quad \nu_{N_+ + l} = 2\sin(c)\exp(p_l^-)V_l^-. \tag{3.10}$$

In order to make the connection to Section 2 in [11], we also define

$$\mathcal{D}_e \equiv \mathrm{diag}(e_1, \ldots, e_N), \tag{3.11}$$

where the vector e is defined by

$$e_j \equiv \exp[(x_j^+ + p_j^+)/2](V_j^+)^{1/2}, \tag{3.12}$$

$$e_{N_+ + l} \equiv i\exp[(x_l^- + p_l^-)/2](V_l^-)^{1/2}. \tag{3.13}$$

Then it is straightforward to verify the relation (recall (2.6)–(2.8))

$$D(a, b, \mu; x) = \frac{\exp(2cx + ic)}{2i\sin c}\mathcal{D}_e^{-2}. \tag{3.14}$$

Moreover, introducing

$$\begin{aligned} L &\equiv e^{2cx+ic}\mathcal{D}_e^{-1}(D(x)^{-1}C)\mathcal{D}_e \\ &= 2i\sin(c)\mathcal{D}_e C\mathcal{D}_e, \end{aligned} \tag{3.15}$$

we readily calculate

$$L_{jk} = i\sin(c)(V_j^+V_k^+)^{1/2}\exp[(p_j^+ + p_k^+)/2]/\sinh[(x_j^+ - x_k^+ + 2ic)/2], \tag{3.16}$$

$$L_{N_++l,N_++m} = i\sin(c)(V_l^- V_m^-)^{1/2}\exp[(p_l^- + p_m^-)/2]/\sinh[(x_l^- - x_m^- + 2ic)/2], \quad (3.17)$$

$$L_{j,N_++m} = -\sin(c)(V_j^+ V_m^-)^{1/2}\exp[(p_j^+ + p_m^-)/2]/\cosh[(x_j^+ - x_m^- + 2ic)/2], \quad (3.18)$$

$$L_{N_++l,k} = \sin(c)(V_l^- V_k^+)^{1/2}\exp[(p_l^- + p_k^+)/2]/\cosh[(x_l^- - x_k^+ + 2ic)/2]. \quad (3.19)$$

Comparing (3.12)–(3.13) and (3.16)–(3.19) to (2.69)–(2.70) in [11], we see that we obtain equality to the vector e (2.69) and Lax matrix L (2.70) when we substitute

$$\beta \to 1, \quad \mu \to 1, \quad \tau = \beta\mu g/2 \to c. \quad (3.20)$$

At this stage this correspondence may seem intriguing at best. But now we proceed to exploit it to rewrite the wave function $\mathcal{W}(x,p)$ (1.1). First, we set

$$\mathcal{W}(a,b,\mu;x,p) = e^{ixp}[1 - K(a,b,\mu;x,p)], \quad (3.21)$$

so that

$$K(a,b,\mu;x,p) = \sum_{n=1}^{N} \frac{R_n(a,b,\mu;x)}{e^p - b_n}. \quad (3.22)$$

Now from (2.10) we deduce

$$R(x) = [D(x) + C]^{-1}\zeta, \quad (3.23)$$

so that we may write K as the inner product

$$K(x,p) = (\zeta, [e^p \mathbf{1}_N - \mathcal{D}_b]^{-1}[D(x) + C]^{-1}\zeta), \quad \mathcal{D}_b \equiv \mathrm{diag}(b_1,\ldots,b_N). \quad (3.24)$$

Introducing the dual Lax matrix

$$A \equiv \mathrm{diag}(\exp(x_1^+),\ldots,\exp(x_{N_+}^+), -\exp(x_1^-),\ldots,-\exp(x_{N_-}^-)) \quad (3.25)$$

(cf. [11] (2.6)), we have

$$\mathcal{D}_b = e^{-ic}A \quad (3.26)$$

for the case at hand, cf. (3.1) and (3.5). Using also (3.14), (3.15) and (3.11) we can rewrite (3.24) as

$$K(x,p) = 2ie^{-ic}\sin(c)\,(\overline{e}, [e^p\mathbf{1}_N - e^{-ic}A]^{-1}[e^{2cx}\mathbf{1}_N + e^{-ic}L]^{-1}e). \quad (3.27)$$

Therefore K is now expressed in terms of the vector e and the matrices A and L from [11].

As a result we are in the position to invoke Section 2 in [11]. (In (2.71) there is an inconsequential error: the factors 2μ should be replaced by μ.) We restrict attention to points in the phase space Ω of the $\mathrm{II}_{\mathrm{rel}}(c,N_+,N_-)$ system where L has spectrum of the form

$$\sigma(L) = \{e^{r_1},\ldots,e^{r_N}\}, \quad r_{N_+} < \cdots < r_1, r_N < \cdots . < r_{N_++1}. \quad (3.28)$$

(For $N_+ = 0$ or $N_- = 0$ this is true on all of Ω; we should also add that we sometimes deviate from the notation used in [11] to prevent ambiguities.) Setting

$$\hat{L} = \mathrm{diag}(e^{r_1}, \dots, e^{r_N}), \tag{3.29}$$

we recall from Subsection 2A in [11] that there exists a $\mathcal{J}$-unitary U such that

$$\hat{L} = U^{-1} L U. \tag{3.30}$$

Introducing

$$\hat{A} \equiv U^{-1} A U, \quad \hat{e} \equiv U^{-1} e, \quad \tilde{e} \equiv U^t e \tag{3.31}$$

(cf. [11] (2.8)–(2.9)), we can transform (3.27) into

$$K(x,p) = 2ie^{-ic}\sin(c)\,(\overline{\tilde{e}}, [e^p \mathbf{1}_N - e^{-ic}\hat{A}]^{-1}[e^{2cx}\mathbf{1}_N + e^{-ic}\hat{L}]^{-1}\hat{e}). \tag{3.32}$$

Defining

$$\hat{b}_n \equiv -\exp(r_n - ic), \quad \hat{a}_n \equiv \overline{\hat{b}}_n, \qquad n = 1, \dots, N, \tag{3.33}$$

$$\hat{R}_n(x) \equiv 2ie^{-ic}\sin(c)\hat{e}_n \sum_{i=1}^{N} [e^{2cx}\mathbf{1}_N - e^{-ic}\hat{A}]^{-1}_{in}\tilde{e}_i, \tag{3.34}$$

we can now rewrite (3.32) as

$$K(a,b,\mu;x,p) = \sum_{n=1}^{N} \frac{\hat{R}_n(p/2c)}{e^{2cx} - \hat{b}_n}. \tag{3.35}$$

We proceed to recall [11] (2.80):

$$\hat{A}_{in} = \hat{e}_i C(1,1,-2c;r,r)_{in}\tilde{e}_n. \tag{3.36}$$

Putting

$$F_{in} \equiv \tilde{e}_i^{-1}(e^p\mathbf{1}_N - e^{-ic}\hat{A})_{ni}\hat{e}_n^{-1}, \tag{3.37}$$

we deduce on the one hand from (3.36)

$$F_{in} = \tilde{e}_i^{-1} e^p \hat{e}_n^{-1}\delta_{in} - e^{-ic}C(1,1,-2c;r,r)_{ni} \tag{3.38}$$

and on the other hand from (3.34)

$$\sum_{n=1}^{N} F_{in}\hat{R}_n(p/2c) = 2ie^{-ic}\sin c, \qquad i = 1, \dots, N. \tag{3.39}$$

Thus we have

$$\hat{R}(x) = [\hat{D}(x) + \hat{C}]^{-1}\zeta, \tag{3.40}$$

where we have introduced

$$\hat{D}(x) \equiv (2i\sin c)^{-1} e^{2cx+ic} \mathrm{diag}((\tilde{e}_1\hat{e}_1)^{-1}, \ldots, (\tilde{e}_N\hat{e}_N)^{-1}), \tag{3.41}$$

$$\hat{C} \equiv \frac{i}{2\sin c} C(1,1,-2c;r,r)^t. \tag{3.42}$$

We now use (B1) in [11] to calculate (recall (3.33))

$$\hat{C}_{in} = \frac{1}{\hat{a}_i - \hat{b}_n}. \tag{3.43}$$

Also, (2.40) and (2.42) in [11] yield

$$\tilde{e}_j\hat{e}_j \in (0,\infty), \qquad \tilde{e}_{N_++l}\hat{e}_{N_++l} \in (-\infty, 0). \tag{3.44}$$

Thus we may reparametrize $\hat{D}(x)$ (3.41) as (cf. (2.6)–(2.8))

$$\hat{D}(x) = D(\hat{a}, \hat{b}, \hat{\mu}; x), \tag{3.45}$$

where

$$\hat{\mu}_n \equiv (i\hat{b}_n\hat{\nu}_n)^{-1}, \qquad \hat{\nu}_n \equiv -2\sin(c)e^{-r_n}\tilde{e}_n\hat{e}_n \in \mathbb{R}^*, \quad n = 1, \ldots, N. \tag{3.46}$$

Comparing (3.23) and (2.5) with (3.40) and (3.43), resp., we deduce that

$$\hat{R}(x) = R(\hat{a}, \hat{b}, \hat{\mu}; x). \tag{3.47}$$

The crux is now that by (1.1), (3.47) and (3.35) we have

$$\begin{aligned} \mathcal{W}(\hat{a}, \hat{b}, \hat{\mu}; x, p) &= e^{ixp}\left(1 - \sum_{n=1}^{N} \frac{\hat{R}_n(x)}{e^p - \hat{b}_n}\right) \\ &= e^{ixp}(1 - K(a, b, \mu; p/2c, 2cx)) \\ &= \mathcal{W}(a, b, \mu; p/2c, 2cx). \end{aligned} \tag{3.48}$$

With these algebraic results at our disposal, we can turn to analysis.

