Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 13–31 **Birthday Issue**

Some Symmetry Classifications of Hyperbolic Vector Evolution Equations

Stephen C ANCO † *and Thomas WOLF* ‡

Department of Mathematics, Brock University, Canada L2S 3A1
E-mail: †*sanco@brocku.ca* ,‡*twolf@brocku.ca*

This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

Motivated by recent work on integrable flows of curves and 1+1 dimensional sigma models, several $O(N)$-invariant classes of hyperbolic equations $U_{tx} = f(U, U_t, U_x)$ for an N-component vector $U(t, x)$ are considered. In each class we find all scaling-homogeneous equations admitting a higher symmetry of least possible scaling weight. Sigma model interpretations of these equations are presented.

1 Introduction

Integrability theory of nonlinear partial differential equations (PDEs) has many different aspects which have developed over the years — Bäcklund transformations, Lax pairs, inverse scattering transform and soliton solutions (Calogero's "S-integrability"), linearizing transformations ("C-integrability"), reduction to Painlevé transcendents, hierarchies of higher symmetries and conservation laws, master symmetries, recursion (Nijenhuis) operators, bi-Hamiltonian structures, and others. To date the most computationally direct and effective test of integrability has proved to be the condition that a PDE system should possess sufficiently many higher symmetries [14, 9] (see also [25, 17, 10, 18]). In particular, for all currently known examples of nonlinear scalar PDEs, the existence of one higher symmetry implies the existence of infinitely many, i.e. a symmetry hierarchy.

A rigorous proof of the observation "one symmetry implies infinitely many" was established in [22, 5] for a wide class of semilinear scalar evolutionary PDEs

$$u_t = u_{nx} + f(u, u_x, \dots, u_{n-1x}), \quad n > 1, \tag{1.1}$$

under a homogeneity restriction with respect to a scaling symmetry on t, x, u (so-called λ-homogeneous form). The proof uses the symbolic method of Gel'fand-Dikii, combined with computer algebra computations. As a main result, it was shown that any such symmetry-integrable scalar polynomial PDE with positive scaling weight ($\lambda > 0$) for u belongs to one of the well known hierarchies [1, 13]: $n = 2$ Burger's ($\lambda = 1$); $n = 3$ Korteweg de Vries (KdV) ($\lambda = 2$), modified Korteweg de Vries (mKdV) ($\lambda = 1$), Ibragimov-Shabat (IS) ($\lambda = 1/2$); $n = 5$ Kaup-Kupershmidt (KK) ($\lambda = 2$), Sawada-Kotera (SK) ($\lambda = 2$), Kupershmidt (K) ($\lambda = 1$), potential Kaup-Kupershmidt ($\lambda = 1$), and potential Sawada-Kotera ($\lambda = 1$). The main evolutionary PDEs of semilinear form (1.1) not covered by this classification are the hierarchy of the nonlinear Schrödinger (NLS) equation and its various derivative-type generalizations [1], in which u is complex-valued so its real and imaginary parts satisfy a pair of coupled scalar evolutionary PDEs. There has been much work and interest in finding multi-component generalizations of all these hierarchies, particularly where u is replaced by a vector U or scalar-vector pair (u, U) or matrix $\mathcal{U}$. Recent symmetry-integrability classification results in this direction have been obtained by extensive computer algebra computations [19, 27, 29]. These classifications test for the existence of a higher symmetry, [1] in analogy with the scalar case.

Comparatively less work has been devoted to investigating symmetry-integrability of hyperbolic PDE systems [33], apart from the sine-Gordon (SG) equation and its scalar variants related to the S^2 sigma model, which both have the wave equation form $u_{tx} = f(u, u_t, u_x)$ and possess zero scaling-weight for u.

In this paper we present some symmetry-integrability classifications of hyperbolic vector PDEs. The classifications are motivated by the following three types of examples of scalar wave equations:

(1) the SG equation $u_{tx} = \sin u$, which is distinguished by scaling weight $\lambda = 0$ for u;

(2a) the variant SG equation $u_{tx} = u\sqrt{1 - u_t^2}$ coming from the transformation $u \to \sin^{-1}(u_t)$, which has scaling weight $\lambda = 1$ for u and shares the same higher symmetries as the mKdV hierarchy;

(2b) the complex-valued version $u_{tx} = u\sqrt{1 - |u_t|^2}$, which shares the higher symmetries of the NLS hierarchy;

(3) the mKdV equation in potential form $u_{tx} = u_{4x} + u_x^2 u_{2x}$ and the NLS equation in potential form $\mathrm{i}u_{tx} = u_{3x} + |u_x|^2 u_x$.

Our results give interesting vector counterparts of the first two types of wave equations. We show that these types of hyperbolic vector equations all have a natural geometric origin in terms of integrable nonlinear sigma models and integrable inverse flows of curves in Riemannian manifolds. For the third type of wave equation, we find a null result that no vector counterparts exist.

2 Classification results

For an N-component vector $U(t, x)$, with N arbitrary, consider an $O(N)$-invariant vector PDE of generalized evolutionary form

$$U_{t\,mx} = f(U, U_x, \ldots, U_{nx}, U_t, \ldots, U_{t\,m-1x}), \quad n, m \geq 0 \tag{2.1}$$

namely f is a sum of terms proportional to its vector arguments with scalar coefficients depending on dot products $\langle \cdot, \cdot \rangle$ of these vectors. Such a vector PDE is strictly *evolutionary*

[1] It should be noted that there are special classes of Bakirov-type multicomponent evolutionary PDE systems that, surprisingly, admit only finitely many higher symmetries [4]. However, all these systems are of a triangular form in which one PDE is a decoupled linear evolution equation. In contrast, the classifications in [19, 27, 29] in addition to our results deal with fully coupled, nonlinear multicomponent PDE systems, describing a single vector or matrix nonlinear evolution equation for U or $\mathcal{U}$, or a non-triangular scalar-vector pair of nonlinear evolution equations for (u, U).

if $m = 0, n \geq 1$, or is *hyperbolic* if $m = 1, n \leq 1$; we call it a *wave equation* if $m = 1, n \geq 0$. In all cases it is *semilinear* whenever $n \leq m$ or if f is linear in U_{nx} when $n > m$. We say it is of *minimal differential order* (m, n) provided f is not a total derivative with respect to x. The order of the vector PDE (2.1) in the ordinary sense is $\max(n, m + 1)$.

If we view a vector PDE (2.1) as formally defining the generator of a flow $D_t D_x^m$ in the jet space $J^{(\infty)} = (t, x, U, U_t, U_x, \ldots, U_{kt\,lx}, \ldots)$, then a *higher symmetry* on the solution jet space $R^{(\infty)} \subset J^{(\infty)}$ (i.e. modulo equation (2.1) and differential consequences) will be a commuting flow D_τ whose generator is of the form

$$U_\tau = g(U, U_x, \ldots, U_{rx}, U_t, \ldots, U_{t\,m-1x}) \tag{2.2}$$

for some order $r > \max(n, m + 1)$, where g is a sum of terms proportional to its vector arguments with scalar coefficients depending on dot products $\langle\cdot,\cdot\rangle$ of these vectors. Thus the higher symmetry condition is

$$[D_\tau,\ D_t D_x^m] = 0 \quad \text{ on } R^{(\infty)}\ . \tag{2.3}$$

A flow (2.1) is said to be λ-homogeneous of weight (λ, μ) if it admits a scaling symmetry group (with parameter $\epsilon \in \mathbb{R}$)

$$(t, x, U) \to (e^{\mu\epsilon} t, e^{\epsilon} x, e^{-\lambda\epsilon} U). \tag{2.4}$$

Note, with this convention for scaling weights, the order of a semilinear flow (2.1) is equal to the weight μ in the evolutionary case and $\mu + 1$ in the hyperbolic case. Without loss of generality, higher symmetries (2.2) for such flows can be assumed to be homogeneous under the same scaling symmetry group (2.4) (since any symmetry necessarily decomposes into a sum of terms with homogeneous forms which themselves will define symmetries). Correspondingly, we will refer to higher symmetries (2.2) by their weight as defined through the scaling

$$\tau \to e^{\nu\epsilon}\tau \tag{2.5}$$

induced by the scaling symmetry group (2.4). Note this weight ν is equal to the order r precisely when a higher symmetry is of semilinear form, i.e. linear in U_{rx}.