Theorem 3.1. *For all of the data a, b and μ given by (3.1)–(3.4), the operator $\mathcal{F}$ is an isometry.*

Proof. The matrix $e^{2cx}\mathbf{1}_N - e^{-ic}A$ is manifestly invertible for real x (cf. (3.25)), so it is clear from (3.34) that $R(\hat{a}, \hat{b}, \hat{\mu}; x)$ has no poles for real x. As a consequence, the data $\hat{a}, \hat{b}, \hat{\mu}$ satisfy the assumptions of Theorem 2.5 with $N_+ = 0$ (since (3.33) says all $\hat{b}_n$ belong to the UHP). Using (2.27) we can now write out (2.34) and substitute (3.48) to get

$$\begin{aligned} 2\pi\int_{-\infty}^{\infty} dx \overline{f}_1(x) f_2(x) &= \int_{-\infty}^{\infty} dp \left[\int_{-\infty}^{\infty} dx_1 \mathcal{W}(a, b, \mu; p/2c, 2cx_1)\overline{f}_1(x_1)\right] \\ &\quad \times \left[\int_{-\infty}^{\infty} dx_2 \overline{\mathcal{W}}(a, b, \mu; p/2c, 2cx_2) f_2(x_2)\right]. \end{aligned} \tag{3.49}$$

Next we change variables

$$x, x_j \to q/2c, q_j/2c, \quad j = 1, 2, \qquad p \to 2cy \tag{3.50}$$

and set

$$g_j(p) \equiv \overline{f}_j(p/2c), \quad j = 1, 2. \tag{3.51}$$

Thus we obtain

$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} dq \overline{g}_2(q) g_1(q) &= \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{\infty} dq_1 \mathcal{W}(a, b, \mu; y, q_1) g_1(q_1) \right] \\ &\quad \times \left[\int_{-\infty}^{\infty} dq_2 \mathcal{W}(a, b, \mu; y, q_2) g_2(q_2) \right]^{-}, \end{aligned} \tag{3.52}$$

with $[\cdots]^-$ denoting the complex-conjugate of $[\cdots]$. Clearly, this can be rewritten as

$$(g_2, g_1) = (\mathcal{F} g_2, \mathcal{F} g_1). \tag{3.53}$$

This holds for all $g_1, g_2 \in C_0^\infty(\mathbb{R})$, so that $\mathcal{F}$ is isometric, as claimed.

For $N_+ = 0$ or $N_- = 0$, (3.28) is valid for all of the above data, so that the theorem now follows for these special cases. Assuming from now on that $N_+ N_- > 0$, the spectrum $\sigma(L)$ is given by (3.28) on a non-empty open subset Ω_+ of the relevant phase space Ω, but Ω_+ is smaller than Ω [11]. Thus we need additional arguments to obtain isometry on all of Ω.

Our reasoning exploits real-analyticity in the particle/antiparticle positions and momenta $(x^+, x^-, p^+, p^-) \in \Omega$. Turning to the details we begin by noting that real-analyticity is clear by inspection for the matrices $D(x), D(x)^{-1}$ and C. Therefore $\tau(x)$ (2.11) is real-analytic too. Fixing $x \in \mathbb{R}$ we may invoke Theorem 2.7 to infer that $\tau(x)$ cannot vanish on Ω. (Alternatively we may derive this in an illuminating way from (3.15). Indeed, it entails

$$\tau(x) = |\mathbf{1}_N + e^{-2cx} e^{-ic} L|. \tag{3.54}$$

Since $\sigma(L)$ belongs to the open right half plane on all of Ω [11], we deduce $\tau(x) \neq 0$ for real x.) Hence $R(x)$ is real-analytic on all of Ω. Thus $\mathcal{W}(x, p)$ (with $p \in \mathbb{R}$ also fixed) is real-analytic, and so is $\overline{\mathcal{W}}(x, p)$.

Consider now (3.52). Since $g_1, g_2 \in C_0^\infty(\mathbb{R})$, the integrals in brackets yield functions that are real-analytic on Ω. Now we split up the y-integral into three integrals over $(-R, R), (R, \infty)$ and $(-\infty, -R)$, where $R > 0$ is at our disposal. Obviously the first integral is real-analytic on Ω. The second one we rewrite as

$$\begin{aligned} &\int_R^\infty dy \left[\int_{-\infty}^{\infty} dp e^{iyp} g_1(p) - \sum_{n=1}^{N} R_n(a, b, \mu; y) \int_{-\infty}^{\infty} dp \frac{e^{iyp} g_1(p)}{e^p - b_n} \right] \\ &\quad \times \left[\int_{-\infty}^{\infty} dq e^{iyq} g_2(q) - \sum_{n=1}^{N} R_n(a, b, \mu; y) \int_{-\infty}^{\infty} dq \frac{e^{iyq} g_2(q)}{e^q - b_n} \right]^{-}. \end{aligned} \tag{3.55}$$

On account of the uniform asymptotics (2.15) it now follows that (3.55) is also real-analytic on Ω.

To handle the third integral we use the identity (2.19) to obtain

$$\int_{-\infty}^{-R} dy \left[\int_{-\infty}^{\infty} dp e^{iyp} a(p) g_1(p) - \sum_{n=1}^{N} \left(R_n(y) - (C^{-1}\zeta)_n \right) \int_{-\infty}^{\infty} dp \frac{e^{iyp} g_1(p)}{e^p - b_n} \right] \times \left[\int_{-\infty}^{\infty} dq e^{iyq} a(q) g_2(q) - \sum_{n=1}^{N} \left(R_n(y) - (C^{-1}\zeta)_n \right) \int_{-\infty}^{\infty} dq \frac{e^{iyq} g_2(q)}{e^q - b_n} \right]^{-}. \tag{3.56}$$

By the uniform asymptotics (2.16) the third integral (3.56) is real-analytic on Ω as well.

We have now shown that the rhs of (3.52) is real-analytic on Ω. As a consequence (3.52) holds true on the connected components of Ω that contain subsets of Ω_+. For $c \in (0, \pi/2)$ Ω is connected, so that (3.52) holds true on all of Ω. For $c = \pi/2$ the restriction $x_j^+ \neq x_l^-$ (cf. (3.4)) separates Ω into a finite number of components with distinct position orderings. Even so, each of these contains subsets of Ω_+ [11], so that (3.52) holds true on all of Ω for $c = \pi/2$ as well. (Alternatively we can fix x^+, x^- such that $x_j^+ \neq x_l^-$ and obtain (3.52) for $c = \pi/2$ by continuity in c from the limit $c \uparrow \pi/2$.) □

Corollary 3.2. *For all of the data (3.1)–(3.4) the operator* $\mathcal{F}$ *satisfies*

$$\mathcal{F}^*\mathcal{F} = \mathbf{1}, \qquad \mathcal{F}\mathcal{F}^* = \mathbf{1} - \sum_{j=1}^{N_+} \nu_j \psi_j \otimes \overline{\psi}_j, \tag{3.57}$$

with

$$(\psi_j, \psi_k) = \nu_k^{-1} \delta_{jk}. \tag{3.58}$$

Proof. This follows upon combining Theorem 3.1 with Theorems 2.5–2.7. □

To conclude this section we add some remarks. First, we note that $\mathcal{F}$ is unitary for $N_+ = 0$, in accordance with the parameters and dual parameters having the same characteristics. (Indeed, we have

$$b_l = -\exp(x_l^- - ic), \quad \hat{b}_l = -\exp(r_l - ic), \tag{3.59}$$

cf. (3.1), (3.5) and (3.33), whereas

$$\nu_l > 0, \quad \hat{\nu}_l > 0, \tag{3.60}$$

cf. (3.3) and (3.44) –(3.46), resp.)

Second, we point out that for $N_+N_- > 0$ the spectrum of L is not simple on all of Ω; in fact, for the special case $N_+ = N_-$ there are phase space points where $\sigma(L)$ consists of only one point [11]. For $N_+N_- > 0$ there is therefore no straightforward generalization of the key identity (3.48) to all of Ω. Below, though, we obtain (A.10)–(A.12) as a substitute.

Third, it is obvious from (3.27) that $K(x,p)$ is $i\pi/c$-periodic in x. Therefore, for any AΔO of the form

$$A_c = \sum_{n=-L}^{K} \gamma_n \exp(-in\pi c^{-1} d/dx), \quad \gamma_n \in \mathbb{R}, \quad K, L \in \mathbb{N}, \tag{3.61}$$

we have an eigenvalue equation

$$A_c\mathcal{W}(x,p) = E_c(p)\mathcal{W}(x,p), \quad E_c(p) \equiv \sum_{n=-L}^{K} \gamma_n \exp(n\pi p/c). \tag{3.62}$$

Since we have

$$\psi_j(x) = \mathcal{W}(x, x_j^+ + ic) \tag{3.63}$$

(recall (2.29)), we deduce that

$$A_c\psi_j(x) = E_{c,j}\psi_j(x), \quad E_{c,j} \equiv \sum_{n=-L}^{K} (-)^n \gamma_n \exp(n\pi x_j^+/c). \tag{3.64}$$

In particular, this yields

$$E_c(p) \in \mathbb{R}, \quad p \in \mathbb{R}, \quad E_{c,1}, \ldots, E_{c,N_+} \in \mathbb{R}. \tag{3.65}$$

It easily follows that we may reinterpret A_c as a self-adjoint operator on $\mathcal{H}_x$, whose action on the core (domain of essential self-adjointness [13])

$$\mathcal{C} \equiv \mathcal{F}C_0^\infty(\mathbb{R}) \oplus \mathcal{H}_{bs} \tag{3.66}$$

coincides with the AΔO-action. In Section 5 we show that $\mathcal{W}(x,p)$ and $\mathcal{F}$ play a similar role for an algebra of higher-order AΔOs with *nonconstant* coefficients. (For the latter the step size π/c in (3.61) is replaced by 1.)

4 Isometry for general parameters

As we already mentioned, the assumptions of Theorem 2.7 imply that $\mathcal{F}^*$ is a partial isometry satisfying (2.34). To prove that $\mathcal{F}$ is isometric under the same assumptions it therefore suffices to show that $\mathcal{F}\phi = 0, \phi \in \mathcal{H}_p$, entails $\phi = 0$. But to prove this directly appears intractable, so that we proceed differently (just as we did in the previous section).

Since the details of our proof are quite substantial, we first explain the main steps in general terms lest the ideas be drowned by technicalities. We henceforth denote the set of parameters $(a, b, \mu) \in \mathcal{P}(N)$ (given by (2.37)–(2.39)) that satisfy the restriction (2.42) of Theorem 2.7 by $\mathcal{P}^{(r)}(N)$.

We first focus on data in $\mathcal{P}^{(r)}(N)$ for which $R(a, b, \mu; x)$ is $i\pi/c$-periodic with $c \in (0, \pi/2]$. We denote this subset by $\mathcal{P}_c^{(r)}(N)$. At face value this periodicity requirement seems very drastic. As we have seen above, the parameters in Section 3 belong to $\mathcal{P}_c^{(r)}(N)$, but, when we fix N and c, we can only vary $2N$ real parameters (e.g., $\rho_1, \ldots, \rho_N$ and $\nu_1, \ldots, \nu_N$, cf. (3.1)–(3.4)). By contrast, $\mathcal{P}^{(r)}(N)$ involves $3N$ real parameters (e.g., $\mathrm{Re}\, b_n, \mathrm{Im}\, b_n$, and ν_n, $n = 1, \ldots, N$).

In fact, however, the parameter set

$$\mathcal{P}_p^{(r)}(N) \equiv \cup_{c\in(0,\pi/2]}\mathcal{P}_c^{(r)}(N) \tag{4.1}$$

of 'periodic' parameters is *dense* in $\mathcal{P}^{(r)}(N)$ as we now show. Consider parameters of the form

$$b_j = \begin{cases} \rho_j e^{-i\phi_j}, & j = 1, \dots, N_+, \\ -\rho_j e^{-i\phi_j}, & j = N_+ + 1, \dots, N, \end{cases}, \quad a_j = \overline{b}_j, \ \ \mu_j = (ib_j\nu_j)^{-1}, \ \ j = 1, \dots, N, \tag{4.2}$$

further restricted by

$$0 < \rho_{N_+} < \cdots < \rho_1, \quad 0 < \rho_N < \cdots < \rho_{N_+ + 1}, \quad \nu \in (0, \infty)^N. \tag{4.3}$$

It is immediate that, when we also demand $\phi_j \in (0, \pi/2), j = 1, \dots, N$, we obtain a dense subset of $\mathcal{P}^{(r)}(N)$. Instead we now require that

$$\phi_j = c_j, \quad c_j \equiv n_j c, \quad n_j \in \mathbb{N}^*, \quad j = 1, \dots, N, \tag{4.4}$$

with

$$c \in (0, c_{\max}), \quad c_{\max} \equiv \frac{\pi}{2\max(n_1, \dots, n_N)}. \tag{4.5}$$

Clearly this entails not only $\phi_j \in (0, \pi/2)$ but also that the data belong to $\mathcal{P}_c^{(r)}(N)$. (Indeed, $i\pi/c$-periodicity of $D(x)$, hence of $R(x)$, follows from (2.6)–(2.8).) The point is now that, when we let $n_1, \dots, n_N$ vary over $\mathbb{N}^*$ and c over $(0, c_{\max})$, the resulting vectors $\phi \equiv (\phi_1, \dots, \phi_N)$ are dense in $(0, \pi/2)^N$, as is easily seen. Thus it follows that $\mathcal{P}_p^{(r)}(N)$ is dense in $\mathcal{P}^{(r)}(N)$, as announced.

So why is $i\pi/c$-periodicity of $R(x)$ crucial? This is because of the following lemma, whose proof is relegated to Appendix A. (To some extent our proof is adapted from the proof of Theorem 4.2 in [1].)

Lemma 4.1. *Assume that (a, b, μ) belongs to $\mathcal{P}_c^{(r)}(N)$ for some $c \in (0, \pi/2]$. Let $\mathcal{C}_R$ denote the rectangular contour with corners $-R, R, R + i\pi/c, -R + i\pi/c$. Set*

$$\mathcal{I}_R \equiv \int_{\mathcal{C}_R} dx \mathcal{W}^*(a, b, \mu; x, q)\mathcal{W}(a, b, \mu; x, p), \quad q, p \in \mathbb{R}, \tag{4.6}$$

with

$$\mathcal{W}^*(x, q) \equiv \overline{\mathcal{W}(\overline{x}, q)}, \quad x \in \mathbb{C}, \quad q \in \mathbb{R}, \tag{4.7}$$

and suppose that $\mathcal{I}_R$ vanishes for sufficiently large R. Then $\mathcal{F}$ is isometric. Moreover, when a, b and μ satisfy the restrictions (3.1)–(3.4), we have

$$\mathcal{I}_R = 0 \qquad (R \text{ large}). \tag{4.8}$$

Of course we have already shown isometry of $\mathcal{F}$ for a, b and μ obeying (3.1)–(3.4), cf. Theorem 3.1. Thus it may seem odd that we would like to reformulate this as (4.8). The crux is, however, that we can use (4.8) as a starting point for an argument involving analytic continuation to show that (4.8) is valid for all parameters given by (4.2)–(4.5). (By contrast, the use of (3.52) as a starting point would not be viable, as becomes clear shortly.) From the first part of Lemma 4.1 we then deduce isometry of $\mathcal{F}$ for a dense subset of $\mathcal{P}^{(r)}(N)$. Rewriting isometry as (3.52) (with, as before, $g_1, g_2 \in C_0^\infty(\mathbb{R})$), we conclude just as in the previous special case (cf. (3.55)–(3.56)) that the rhs is real-analytic on $\mathcal{P}^{(r)}(N)$ in the $3N$ real variables $\operatorname{Re} b_n, \operatorname{Im} b_n$, and $\nu_n, n = 1, \ldots, N$. Likewise we obtain not only isometry on $\mathcal{P}^{(r)}(N)$, but also on the connected component of $\mathcal{P}(N)$ determined by the tau-function restriction (2.40). (We recall this yields all of $\mathcal{P}(N)$ for $N_- = 0$ and for $N_+ = 0$.) Thus we obtain the main result of this section, Theorem 4.3.

After this outline of our strategy we prepare the ground for Lemma 4.2. We are going to deduce the vanishing of $\mathcal{I}_R$ (4.6) for large R and parameters in $\mathcal{P}_c^{(r)}(N)$ of the form (4.2)–(4.5) (with further restrictions detailed shortly) from its vanishing on a tiny subset of $\mathcal{P}_c^{(r)}(M)$, where

$$M \equiv \sum_{i=1}^{N} n_i. \tag{4.9}$$

The latter is given by data of the form studied in Section 3, but with N replaced by $M > N \geq 1$. (The case $M = N$ is irrelevant.) Hence (4.8) holds true on the subset.

We arrive at the wave function for the above points in $\mathcal{P}_c^{(r)}(N)$ by starting with wave functions for the special points in $\mathcal{P}_c^{(r)}(M)$, and defining a suitable path for analytic continuation between these two types of wave functions. We continue by specifying the starting points in $\mathcal{P}_c^{(r)}(M)$. First, we set

$$n_i^+ \equiv n_i, \quad i = 1, \ldots, N_+, \quad n_i^- \equiv n_{N_+ + i}, \quad i = 1, \ldots, N_-, \tag{4.10}$$

$$M_+ \equiv \sum_{i=1}^{N_+} n_i^+, \quad M_- \equiv \sum_{i=1}^{N_-} n_i^-, \tag{4.11}$$

and choose parameters

$$b_j = \begin{cases} \rho_j e^{-ic}, & j = 1, \ldots, M_+, \\ -\rho_j e^{-ic}, & j = M_+ + 1, \ldots, M, \end{cases}, \quad a_j = \overline{b}_j, \ \ \mu_j = (i b_j \nu_j)^{-1}, \ \ j = 1, \ldots, M, \tag{4.12}$$

where

$$c \in (0, \pi/4M^2), \tag{4.13}$$

$$0 < \rho_{M_+} < \cdots < \rho_1, \quad 0 < \rho_M < \cdots < \rho_{M_+ + 1}, \quad \nu_1, \ldots, \nu_M \in (0, \infty), \tag{4.14}$$

cf. (3.1)–(3.3). Now we define particle and antiparticle positions by (3.5) with $N_+, N_- \to M_+, M_-$, and then specialize to clusters of the form

$$x^+_{n_1^+ + \cdots + n_{j-1}^+ + k} = \eta_j^+ + (n_j^+ + 1 - 2k)c, \quad j = 1, \ldots, N_+, \quad k = 1, \ldots, n_j^+, \tag{4.15}$$

$$x^-_{n_1^- + \cdots + n_{l-1}^- + m} = \eta_l^- + (n_l^- + 1 - 2m)c, \quad l = 1, \ldots, N_-, \quad m = 1, \ldots, n_l^-. \tag{4.16}$$

Here we choose cluster centers

$$\eta^+_{N_+} << \cdots << \eta_1^+, \quad \eta^-_{N_-} << \cdots << \eta_1^-, \tag{4.17}$$

where << denotes distances large enough so that the clusters are separated. (In view of (4.13) this is already true for distances larger than $\pi/2M$.) Defining particle and antiparticle momenta via (3.10) with $N_+, N_- \to M_+, M_-$, we indeed obtain special points in the phase space of the $\mathrm{II}_{\mathrm{rel}}(c, M_+, M_-)$ system, as announced.

Next we describe the path in general terms. (We postpone the precise definition to ease the exposition.) It involves a continuous function $z(t) \in \mathbb{C}, t \in [0,1]$, with $z(0) = 1, z(1) = i$; $z(t), t \in (0,1)$, stays in the first quadrant and satisfies $|z(t)| > 1$. This function enters into (4.15) and (4.16) upon replacing c by $cz(t)$. Thus we obtain complex $x_\alpha^+(t), \alpha = 1, \ldots, M_+$, and $x_\beta^-(t), \beta = 1, \ldots, M_-$. The corresponding path in the parameter space is then given by

$$b_\alpha(t) \equiv \exp(x_\alpha^+(t))e^{-ic}, \quad b_{M_++\beta}(t) \equiv -\exp(x_\beta^-(t))e^{-ic}, \tag{4.18}$$

$$a_j(t) \equiv b_j(t)e^{2ic}, \quad j = 1, \ldots, M, \tag{4.19}$$

$$\mu_j(t) \equiv (ib_j(t)\nu_j(t))^{-1}, \quad j = 1, \ldots, M, \tag{4.20}$$

where

$$\nu_\alpha(t) = 2\sin(c)\exp(p_\alpha^+)V_\alpha^+(x^+(t), x^-(t)), \quad \alpha = 1, \ldots, M_+, \tag{4.21}$$

$$\nu_{M_++\beta}(t) = 2\sin(c)\exp(p_\beta^-)V_\beta^-(x^+(t), x^-(t)), \quad \beta = 1, \ldots, M_-. \tag{4.22}$$

It already follows from the properties of $z(t)$ described thus far that the parameters $(a(1), b(1))$ of the endpoint do not belong to $\Pi_-(M)$ (given by (2.2)) and that, when $n_j^+ > 2$ for some j, we must have $(a(t_0), b(t_0)) \notin \Pi_-(M)$ for some $t_0 \in (0,1)$. To understand the first assertion note that our standing assumption $M > N$ entails that at least one nontrivial cluster is present. Letting e.g. $n_1^+ > 1$ we get

$$\begin{array}{cc} b_1(1) = \exp(\eta_1^+ + i(n_1^+ - 2)c), & a_1(1) = \exp(\eta_1^+ + in_1^+c), \\ \vdots & \vdots \\ b_{n_1^+}(1) = \exp(\eta_1^+ - in_1^+c), & a_{n_1^+}(1) = \exp(\eta_1^+ - i(n_1^+ - 2)c), \end{array} \tag{4.23}$$

so that

$$b_1(1) = a_2(1), \ldots, b_{n_1^+-1}(1) = a_{n_1^+}(1). \tag{4.24}$$

Hence the distinctness requirement of $\Pi_-(M)$ is violated. To see why the second assertion holds true, let $n_1^+ > 2$ and note that the path from the number $b_1(0)$ in the lower half plane to the number $b_1(1)$ in the upper half plane must have crossed $(0, \infty)$; at this t-value t_0 (say) the requirement $b_1(t_0) \in \mathbb{C}_-$ is violated, cf. (2.1)–(2.2). (The latter type of pole crossing is the main reason why controlling analytic continuation of (3.52) would be intractable.)