Throughout, we will assume a non-strict polynomial form for λ-homogeneous vector PDEs (2.1) and higher symmetries (2.2). More specifically, as explained in Sec. 3, f and g will be linear in their vector arguments and polynomial in all nonnegative-weight dot products of these vectors, with undetermined coefficients that either are functions of any zero-weight dot products or otherwise are constants. We refer to such a form as the *nonnegatively-weighted polynomial class* of vector PDEs and higher symmetries. Hereafter we restrict attention to scaling weights $\lambda = 0, \frac{1}{2}, 1, 2$ in analogy with the classes of symmetry-integrable scalar evolutionary PDEs summarized in Sec. 1.

In the case of evolutionary vector PDEs, the following classification was obtained in [27]. Up to scalings of t, x, U, the only vector counterparts of λ-homogeneous evolutionary scalar polynomial PDEs (1.1) with scaling weights $\lambda = \frac{1}{2}, 1, 2$ possessing a homogeneous higher symmetry of least possible weight are given by, in the case of real-valued U:

- vector mKdV equations $\lambda = 1, \mu = 3$,

$$U_t = U_{3x} + \langle U, U\rangle U_x, \tag{2.6}$$

$$U_t = U_{3x} + \langle U, U\rangle U_x + \langle U, U_x\rangle U; \tag{2.7}$$

- vector IS equation $\lambda = \frac{1}{2}, \mu = 3$,

$$U_t = U_{3x} + 3\langle U, U\rangle U_{2x} + 6\langle U, U_x\rangle U + 3\langle U, U\rangle^2 U_x + 3\langle U_x, U_x\rangle U; \tag{2.8}$$

and in the case of complex-valued U:

- vector NLS equations $\lambda = 1, \mu = 2$,

$$\mathrm{i}U_t = U_{2x} \pm \langle U, \bar{U}\rangle U, \tag{2.9}$$
$$\mathrm{i}U_t = U_{2x} \pm 2\langle U, \bar{U}\rangle U \mp \langle U, U\rangle \bar{U}; \tag{2.10}$$

- vector derivative-NLS equations $\lambda = \frac{1}{2}, \mu = 2$,

$$\begin{aligned} \mathrm{i}U_t \;=\; & U_{2x} + a_1\langle U, \bar{U}\rangle U_x + a_2\langle U, U\rangle \bar{U}_x + a_3\langle U, \bar{U}_x\rangle U + a_4\langle \bar{U}, U_x\rangle U + a_5\langle U, U_x\rangle \bar{U} \\ & + a_6\langle U, U\rangle\langle \bar{U}, \bar{U}\rangle U + a_7\langle U, U\rangle\langle U, \bar{U}\rangle \bar{U} + a_8\langle U, \bar{U}\rangle^2 U \end{aligned} \tag{2.11}$$

whose coefficients $a_1, \ldots, a_8$ depend on two constant parameters, falling into 6 classes, as listed in [27]. We remark that this classification does not consider the complex-valued case for $\mu = 3$ or $\mu = 5$, i.e. vector analogs of complexly-coupled mKdV, IS, K, KK, SK, and potential KK, SK equations, other than ones contained in the NLS hierarchy. Hereafter, in the case of complexed-valued U, we allow $t \to \mathrm{i}t$, $\tau \to \mathrm{i}\tau$, and impose invariance under the phase symmetry $U \to e^{\mathrm{i}\epsilon}U$ ($\epsilon \in \mathbb{R}$), as displayed by NLS type systems.