Since all of our functions in Section 2 are defined solely for parameters in $\Pi_-(M)$, we are not entitled to use the notation $R(a(t), b(t), \mu(t); x)$ without further ado. In fact, we now abandon this parametrization and the use of the linear system (2.10) in favor of a new parametrization of the wave function that hinges on the alternative representation (2.24).

To detail this we first rewrite (2.24) for parameters in $\mathcal{P}(M)$ by using Cauchy's identity to expand the determinants. (We refer to Sections 1 and 2 of [12] for more details on the combinatorics occurring in the sequel.) Since we use (2.24) only for the case

$$a_j = b_j e^{2ic}, \quad j = 1, \ldots, M, \tag{4.25}$$

we specialize to points in $\mathcal{P}(M)$ that correspond to the $\mathrm{II}_{\mathrm{rel}}(c, M_+, M_-)$ system (just as the initial points). We write the result of the expansions as

$$\mathcal{W}_c(b, \nu; x, p) = e^{ixp}\tau(c, b, \kappa(p); x)/\tau(c, b, \nu; x). \tag{4.26}$$

Here we have

$$\tau(c, b, \nu; x) \equiv \sum_{s_1,\ldots,s_M=0,1} \exp\left(\sum_{1\le j<k\le M} s_j s_k B_{jk} + \sum_{1\le j\le M} s_j\left[\xi_j^0 - 2cx - ic\right]\right), \tag{4.27}$$

with

$$\exp(B_{jk}) \equiv \frac{e^{2ic}(b_j - b_k)^2}{(b_j e^{2ic} - b_k)(b_j - b_k e^{2ic})}, \quad j, k = 1, \ldots, M, \tag{4.28}$$

$$\exp(\xi_j^0) \equiv \nu_j/2\sin(c). \tag{4.29}$$

(To check this equals $|\mathbf{1}_N + D(x)^{-1}C|$ for a_j and b_j related by (4.25) and $\mu_j = (ib_j\nu_j)^{-1}$, note we have

$$\mu_j^{-1}\exp(ix\mathrm{Ln}(a_j/b_j)) = \frac{\nu_j(a_j - b_j)}{2\sin(c)}\exp(-2cx - ic), \quad j = 1, \ldots, M, \tag{4.30}$$

cf. (2.6)–(2.7).) Moreover, $\tau(c, b, \kappa(p); x)$ is given by the rhs of (4.27), with ν_j replaced by

$$\kappa_j(p) \equiv \nu_j\left(\frac{e^p - b_j e^{2ic}}{e^p - b_j}\right), \quad j = 1, \ldots, M. \tag{4.31}$$

From this representation we deduce the following analyticity properties in the parameters b and ν. First, $\tau(c, b, \nu; x)$ is analytic for $\nu \in \mathbb{C}^M$ and all $b \in \mathbb{C}^M$ such that the numbers $b_1e^{2ic}, \ldots, b_Me^{2ic}, b_1, \ldots, b_M$ are distinct. Second, the same is true for $\tau(c, b, \kappa(p); x)$ provided that in addition $b_j \ne e^p, j = 1, \ldots, M$. When we now inspect the above path, we see that in the initial point we have

$$\mathcal{W}(a(0), b(0), \mu(0); x, p) = \mathcal{W}_c(b(0), \nu(0); x, p). \tag{4.32}$$

Next we define an 'adjoint wave function' by setting

$$\mathcal{W}_c^\dagger(b, \nu; x, q) \equiv e^{-ixq}\tau(c, b, \kappa^\dagger(q); x - i)/\tau(c, b, \nu; x - i), \quad q \in \mathbb{R}. \tag{4.33}$$

Here we have introduced

$$\kappa_j^\dagger(q) \equiv \nu_j\left(\frac{e^q - b_j}{e^q - b_j e^{2ic}}\right), \quad j = 1, \ldots, M. \tag{4.34}$$

Thus the analyticity features of $\mathcal{W}_c^\dagger$ in b and ν are the same as those of $\mathcal{W}_c$, except that we now need $b_j \neq e^{q-2ic}, j = 1, \ldots, M$. Moreover, from the tau-function property (2.22) and reality of $\exp(B_{jk})$ for parameters given by (3.1)–(3.3) (with $N \to M$) we deduce that

$$\mathcal{W}^*(a(0), b(0), \mu(0); x, q) = \mathcal{W}_c^\dagger(b(0), \nu(0); x, q), \quad x \in \mathbb{C}, \quad q \in \mathbb{R}. \tag{4.35}$$

We can now rewrite (4.8) for a, b and μ given by (3.1) and (3.3) (with $N \to M$ and $c \in (0, \pi/2)$) as

$$\int_{\mathcal{C}_R} dx \mathcal{W}_c^\dagger(b, \nu; x, q) \mathcal{W}_c(b, \nu; x, p) = 0, \quad q, p \in \mathbb{R} \quad (R \text{ large}). \tag{4.36}$$

When (x^+, x^-, p^+, p^-) varies over the phase space of the $\mathrm{II}_{\mathrm{rel}}(c, M_+, M_-)$ system, the parameters be^{ic} and ν vary over open subsets of $\mathbb{R}^M$. By virtue of the above-mentioned analyticity properties in b and ν it then follows that (4.36) still holds true for all $(b, \nu) \in \mathbb{C}^{2M}$ that can be reached by a path such that we have not only

$$b_1, \ldots, b_M, b_1 e^{2ic}, \ldots, b_M e^{2ic} \text{ distinct}, \tag{4.37}$$

$$b_j \neq e^p, \quad b_j \neq e^{q-2ic}, \quad j = 1, \ldots, M, \tag{4.38}$$

along the path, but also no poles crossing the contour $\mathcal{C}_R$.

Returning to the above special path involving $z(t)$, we see that (4.37) can be easily satisfied for $t \in [0, 1)$. Once $z(t)$ is fixed, it is also clear that we can satisfy (4.38) along the path by choosing q and p sufficiently large. Thus the main problem in continuing (4.36) consists in showing that no poles cross the contour. More specifically we cannot have pole crossings along the vertical parts if we choose R large enough, but the difficulty is to show that $\tau(c, b, \nu; x)$ has no real zeros along the path.

In the proof of Lemma 4.2 we show that $z(t)$ can indeed be defined such that for $t \in [0, 1)$ we have (4.37)–(4.38) for $b(t)$ and no real zeros for $\tau(c, b(t), \nu(t); x)$. (This involves some analysis.) Therefore (4.36) continues to hold for $t \in (0, 1)$. But as we have already seen, the distinctness requirement (4.37) is *violated* in the endpoint, since nontrivial clusters are present (recall (4.23)–(4.24)). Thus we do *not* retain analyticity in the endpoint.

The proof that (4.36) does make sense and holds true in the endpoint $z(1) = i$ involves a straightforward adaptation of the fusion formulae in Section 2 of [12]. Briefly, these formulae entail that for $t \uparrow 1$ the functions $\mathcal{W}_c^\dagger(b(t), \nu(t); x, q)$ and $\mathcal{W}_c(b(t), \nu(t); x, p)$ converge to $\mathcal{W}^*(\alpha, \beta, m; x, q)$ and $\mathcal{W}(\alpha, \beta, m; x, p)$ for certain (α, β, m) of the form (4.2)–(4.5). Hence we are able to deduce that $\mathcal{I}_R$ vanishes for points in $\mathcal{P}_c^{(r)}(N)$.

We proceed to present the details of the relevant fusion formulae. The first step is to rewrite the tau-function expansion (4.27) in terms of the variables x^+, x^-, p^+ and p^- via (3.5)–(3.10) (with N replaced by M of course). To avoid unwieldy formulae, however, it is expedient to switch to slightly different variables given by

$$y_j \equiv x_j^+, \quad j = 1, \ldots, M_+, \quad y_{M_+ + l} \equiv x_l^- + i\pi, \quad l = 1, \ldots, M_-, \tag{4.39}$$

$$p_j \equiv p_j^+, \quad j = 1, \ldots, M_+, \quad p_{M_+ + l} \equiv p_l^-, \quad l = 1, \ldots, M_-, \tag{4.40}$$

$$\eta_k \equiv \eta_k^+, \quad k = 1, \ldots, N_+, \quad \eta_{N_+ + m} \equiv \eta_m^- + i\pi, \quad m = 1, \ldots, N_-. \tag{4.41}$$

Then we can rewrite (4.28) as (recall (3.6))

$$\exp(B_{jk}) = 1/f^2(c; y_j - y_k), \quad j,k = 1,\dots,M, \tag{4.42}$$

and accordingly (4.27) becomes

$$\tau(c,b,\nu;x) = \sum_{l=0}^{M} \sum_{\substack{I\subset\{1,\dots,M\}\\ |I|=l}} \exp\left(\sum_{k\in I}[p_k - 2cx - ic]\right) \prod_{\substack{m\in I\\ n\notin I}} f(c; y_m - y_n). \tag{4.43}$$

Moreover, $\tau(c,b,\kappa(p);x)$ and $\tau(c,b,\kappa^\dagger(q);x)$ are given by the rhs of (4.43) with

$$p_k \to p_k + \ln((e^p - e^{y_k+ic})/(e^p - e^{y_k-ic})), \tag{4.44}$$

$$p_k \to p_k + \ln((e^q - e^{y_k-ic})/(e^q - e^{y_k+ic})), \tag{4.45}$$

cf. (4.31) and (4.34), resp.

We are now prepared to return to the special cluster variables (4.15)–(4.16) and the path obtained by replacing c in these formulae by $cz(t)$. Along this path $\eta_1,\dots,\eta_N$ and $p_1,\dots,p_M$ are constant, whereas $y_1,\dots,y_M$ depend on t. Specifically we have

$$y_{n_1+\cdots+n_{j-1}+k}(t) = \eta_j + (n_j + 1 - 2k)cz(t), \quad j = 1,\dots,N, \quad k = 1,\dots,n_j. \tag{4.46}$$

Substituting this into the rhs of (4.43)–(4.45), we can not only verify the asserted continuity for $t \in [0,1]$ but also calculate the $t \uparrow 1$ limit explicitly by following the reasoning in [12] leading from (2.21) to (2.27); in the present case we obtain in this way

$$\lim_{t\uparrow 1} \tau(c,b(t),\nu(t);x) = \sum_{m=0}^{N} \sum_{\substack{J\subset\{1,\dots,N\}\\ |J|=m}} \exp\left(\sum_{j\in J}[P_j - 2c_j x - ic_j]\right) \prod_{\substack{j\in J\\ k\notin J}} F_{jk}(\eta_j - \eta_k), \tag{4.47}$$

with

$$P_j \equiv \sum_{k=1}^{n_j} p_{n_1+\cdots+n_{j-1}+k}, \quad j = 1,\dots,N, \tag{4.48}$$

$$F_{jk}(\eta) \equiv \left(\frac{\sinh^2(\eta/2) + \sin^2(c_j + c_k)/2}{\sinh^2(\eta/2) + \sin^2(c_j - c_k)/2}\right)^{1/2}, \tag{4.49}$$

and the positive square root understood. The $t \uparrow 1$ limits of $\tau(c,b(t),\kappa(p,t);x)$ and $\tau(c,b(t),\kappa^\dagger(q,t);x)$ are given by the rhs of (4.47) with

$$P_j \to P_j + \ln((e^p - e^{\eta_j+ic_j})/(e^p - e^{\eta_j-ic_j})), \tag{4.50}$$

$$P_j \to P_j + \ln((e^q - e^{\eta_j-ic_j})/(e^q - e^{\eta_j+ic_j})), \tag{4.51}$$

resp.

If we now set

$$\beta_j \equiv \exp(\eta_j - ic_j), \quad \alpha_j \equiv \overline{\beta_j}, \quad j = 1,\dots,N, \tag{4.52}$$

$$\zeta_j \equiv 2\sin(c_j)\exp(P_j)\prod_{j\neq k} F_{jk}(\eta_j-\eta_k), \quad m_j \equiv (i\beta_j\zeta_j)^{-1}, \quad j=1,\ldots,N, \tag{4.53}$$

then we obtain parameters (α,β,m) of the form (4.2)–(4.5) such that we also have

$$\lim_{t\uparrow 1} \mathcal{W}_c(b(t),\nu(t);x,p) = \mathcal{W}(\alpha,\beta,m;x,p), \tag{4.54}$$

$$\lim_{t\uparrow 1} \mathcal{W}_c^{\dagger}(b(t),\nu(t);x,q) = \mathcal{W}^*(\alpha,\beta,m;x,q). \tag{4.55}$$

To verify this one should start from the representation (2.24) for the wave function and expand the determinants as before. (The key identities to be used are

$$\frac{1}{F_{jk}(\eta_j-\eta_k)^2} = \begin{vmatrix} 1 & \frac{\alpha_j-\beta_j}{\alpha_j-\beta_k} \\ \frac{\alpha_k-\beta_k}{\alpha_k-\beta_j} & 1 \end{vmatrix} \tag{4.56}$$

and (4.30) with c, a_j, b_j, ν_j and μ_j replaced by $c_j, \alpha_j, \beta_j, \zeta_j$ and m_j, resp.)

We now have all of the algebraic ingredients in hand; hence it remains to supply the analytic details that were omitted in the above road map. First we define a path $z(t)$ along which the tau-function has no real zeros and (4.37) holds true.

Lemma 4.2. *Fix $p^+ \in \mathbb{R}^{M_+}, p^- \in \mathbb{R}^{M_-}, \eta^+ \in \mathbb{R}^{N_+}$ and $\eta^- \in \mathbb{R}^{N_-}$ satisfying*

$$\eta^+_{j-1} - \eta^+_j > 4, \quad j=2,\ldots,N_+, \quad \eta^-_{l-1} - \eta^-_l > 4, \quad l=2,\ldots,N_-, \tag{4.57}$$

and define $y(t)$ via (4.46) and (4.41), with $c \in (0,\pi/4M^2)$. Then the tau-function

$$\sum_{l=0}^{M} \sum_{\substack{I\subset\{1,\ldots,M\} \\ |I|=l}} \exp\left(\sum_{k\in I}[p_k - 2cx - ic]\right) \prod_{\substack{m\in I \\ n\notin I}} f(c; y_m(t) - y_n(t)) \tag{4.58}$$

has no zeros for real x along the following path $z(t), t\in[0,1]$: (a) as t goes from 0 to 1/3, $z(t)$ moves from 1 to M along the real axis; (b) as t goes from 1/3 to 2/3, $z(t)$ moves from M to Mi along the arc $|z|=M$; (c) as t goes from 2/3 to 1, $z(t)$ descends along the imaginary axis from Mi to i. Moreover, for $t\in[0,1)$ the numbers

$$\exp(y_j(t)-ic), \quad \exp(y_j(t)+ic), \quad j=1,\ldots,M, \tag{4.59}$$

are distinct.

The proof of Lemma 4.2 can be found in Appendix A. With Lemmas 4.1 and 4.2 at our disposal we are in the position to obtain the main result of this section (and of the paper). We define

$$\mathcal{P}_{\neq}(N) \equiv \{(a,b,\mu)\in\mathcal{P}(N) \mid \tau(a,b,\mu;x)\neq 0,\ \forall x\in\mathbb{R}\} \tag{4.60}$$

and recall that $\mathcal{P}^{(r)}(N)$ is a subset of $\mathcal{P}_{\neq}(N)$.

Theorem 4.3. *The operator $\mathcal{F}$ is isometric on the connected component $\mathcal{P}^{(r)}_{\neq}(N)$ of $\mathcal{P}_{\neq}(N)$ that contains $\mathcal{P}^{(r)}(N)$.*

Proof. The initial points in Lemma 4.2 belong to the subset of $\mathcal{P}^{(r)}(M)$ that corresponds to the $\mathrm{II}_{\mathrm{rel}}(c, M_+, M_-)$ system. Thus the contour integral $\mathcal{I}_R$ vanishes for large R, cf. (4.8). Rewriting this as (4.36), we have shown in Lemma 4.2 that for $t \in [0, 1)$ the path yielding $(b(t), \nu(t))$ stays in the region of $\mathbb{C}^{2M}$ where (4.37) holds true. Clearly, we can satisfy (4.38) along the path by choosing p and q sufficiently large. Likewise, if we take R large enough, there are no poles crossing the vertical parts of $\mathcal{C}_R$. By Lemma 4.2 there are no poles crossing the horizontal parts either. Therefore we have

$$\int_{\mathcal{C}_R} dx \mathcal{W}_c^\dagger(b(t), \nu(t); x, q)\mathcal{W}_c(b(t), \nu(t); x, p) = 0, \quad t \in [0, 1), \tag{4.61}$$

for p, q and R sufficiently large.

We now invoke the limits (4.54)–(4.55). Since the denominator tau-functions stay away from 0 on $\mathcal{C}_R$ as $t \uparrow 1$, they entail

$$\int_{\mathcal{C}_R} dx \mathcal{W}^*(\alpha, \beta, m; x, q)\mathcal{W}(\alpha, \beta, m; x, p) = 0. \tag{4.62}$$

This result is obtained under various restrictions that we remove next. First, we use real-analyticity in p and q to deduce (4.62) for all $(p, q) \in \mathbb{R}^2$. Second, since the denominator tau-functions stay away from 0 for $c \in (0, c_{\mathrm{max}})$, we may use real-analyticity in c to extend (4.62) from $c \in (0, \pi/4M^2)$ to $c \in (0, c_{\mathrm{max}})$. Third, we use real-analyticity in η^+ and η^- to extend the validity of (4.62) to all $(\eta^+, \eta^-) \in \mathbb{R}^N$ given by (4.17) with $<<$ replaced by $<$. Since we have not imposed a restriction on p^+ and p^-, there is no restriction on P_j (4.48) either. Therefore all ζ_j (given by (4.53)) vary over $(0, \infty)$.

The upshot is that we obtain

$$\int_{\mathcal{C}_R} dx \mathcal{W}^*(a, b, \mu; x, q)\mathcal{W}(a, b, \mu; x, p) = 0 \tag{4.63}$$

for all parameters of the form (4.2)–(4.5). From the first part of Lemma 4.1 we now deduce that $\mathcal{F}$ is an isometry for these parameters. Rewriting isometry as (3.52) (with $g_1, g_2 \in C_0^\infty(\mathbb{R})$) and recalling that the pertinent parameters form a dense subset of $\mathcal{P}^{(r)}(N)$, we can use real-analyticity in $\mathrm{Re}\, b_j, \mathrm{Im}\, b_j$ and ν_j, $j = 1, \dots, N$, to deduce isometry on all of $\mathcal{P}^{(r)}(N)$ (by the reasoning in the proof of Theorem 4.3, cf. (3.55)–(3.56)). Likewise we deduce isometry on $\mathcal{P}^{(r)}_{\neq}(N)$. $\square$

Corollary 4.4. *For all* $(a, b, \mu) \in \mathcal{P}^{(r)}_{\neq}(N)$ *the operator* $\mathcal{F}$ *satisfies*

$$\mathcal{F}^*\mathcal{F} = \mathbf{1}, \qquad \mathcal{F}\mathcal{F}^* = \mathbf{1} - \sum_{j=1}^{N_+} \nu_j \psi_j \otimes \overline{\psi}_j, \tag{4.64}$$

where

$$(\psi_j, \psi_k) = \nu_k^{-1}\delta_{jk}. \tag{4.65}$$

Proof. Clear from Theorem 4.3 and Theorems 2.5–2.7. $\square$

5 Eigenfunction properties

As we mentioned above, we believe that, when the functions $\mu_1(x), \ldots, \mu_N(x)$ are not constant, $\mathcal{F}$ cannot be isometric. Therefore the reader may wonder why we did not require $\mu \in \mathbb{C}^{*N}$ from the outset. Our reason for not doing so is that the principal result of this section (Theorem 5.1) holds true when we only require (2.4). (Actually, though i-periodicity is indispensable, the space $\mathcal{R}(c)$ (2.3) could be further enlarged for this to be the case.) Theorem 5.1 involves Laurent polynomials of the form

$$\mathcal{L}(z) = \sum_{n=-L}^{K} \gamma_n z^n, \qquad \gamma_0 \equiv 0, \quad K, L \in \mathbb{N}, \tag{5.1}$$

with $\gamma_K \neq 0$ for $K > 0$ and $\gamma_{-L} \neq 0$ for $L > 0$. Its proof is partly adapted from the proof of Theorem 2.5 in [1].