2.1 Vector-potential wave equations

We now proceed to discuss symmetry-integrability classifications of vector U_{tx} equations. Our results in theorems 1–3 are established by computer algebra computations, as outlined in detail in Sec. 3. In each case we explicitly determine all equations admitting a higher symmetry of least weight.

The first class we will consider consists of potential equations related to the integrable evolutionary vector equations (2.6)–(2.11) via $U = V_x$, with scaling weight $\lambda_V = \lambda - 1$. Note that the only ones whose potential form has non-negative scaling weight λ_V are the vector mKdV equations and NLS equations with $\lambda_V = 0$. A natural question is whether they still possess higher symmetries in potential form. We will settle a slightly more general classification problem by considering λ_V-homogeneous vector potential wave equations

$$V_{tx} = f(V, V_x, \ldots, V_{nx}), \quad n \geq 0 \tag{2.12}$$

where f is a sum of terms proportional to the vectors V_{lx}, $0 \leq l \leq n$, with coefficients depending on the dot products $\langle V_{jx}, V_{kx}\rangle$.

Theorem 1. *In the nonnegatively-weighted polynomial class for the potential-mKdV weight $(\lambda_V, \mu) = (0, 3)$, every λ_V-homogeneous vector wave equation (2.12) that possesses a homogeneous higher symmetry of weight $\nu = 5$ containing no t derivatives is linear. The same result is true for the potential-NLS weight $(\lambda_V, \mu) = (0, 2)$ with $\nu = 3$.*

In contrast to this null result, there are many nonlinear *scalar* u_{tx} equations possessing a higher symmetry: for example, the potential-mKdV equation $u_{tx} = u_{4x} + u^2 u_x$, and the 3rd-order Burger's equation in potential form $u_{tx} = u_{4x} + u_x u_{3x} + {u_{2x}}^2 + \frac{1}{3}{u_x}^2 u_{2x}$.

2.2 Vector sine-Gordon equations

The next class of vector wave equations we investigate will be hyperbolic vector variants of the SG equation $u_{tx} = \sin u$, distinguished by the scaling symmetry $(t, x, u) \to (e^{-\epsilon}t, e^{\epsilon}x, u)$. Although the transcendental form of nonlinearity in the SG equation precludes any obvious vector analog, we note there are other integrable hyperbolic scalar equations with the same scaling symmetry yet compatible with a vector nature for u, such as the equation $u_{tx} = -u(1-u^2)^{-1}u_t u_x$ which is related to the scalar wave equation $u_{tx} = 0$ by the transformation $u \to \sin u$. We now classify analogous hyperbolic vector equations possessing higher symmetries with the same weight as the ones in both the SG hierarchy and the hierarchy connected with the scalar wave equation.

Theorem 2. *In the nonnegatively-weighted polynomial class for the SG weight* $(\lambda, \mu) = (0, -1)$*, every* λ*-homogeneous nonlinear hyperbolic vector evolution equation (2.1) that possesses a higher symmetry of weight* $\nu = 3$ *is given by (up to scalings of* t, x, U*)*

$$U_{tx} = -\langle U_t, U_x\rangle(\langle U, U\rangle + \alpha)^{-1}U, \quad \alpha = -1, 0, 1 \tag{2.13}$$

$$U_\tau = \langle U, U_x\rangle^2 U_x. \tag{2.14}$$

This result extends at least to weights $\nu = 4, 5, 6, 7$*, and in particular*

$$U_\tau = (\langle U, U_{3x}\rangle + 3\langle U_x, U_{2x}\rangle)\langle U, U_x\rangle U_x + (\langle U, U_{2x}\rangle + \langle U_x, U_x\rangle)^2 U_x \\ +\alpha(\langle U_x, U_{3x}\rangle + \langle U_{2x}, U_{2x}\rangle)U_x \tag{2.15}$$

is an admitted higher symmetry of order 3 *for weight* $\nu = 5$.

We will give a geometrical interpretation for this wave equation in terms of a N-dimensional sigma model based on a paraboloid Riemannian space in Sec. 4, where we also discuss the question of its integrability.