Theorem 5.1. *Suppose $(a, b, \mu(x))$ satisfy (2.2)–(2.4). Let $\mathcal{L}(z)$ be a Laurent polynomial of the form (5.1) that satisfies*

$$\mathcal{L}(a_j) = \mathcal{L}(b_j), \quad j = 1, \ldots, N. \tag{5.2}$$

Then there exists one and only one $A\Delta O$ of the form

$$A = \sum_{n=-L'}^{K'} V_n(x) \exp(-ind/dx), \qquad K', L' \in \mathbb{N}, \qquad V_n \in \mathcal{M}, \tag{5.3}$$

such that

$$A\mathcal{W}(a, b, \mu; x, p) = \mathcal{L}(e^p)\mathcal{W}(a, b, \mu; x, p). \tag{5.4}$$

More specifically we have $K' = K, L' = L$ and

$$K > 0 \Rightarrow V_K(x) = \gamma_K, \tag{5.5}$$

$$L > 0 \Rightarrow V_{-L}(x) \in \mathcal{M}^*. \tag{5.6}$$

Proof. We firstly assume that A is an AΔO of the form (5.3) that satisfies (5.4) and prove that the coefficients $V_n(x)$ are uniquely determined. It is convenient to introduce the auxiliary wave function

$$\mathcal{A}(x, p) \equiv \prod_{n=1}^{N} (e^p - b_n) \cdot \mathcal{W}(x, p). \tag{5.7}$$

Hence $\exp(-ixp)\mathcal{A}(x, p)$ is a polynomial in e^p, cf. (1.1); more specifically we may write

$$\mathcal{A}(x, p) = e^{ixp} \sum_{k=0}^{N} c_k(x) e^{kp}, \qquad c_0(x) \in \mathcal{M}^*, \quad c_N(x) = 1. \tag{5.8}$$

(It is not immediate that c_0 cannot vanish identically, but this can be verified by using (2.13).)

Obviously (5.4) entails

$$A\mathcal{A}(x,p) = \mathcal{L}(e^p)\mathcal{A}(x,p), \tag{5.9}$$

so that we must have equality of coefficients of powers e^{lp}. Comparing coefficients of $e^{(N+K)p}$, we see that K' cannot be smaller than K. Assuming K' is larger than K, we compare coefficients of e^{lp} for $l = K', K'-1, \ldots, K+1$, to obtain $V_l(x) = 0$. Thus we have $K' = K$, and (5.5) follows from equality of $e^{(N+K)p}$-coefficients. More generally, equality of $e^{(N+K-m)p}$-coefficients for $m = 0, \ldots, K-1$, yields

$$\sum_{l=0}^{m} V_{K-l}(x) c_{N+l-m}(x - i(K-l)) = \sum_{l=0}^{m} \gamma_{K-l} c_{N+l-m}(x). \tag{5.10}$$

Likewise we deduce $L' = L$ and

$$L > 0 \Rightarrow V_{-L}(x) = \gamma_{-L} c_0(x)/c_0(x+iL) \in \mathcal{M}^* \tag{5.11}$$

as asserted. Equality of $e^{(-L+j)p}$-coefficients for $j = 0, \ldots, L$, yields

$$\sum_{n=0}^{j} V_{-L+n}(x) c_{j-n}(x + i(L-n)) = \sum_{n=0}^{j} \gamma_{-L+n} c_{j-n}(x). \tag{5.12}$$

We now show uniqueness of the coefficients $V_K(x), \ldots, V_{-L}(x)$, in several steps. First, in the trivial case $L, K = 0$ we have $\mathcal{L} = 0$, $V_0(x) = 0$ and $A = 0$. Second, let $L = 0$ and $K > 0$. Then we use $c_N(x) = 1$ and (5.10) for $m = 0, \ldots, K-1$, to deduce recursively that the coefficients

$$V_K(x), \ldots, V_1(x) \tag{5.13}$$

are uniquely determined. Comparing the coefficients of e^{lp} with $l = 0$ and using $c_0 \in \mathcal{M}^*$, we deduce that

$$V_0(x) = 0 \quad (L = 0). \tag{5.14}$$

Third, let $L > 0$ and $K = 0$. Then we can use $c_0 \in \mathcal{M}^*$ and (5.12) for $j = 0, \ldots, L$, to deduce recursively that

$$V_{-L}(x), \ldots, V_0(x) \tag{5.15}$$

are uniquely determined.

Finally, for $L > 0$ and $K > 0$ we first invoke (5.10) with $m = 0, \ldots, K-1$, to see that the coefficients (5.13) are unique and then invoke (5.12) with $j = 0, \ldots, L$, to obtain uniqueness of the coefficients (5.15).

The upshot is that existence implies uniqueness. To prove existence we follow the reasoning in the proof of Theorem 2.5 in [1]. It applies as it stands until we reach the paragraph containing (2.52). (This is the first place where the Shabat AΔO A_S enters the proof.) Now we *define* A by determining the coefficients $V_n(x)$ recursively, as already detailed. Then the function

$$\mathcal{D}(x,p) \equiv A\mathcal{A}(x,p) - \mathcal{L}(e^p)\mathcal{A}(x,p) \tag{5.16}$$

is manifestly of the form (2.58). Hence we can complete the proof as before, the assumption (5.2) playing the same role as (2.61). □

Rewriting the condition (5.2) as

$$\sum_{n=-L}^{K} \gamma_n v_n^{(j)} = 0, \qquad v_n^{(j)} \equiv a_j^n - b_j^n, \qquad j = 1, \ldots, N, \tag{5.17}$$

it becomes clear that for given $(a, b) \in \Pi_-(N)$ there exist a great many Laurent polynomials of 'degree' $K + L$ greater than N satisfying (5.2); obviously these polynomials form an algebra. Likewise it follows that for generic $(a, b) \in \Pi_-(N)$ no Laurent polynomial with $K + L \leq N$ satisfying (5.2) exists. Thus any wave function $\mathcal{W}(a, b, \mu; x, p)$ is a joint eigenfunction for an algebra $\mathcal{A}$ of AΔOs of the form (5.3), whose order $K+L$ is generically greater than N. Moreover, any pair $A_1, A_2 \in \mathcal{A}$ commutes because $[A_1, A_2]\mathcal{W} = 0$ implies $A = 0$ by virtue of Theorem 5.1.

If we restrict attention to $(a, b, \mu) \in \mathcal{P}_{\neq}^{(r)}(N)$ and to Laurent polynomials obeying (5.2) that have real coefficients γ_n, it is easy to see that the associated AΔO A (5.3) can be reinterpreted as a self-adjoint operator on $\mathcal{H}_x$ with a core given by (3.66). Indeed, for these data we may invoke Corollary 4.4, so on $\mathcal{F}C_0^\infty(\mathbb{R})$ we may view A as the pullback of the real-valued multiplication operator $\mathcal{L}(e^p)$ under the isometry $\mathcal{F}$, whilst the eigenvalues $\mathcal{L}(a_j)$ on the pairwise orthogonal bound states

$$\psi_j(x) = \mathcal{W}(x, \mathrm{Ln}(a_j)), \qquad j = 1, \ldots, N_+, \tag{5.18}$$

are also real. (This follows from (5.2), $\gamma_n \in \mathbb{R}$, and equality of b_j and $\overline{a}_j$.) From this description it is also clear that the commutative AΔO algebra $\mathcal{A}$ gives rise to a commutative algebra of self-adjoint operators on $\mathcal{H}_x$.

6 Concluding remarks

(i) (*Real tau-function zeros*) We recall that we have

$$\mathcal{P}_{\neq}^{(r)}(N) = \mathcal{P}(N) \qquad (N_+ = 0 \text{ or } N_- = 0). \tag{6.1}$$

Now assume that $N_+N_- > 0$. We claim that in this case $\mathcal{P}_{\neq}(N)$ (4.60) is a proper subset of $\mathcal{P}(N)$. To show this it suffices to prove the existence of $(a, b, \mu) \in \mathcal{P}(N)$ for which $\tau(a, b, \mu; x)$ has a real zero. To this end we specialize to data satisfying (3.1) and (3.3), but now with $c \in (\pi/2, \pi)$. Then we can proceed as before to obtain (3.5)–(3.20) and

$$\tau(x) = \left|\mathbf{1}_N + e^{-2cx} e^{-ic} L\right|. \tag{6.2}$$

Now from [11] it follows that

$$\sigma(L) \subset \{\lambda \in \mathbb{C}^* \mid \mathrm{Arg}(\lambda) \leq \pi - c\}, \qquad c \in (\pi/2, \pi), \tag{6.3}$$

and that points in the phase space Ω exist for which L has eigenvalues

$$\lambda_\pm = \kappa \exp(\pm i(\pi - c)), \qquad \kappa > 0. \tag{6.4}$$

Thus we need only choose $x \in \mathbb{R}$ such that $\exp(-2cx) = \kappa^{-1}$ to obtain a real tau-function zero.

For the special case $N_+ = N_- = 1$ the tau-zeros corresponding to the above $\mathrm{II}_{\mathrm{rel}}(c, 1, 1)$ points occur if and only if

$$\nu_1 = \nu_2 = \nu, \quad b_1 = -b_2 = \rho e^{-ic}, \quad \nu, \rho \in (0, \infty), \quad c \in (\pi/2, \pi), \tag{6.5}$$

as is readily verified. Thus the associated (a, b, μ) form a 3-dimensional submanifold of the 6-dimensional manifold $\mathcal{P}(2)$.

(ii) (*Some open problems*) As we already mentioned, we do not know whether nonconstant $\mu(x)$ exist for which $\mathcal{F}$ is an isometry. More generally, it is an open question whether isometry is violated for $(a, b, \mu) \notin \mathcal{P}(N)$. From now on we restrict attention to $(a, b, \mu) \in \mathcal{P}(N)$ with $N_+N_- > 0$. Then there are again quite a few questions we cannot answer. Even for the simplest case, namely $N_+ = N_- = 1$, we do not know whether $\tau(x)$ can have zeros in the strip $\mathcal{S}$ (2.41) when b_1 and b_2 are not on the same line through the origin. (When they are on the same line, we can invoke the previous remark to rule out the absence of zeros in the interior of $\mathcal{S}$ and the presence of boundary zeros if and only if (6.5) holds true.)

A remarkable feature of the special case $N_+ = N_- = 1$ is that $R(x)$ does *not* have real zeros when (6.5) holds true. (This is because $E_1(x)$ and $E_2(x)$ in (2.12) have zeros at the same real x as $\tau(x)$.) Hence the associated transform $\mathcal{F}$ is bounded. Even so, it is *not* isometric. This claim can be substantiated by invoking results from [14]; the point is that the wave function amounts to one of the wave functions studied in [14] and shown to be non-isometric. (It fails to be isometric on a 1-dimensional subspace.)

More generally, for $N_+N_- > 0$ and $N > 2$, one may ask whether $\tau(x)$-zeros for x real can give rise to $R(x)$-poles for x real. We do not know whether this ever happens. Also, if $\tau(x)$ has real zeros, whereas $R(x)$ has none, is the associated bounded operator $\mathcal{F}$ always non-isometric?