2.3 Vector hyperbolic flow equations

The final class of vector wave equations we will consider is motivated by the SG variant equation $u_{tx} = u\sqrt{1-u_t^2}$ whose associated flow commutes with the mKdV hierarchy of higher symmetry flows. With respect to this hierarchy, this wave equation represents a -1 flow as it is mapped into the trivial flow $u_\tau = 0$ by the mKdV recursion operator. We recall that the recursion operator generates the mKdV hierarchy starting from the x-translation symmetry $u_\tau = u_x$, referred to as the 0 flow for the hierarchy; the $+1$ flow is the mKdV equation itself $u_\tau = u_{3x} + u^2 u_x$, while its higher symmetries define the $+2$ flow and so on. We now systematically look for hyperbolic vector equations that describe -1 flows for the hierarchies of each integrable vector evolutionary equation (2.6)–(2.11). In general, the -1 flow corresponding to a λ-homogeneous $+1$ flow in the vector case will have weight $(\lambda, -1)$, as we explain next.

Consider first the case of the mKdV and IS hierarchies where the 0 flow is $U_\tau = U_x$ with scaling weight ν_0 equal to 1. Since the scaling weight of the $+1$ flow is $\nu_1 = 3$, the weight difference between successive flows in the hierarchy is $w = \nu_1 - \nu_0 = 2$. Hence if the -1 flow is regarded as a nonlocal evolutionary equation $U_t = D_x^{-1} f(U, U_t, U_x)$ then it will have scaling weight $\mu = \nu_0 - w = 2\nu_0 - \nu_1 = -1$. The remaining case is the NLS and derivative-NLS hierarchies where the 0 flow is $U_\tau = \mathrm{i}U$ with scaling weight

ν_0 equal to 0. The $+1$ flow is then $U_\tau = U_x$, and the (derivative-) NLS equation itself corresponds to the $+2$ flow, with scaling weight $\nu_2 = 2$. The weight difference between flows in this case is $w = \nu_1 - \nu_0 = 1$, and so the -1 flow will then have scaling weight $\mu = 0 - w = 2\nu_0 - \nu_1 = -1$. Therefore in all cases the weight $\mu = -1$ of t for the -1 flow is simply the negative of the weight of x. A similar argument clearly applies to the well-known scalar hierarchies.

We now state classifications that determine all -1 flows in the vector case.

Theorem 3. *(mKdV and NLS -1 flows) For λ-homogeneous nonlinear hyperbolic vector evolution equations (2.1) with weight $(\lambda, \mu) = (1, -1)$ in the nonnegatively-weighted polynomial class, the ones that possess a higher symmetry of mKdV weight $\nu = 3$ are given by (up to scalings of t, x, U)*

$$U_{tx} = \pm\sqrt{1 - \alpha\langle U_t, U_t\rangle}U, \quad \alpha = -1, 1 \tag{2.16}$$

$$U_\tau = U_{3x} + \frac{3}{2}\alpha\langle U, U\rangle U_x, \tag{2.17}$$

and

$$U_{tx} = \langle U, U_t\rangle\langle U_t, U_t\rangle^{-1}(1 + A)U_t - \alpha(1 - A)U, \quad \alpha = -1, 0, 1 \tag{2.18}$$

$$U_\tau = U_{3x} + 3\alpha\langle U, U\rangle U_x + 3\alpha\langle U, U_x\rangle U, \tag{2.19}$$

$$\textit{where } A = \pm\sqrt{1 + \alpha\langle U_t, U_t\rangle}, \tag{2.20}$$

while the ones that possess a higher symmetry of NLS weight $\nu = 2$ are given by (up to scalings of t, x, U)

$$U_{tx} = \langle U, \bar{U}_t\rangle\langle U_t, \bar{U}_t\rangle^{-1}(\alpha + B)U_t - (\alpha - B)U, \quad \alpha = -1, 1 \tag{2.21}$$