We surmise that $\tau(x)$ cannot have zeros in the interior of $\mathcal{S}$ (2.41). More generally, we believe (but were unable to prove) that

$$\mathcal{P}_{\neq}^{(r)}(N) = \mathcal{P}_{\neq}(N) \qquad (?) \tag{6.6}$$

(iii) (*Soliton equations*) The reflectionless second-order AΔOs and wave functions studied in [1] and [4, 5, 6] are closely related to nonlocal soliton evolution equations of Shabat and Toda type, resp. We recall that the associated subsets of $\mathcal{P}(N)$ are characterized by $b_j = 1 - i\kappa_j, \kappa_j \in \mathbb{R}^*$, and $b_j = \exp(ir_j), r_j \in (-\pi, 0) \cup (0, \pi), j = 1, \ldots, N$, resp. Presumably one can tie in the reflectionless higher-order AΔOs and wave functions at issue in this paper with a nonlocal counterpart of the 2D Toda lattice, but we have not pursued this angle. (There might also be a connection with the nonlocal soliton hierarchies studied more than a decade ago, cf. Santini's review [15].) To elaborate slightly on an eventual nonlocal 2D Toda soliton hierarchy, it should be pointed out that the reflectionless solutions cannot naturally be accomodated in fermion Fock spaces, as is the case for the 2D Toda lattice hierarchy [16, 17]. This is because the discrete index $n \in \mathbb{Z}$ of the tau-functions is interpreted as a label for the charge sectors in fermion Fock space. Since we replace $n \in \mathbb{Z}$ by $x \in \mathbb{C}$, no such 'Grassmannian picture' appears to exist.

Another remark of interest to basic soliton theory (for which we recommend [18, 19]) concerns the special points (6.5). If we also take $\rho = 1$, then these data belong to the 4-dimensional Toda submanifold of $\mathcal{P}(2)$. Now the t-dependent data

$$\nu_\pm(t) \equiv \nu \exp(\mp 2t \sin c), \qquad b_\pm \equiv \pm e^{-ic}, \quad \nu \in (0, \infty), \quad c \in (\pi/2, \pi), \tag{6.7}$$

correspond to a right-moving / left-moving 2-soliton solution, cf. [5]. Due to the isometry breakdown of $\mathcal{F}$ for $t = 0$, the asociated Lax operator $A(t)$ (a second-order reflectionless AΔO) fails to be self-adjoint for $t = 0$. This type of instability appears to be a novel phenomenon.

(iv) (*The role of* $\mathrm{II}_{\mathrm{rel}}$ *systems: a comparison*) For the 2D Toda and KP N-soliton solutions and their well-known specializations, the connection to the $\mathrm{II}_{\mathrm{rel}}(c, N_+, N_-)$ systems can be made via the action-angle description of the latter. To be specific, the starting point for the fusion procedure in [12] consists in clusters of action variables $\theta_1, \ldots, \theta_M$ (cf. [12] (2.21)). For the subset of phase space on which no bound states of particles and antiparticles occur, the particle-antiparticle charges are encoded via the sign of the real quantities $\exp(\xi_1^0), \ldots, \exp(\xi_M^0)$ in the tau-function (cf. [12] (1.1) and (1.10)), which involve the generalized angles $q_1, \ldots, q_M$ (cf. [12] (2.16)–(2.17)).

By contrast, in our present context it is essential to start from clusters of particle-antiparticle position coordinates (cf. (4.15)–(4.16)), whereas $\exp(\xi_1^0), \ldots, \exp(\xi_M^0)$ are positive numbers involving the particle-antiparticle momenta (cf. (4.29), (4.21)). More precisely, this change of starting point is necessary for the case $N_+N_- > 0$, since for $N_+ = 0$ or $N_- = 0$ both viewpoints are basically the same (by virtue of the self-duality of these special cases).

In our previous work on nonlocal soliton equations of Shabat and Toda type we already encountered this puzzling necessity to switch from action-angle to position-momentum variables. Indeed, we could only relate soliton solutions consisting of N_+ right-moving and N_- left-moving solitons to the $\mathrm{II}_{\mathrm{rel}}(\pi/2, N_+, N_-)$ systems when we used a (non-obvious) parametrization in terms of positions and momenta, cf. [1] and [5].

Appendix A. Proofs of Lemmas 4.1 and 4.2

Proof of Lemma 4.1. We begin by showing that (4.8) holds for the parameters of Section 3. To this end we first consider a, b and μ that correspond to points in Ω_+. (Recall this is the subset of Ω where (3.28) holds true.) Then we have from (3.48) and (3.47)

$$\mathcal{W}(a, b, \mu; x, p) = e^{ixp} \left(1 - \sum_{n=1}^{N} \frac{R_n(\hat{a}, \hat{b}, \hat{\mu}; p/2c)}{e^{2cx} - \hat{b}_n} \right). \tag{A.1}$$

Choosing R large enough so that the points

$$\hat{b}_n \in \mathrm{UHP}, \quad \overline{\hat{b}}_n \in \mathrm{LHP}, \quad n = 1, \ldots, N, \tag{A.2}$$

are inside $\mathcal{C}_R$, the contour integral $\mathcal{I}_R$ equals $2\pi i$ times the sum of the residues of the integrand at the simple poles (A.2). Since all of $\hat{b}_1, \ldots, \hat{b}_N$ belong to the UHP, this residue sum vanishes. (One need only adapt (C.12)–(C.16) in [1] to verify this.)

As a consequence (4.8) holds on Ω_+. Using real-analyticity in the variables x^+, x^-, p^+ and p^-, we now obtain (4.8) on all of Ω in the same way as in the proof of Theorem 3.1, cf. the paragraph containing (3.54). (We need only choose R large enough so that no poles cross the vertical parts of the contour upon continuation from Ω_+ to Ω; the horizontal parts cannot be crossed, since we have $\tau(x) \neq 0$ for real x and $i\pi/c$-periodicity of $\tau(x)$ on all of Ω.)

We continue to prove the first assertion (isometry of $\mathcal{F}$). Since $\mathcal{P}_c^{(r)}(N)$ is a subset of $\mathcal{P}^{(r)}(N)$, the tau-functions $\tau(x)$ and $\tau^*(x)$ have no real zeros. Since they are $i\pi/c$-periodic, it follows from (1.1) and (2.12) that the integrand in (4.6) has no poles on the horizontal parts of $\mathcal{C}_R$. Furthermore, for sufficiently large R there are no poles on the vertical parts. Hence the contour integral $\mathcal{I}_R$ is well defined. By $i\pi/c$-periodicity of $R(x)$ and $R^*(x)$, the vanishing of $\mathcal{I}_R$ can be rewritten as

$$0 = I_R(q,p) + \mathcal{B}_R^+(q,p) - e^{\pi(p-q)/c} I_R(q,p) + \mathcal{B}_R^-(q,p), \tag{A.3}$$

where we have introduced

$$I_R(q,p) \equiv \int_{-R}^{R} dx \mathcal{W}^*(x,q)\mathcal{W}(x,p), \tag{A.4}$$

$$\mathcal{B}_R^+(q,p) \equiv \int_{R}^{R+i\pi/c} dx \mathcal{W}^*(x,q)\mathcal{W}(x,p), \tag{A.5}$$

$$\mathcal{B}_R^-(q,p) \equiv \int_{-R+i\pi/c}^{-R} dx \mathcal{W}^*(x,q)\mathcal{W}(x,p). \tag{A.6}$$

Next we let $g_1, g_2 \in C_0^\infty(\mathbb{R})$, and note that by Fubini's theorem and dominated convergence we have

$$(\mathcal{F}g_2, \mathcal{F}g_1) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-\infty}^{\infty} dq \overline{g}_2(q) \int_{-\infty}^{\infty} dp g_1(p) I_R(q,p). \tag{A.7}$$

On account of (A.3) it therefore remains to prove

$$\frac{1}{2\pi} \lim_{R\to\infty} \int_{-\infty}^{\infty} dq \overline{g}_2(q) \int_{-\infty}^{\infty} dp g_1(p) \frac{\mathcal{B}_R^+(q,p) + \mathcal{B}_R^-(q,p)}{\exp[\pi(p-q)/c] - 1} = (g_2, g_1). \tag{A.8}$$

To this end we first obtain a new representation for $K(x,p)$, cf. (3.21)–(3.22). Since $R(x)$ is $i\pi/c$-periodic, it follows from the linear system (2.10) that $D(x)$ is $i\pi/c$-periodic. Hence the factors $\exp(-ix\mathrm{Ln}(a_j/b_j)), j = 1, \ldots, N$, are $i\pi/c$-periodic, cf. (2.6)–(2.7). By (2.8) this implies

$$\mathrm{Ln}(a_j/b_j) \in 2ic\mathbb{N}^*, \quad j = 1, \ldots, N. \tag{A.9}$$

It easily follows that $R(x)$ is a rational function of e^{2cx}; likewise, $R^*(x)$ is rational in e^{2cx}, so that $K(x,p)$ is rational in e^{2cx}. Since $K(x,p) \to 0$ as $x \to \infty$, it follows that K is of the form

$$K(a,b,\mu;x,p) = \sum_{j=1}^{J} \sum_{l=1}^{N_j} \frac{c_{jl}(p)}{(e^{2cx} - \hat{b}_j)^l}. \tag{A.10}$$

Moreover, since K has no real x-poles, we have

$$\hat{b}_j \in \mathbb{C}_-, \quad j = 1, \ldots, J. \tag{A.11}$$

Finally, in view of the definition (3.22) of K, the coefficients are of the form

$$c_{jl}(p) = \sum_{n=1}^{N} \frac{c_{jln}}{e^p - b_n}, \quad c_{jln} \in \mathbb{C}. \tag{A.12}$$

It is clear from this representation how large we should choose R so that all poles $\hat{b}_1, \ldots, \hat{b}_J$ are inside $\mathcal{C}_R$: letting

$$M_+ \equiv \max_{n \in \{1,\ldots,J\}} |\hat{b}_n|, \quad M_- \equiv \min_{n \in \{1,\ldots,J\}} |\hat{b}_n|, \tag{A.13}$$

we should choose $R \in [R_0, \infty)$ with R_0 such that

$$\exp(2cR_0) > M_+, \quad \exp(-2cR_0) < M_-. \tag{A.14}$$

It is also obvious that the vanishing of $\mathcal{I}_R$ amounts to the zero residue equation

$$\sum_{j=1}^{J} c_{j1}(p) = 0. \tag{A.15}$$

Next we use the K-representation (A.10) to rewrite the integrand $\mathcal{W}^*(x,q)\mathcal{W}(x,p)$ of $\mathcal{B}_R^+(q,p)$ (A.5). Save for the term $\exp ix(p-q)$, all terms have poles (as functions of e^{2cx}) at one or two of the points

$$\hat{b}_1, \ldots, \hat{b}_J, \bar{\hat{b}}_1, \ldots, \bar{\hat{b}}_J, \tag{A.16}$$

which are not necessarily distinct. Separating product terms into single pole terms (if need be), we wind up with a sum of terms of the form

$$T_l(x) = \frac{e^{ix(p-q)} C(q,p)}{(e^{2cx} - \beta)^l}, \quad l \in \mathbb{N}^*, \tag{A.17}$$

where β denotes one of the points (A.16) and where the coefficients $C(q,p)$ are bounded on $\mathbb{R}^2$.

We proceed to show that all terms of the form (A.17) yield a vanishing contribution to the lhs of (A.8). To this end we set $a \equiv p - q$ and introduce the functions

$$F_{R,l}(a) \equiv \int_R^{R+i\pi/c} dx \frac{e^{iax}}{(e^{2cx} - \beta)^l}, \quad a \in \mathbb{R}, \quad \beta \in \mathbb{C}_-, \quad |\beta| \in [M_-, M_+]. \tag{A.18}$$

Since $R \geq R_0$, we have $\mathrm{Re}\,(1 - \beta e^{-2cx}) > 0$ on the line segment from R to $R + i\pi/c$. Choosing

$$\mathrm{Im}\,(\log(z)) \in (-\pi/2, \pi/2), \quad \mathrm{Re}\, z > 0, \tag{A.19}$$

we write

$$\frac{1}{e^{2cx}-\beta}=\frac{1}{2c\beta}\partial_x\log(1-\beta e^{-2cx}), \tag{A.20}$$

and integrate by parts in the integral with $l=1$ to obtain

$$\begin{aligned}F_{R,1}(a) &= -\frac{ia}{2c\beta}\int_R^{R+i\pi/c}dxe^{iax}\log(1-\beta e^{-2cx}) \\ &\quad +\frac{1}{2c\beta}e^{iaR}(e^{-\pi a/c}-1)\log(1-\beta e^{-2cR}).\end{aligned} \tag{A.21}$$

From this we infer that

$$|F_{R,1}(a)|\leq C_{1,L}|a|e^{-2cR},\quad \forall a\in[-L,L],\quad \forall R\geq R_0. \tag{A.22}$$

Clearly this estimate implies that the contribution of the terms (A.17) with $l=1$ to (A.8) vanishes as $R\to\infty$.

To handle the terms with $l>1$ we note that (A.21) entails

$$\partial_\beta F_{R,1}(a)=-\frac{1}{\beta}F_{R,1}(a)-iaF_{R,1}(a)+\frac{e^{iaR}(e^{-\pi a/c}-1)}{e^{2cR}-\beta}. \tag{A.23}$$

Now it is evident from (A.18) that we have

$$F_{R,l}(a)=\frac{1}{(l-1)!}\partial_\beta^l F_{R,1}(a). \tag{A.24}$$

Iterating (A.23) and using (A.22) we readily deduce

$$|F_{R,l}(a)|\leq C_{l,L}|a|e^{-2cR},\quad \forall a\in[-L,L],\quad \forall R\geq R_0. \tag{A.25}$$

Therefore the $l>1$ terms yield vanishing contribution as well.

Turning to $\mathcal{B}_R^-(q,p)$ we first write, using (2.19),

$$\mathcal{W}(x,p)=e^{ixp}(a(p)-\tilde{K}(x,p)), \tag{A.26}$$

with

$$\tilde{K}(x,p)\equiv\frac{R_n(x)-(C^{-1}\zeta)_n}{e^p-b_n}. \tag{A.27}$$

Since $\tilde{K}\to 0$ as $x\to-\infty$ (by (2.14)), we infer that $\tilde{K}$ is of the form

$$\tilde{K}(a,b,\mu;x,p)=\sum_{j=1}^{J}\sum_{l=1}^{N_j}\frac{\tilde{c}_{jl}(p)}{\left(e^{-2cx}-\hat{b}_j^{-1}\right)^l}. \tag{A.28}$$

Proceeding in the same way for $\mathcal{W}^*(x,q)$, it is not hard to see that our previous estimates can be used to conclude that the term $a(p)\overline{a(q)}\exp ix(p-q)$ is the only one in the expansion of the integrand of $\mathcal{B}_R^-(q,p)$ that yields a nonvanishing contribution to the lhs of (A.8).

It remains to show that the limit

$$\frac{1}{2\pi}\lim_{R\to\infty}\int_{-\infty}^{\infty}dq\overline{g}_2(q)\int_{-\infty}^{\infty}dpg_1(p)\frac{\int_R^{R+i\pi/c}dxe^{ix(p-q)}+a(p)\overline{a(q)}\int_{-R+i\pi/c}^{-R}dxe^{ix(p-q)}}{\exp[\pi(q-p)/c]-1} \tag{A.29}$$

equals (g_2, g_1). This follows in the same way as in the proof of Theorem 4.2 in [1], cf. p. 388 in [1]. □

Proof of Lemma 4.2. We begin by proving the distinctness assertion. Recalling (4.39) we see that for $j = 1, \dots, M_+$ the phase of the numbers (4.59) stays close to 0 (in fact, it stays in $(-\pi/4M, \pi/4M)$). Likewise it stays close to π for $j = M_+ + 1, \dots, M$. Thus we cannot have collisions among these subsets. Due to (4.57) there cannot be collisions involving two distinct clusters either. The points of a cluster have distinct moduli for $t \in [0, 2/3)$, so that it remains to study whether equality of $\exp(y_j(t) - ic)$ and $\exp(y_k(t) + ic)$ for j, k in the same cluster and $t \in [2/3, 1)$ can occur. But this can be excluded by noting that for $t \in [2/3, 1)$ the minimal angular distance within a cluster varies over $(2c, 2Mc]$.

Next we show the absence of real zeros for (4.58). It is convenient to introduce closed sectors

$$S(\phi) \equiv \{z \in \mathbb{C} \mid |\mathrm{Arg}z| \le \phi\}. \tag{A.30}$$

The $l = 0$ term in (4.58) equals 1, so it suffices to prove that for real x all other terms remain in $S(\pi/4)$ along the path. The $l = M$ term equals

$$\exp\left(\sum_{k=1}^{M} p_k - 2Mcx - iMc\right). \tag{A.31}$$

Since c is smaller than $\pi/4M^2$, it belongs to $S(\pi/4M)$. If we fix I with $|I| = l \in \{1, \dots, M-1\}$, it remains to show that we have

$$\exp(-ilc)\prod_{\substack{m\in I\\ n\notin I}} f(c; y_m(t) - y_n(t)) \in S(\pi/4), \quad t \in [0, 1]. \tag{A.32}$$

There are $l(M - l)$ terms in the product. Since $l(M - l) \le M^2/4$, it suffices to prove

$$f(c; y_m(t) - y_n(t))^2 \in S(\pi/M^2), \quad t \in [0, 1], \tag{A.33}$$

for all $m \ne n$. We proceed to do so by a case analysis.

We begin by noting that

$$f(c; y_m(t) - y_n(t))^2 = 1 + \frac{\sin^2 c}{\sinh^2((\eta_j - \eta_k)/2 + lcz(t))}, \tag{A.34}$$

with

$$j, k \in \{1, \dots, N\}, \quad |l| \le \max(n_1, \dots, n_N) \le M. \tag{A.35}$$

Consider first the case $j = k$. Then we should show

$$\gamma_l(t) \equiv 1 + \frac{\sin^2 c}{\sinh^2(lcz(t))} \in S(\pi/M^2), \quad l = 1, \ldots, M, \quad t \in [0, 1]. \tag{A.36}$$

Now we have

$$\gamma_l(t) \in [0, \infty), \quad t \in [0, 1/3] \cup [2/3, 1], \tag{A.37}$$

with equality to zero only for $l = 1$ and $t = 1$. Thus it remains to show (A.36) for $t \in (1/3, 2/3)$. Then we have $|z(t)| = M$, so we can invoke the elementary bounds

$$\sin^2 x > x^2/2, \quad x \in (0, \pi/4], \tag{A.38}$$

$$|\sinh(re^{i\phi})|^2 > r^2/2, \quad r \in (0, \pi/4], \quad \phi \in [0, 2\pi). \tag{A.39}$$

First, from (A.39) we have

$$\left| \frac{\sin^2 c}{\sinh^2(lcz(t))} \right| < \frac{2 \sin^2 c}{l^2 c^2 M^2}, \quad t \in (1/3, 2/3). \tag{A.40}$$

Second, for a number of the form $1 + w, |w| + 1$, to belong to the sector $S(\phi), \phi \in (0, \pi/2)$, it clearly suffices that $|w|$ is smaller than $\sin \phi$. Thus we need only verify

$$\frac{2 \sin^2 c}{l^2 c^2 M^2} < \sin(\pi/M^2). \tag{A.41}$$

Using $\sin c < c$ and (A.38), this is implied by the inequality

$$\frac{2}{l^2 M^2} < (\pi/M^2)/2^{1/2}, \tag{A.42}$$

whose validity is plain.

It remains to consider the case $j \neq k$. As before it suffices to show

$$\frac{\sin^2 c}{|\sinh^2((\eta_j - \eta_k)/2 + lcz(t))|} \leq \sin(\pi/M^2), \quad t \in (1/3, 1]. \tag{A.43}$$

There are two subcases: $\eta_j - \eta_k = 2\rho \pm i\pi, \rho \in \mathbb{R}$, and $\eta_j - \eta_k = \pm 2\rho, \rho > 2$. In the first subcase we use

$$|\cosh(x + iy)|^2 = \cos^2 y + \sinh^2 x, \quad x, y \in \mathbb{R}, \tag{A.44}$$

to get

$$\frac{1}{|\cosh^2(\rho + lcz(t))|} \leq \frac{1}{\cos^2(lcM)} \leq \frac{1}{\cos^2(\pi/4)} = 2. \tag{A.45}$$

Clearly we have

$$2 \sin^2 c < \sin(\pi/M^2), \quad c < \pi/4M^2, \tag{A.46}$$

and so (A.43) follows. In the second subcase we use

$$|\sinh(x + iy)|^2 = \sin^2 y + \sinh^2 x, \quad x, y \in \mathbb{R}, \tag{A.47}$$

to get

$$\frac{1}{|\sinh^2(\rho + lcz(t))|} \leq \frac{1}{\sinh^2(\rho - lcM)} < \frac{1}{\sinh^2(1)} < 1, \tag{A.48}$$

so that (A.43) follows again. $\square$

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