$$\mathrm{i}U_\tau = U_{2x} + \frac{1}{4}\langle U, \bar{U}\rangle U \tag{2.22}$$

$$\textit{where } B = \pm\sqrt{\alpha^2 - \frac{1}{2}\langle U_t, \bar{U}_t\rangle}, \tag{2.23}$$

and

$$U_{tx} = \sqrt{C_2 + \langle U_t, \bar{U}_t\rangle + \alpha}\; C_1^{-1}\;\left(\langle U, U_t\rangle\bar{U}_t - \langle U, \bar{U}_t\rangle U_t\right) + \sqrt{C_2 - \langle U_t, \bar{U}_t\rangle - \alpha}\; U, \tag{2.24}$$

$$\mathrm{i}U_\tau = U_{2x} + \frac{1}{2}\langle U, \bar{U}\rangle U - \frac{1}{4}\langle U, U\rangle\bar{U}, \quad \alpha = -1, 0, 1 \tag{2.25}$$

$$\textit{where } C_1 = \sqrt{\langle U_t, U_t\rangle\langle\bar{U}_t, \bar{U}_t\rangle - \langle U_t, \bar{U}_t\rangle^2}, \tag{2.26}$$

$$C_2 = \pm\sqrt{\alpha^2 + 2\alpha\langle U_t, \bar{U}_t\rangle + \langle U_t, U_t\rangle\langle\bar{U}_t, \bar{U}_t\rangle}.$$

All four of these -1 flow equations are expected to be integrable, as they possess a $+1$ flow of the vector mKdV hierarchies or a $+2$ flow of the vector NLS hierarchies (up to scaling of t, x, U, τ). We will interpret the first two of them geometrically in terms of flat homogeneous $N + 1$-dimensional sigma models in Sec. 4.

Finally, we mention that a classification of IS and derivative-NLS -1 flows analogous to theorem 3 in the nonnegatively-weighted polynomial class with weight $(\lambda, \mu) = (\frac{1}{2}, -1)$ yields a null result. However, such flows do exist in a wider polynomial class, which we will present elsewhere.

3 Computational aspects

Here we describe the algorithmic and implementation issues involved in the computations used to prove the classification theorems in Sec. 2. These computations divide into three parts:

(1) making an ansatz for the PDEs and symmetries with specified homogeneity weights;
(2) computing the symmetry conditions;
(3) solving the system of symmetry conditions.

The first two tasks are handled by the program SVSym (Scalar-Vector-Symmetry) [2] and the third computation is performed by the package Crack described in [31].

The class of PDE systems for investigation is allowed to involve any number of scalar functions u and vector functions U of t, x, which we denote collectively as ϕ^i. An essential limitation currently in the program is that the left-hand side (l.h.s.) of each PDE must consist of a single derivative $\phi^i_{pt\,qx}$ sharing the same order $p, q \geq 0$ for all PDEs in the system.

It is assumed that all derivatives occurring on the right-hand side (r.h.s.) of the PDEs and the symmetries are not equal to any l.h.s. derivatives or their differential consequences. In addition, r.h.s. derivatives must have a lower priority with respect to some total ordering $>_T$ of derivatives than all l.h.s. derivatives, to avoid infinite substitution loops in computing the symmetry condition. For example, the equation $u_{tx} = u_{tt} + u_{xx}$ would lead to an infinite loop of substitutions for differential consequences of u_{tx}, as seen by $u_{txx} \to u_{ttx} + u_{xxx}$, $u_{ttx} \to u_{ttt} + u_{txx}$. This problem is avoided if we adopt (without loss of generality) the lexicographical ordering $t >_T x$. Then because we have $u_{tt} >_T u_{tx}$, this equation would be handled by bringing the highest priority derivatives to the l.h.s., $u_{tt} = u_{xx} - u_{tx}$, which no longer leads to any infinite substitution loops.

3.1 Specifying an Ansatz

Many integrable systems are λ-homogeneous with respect to multiple homogeneity weights. For example, denoting the weight for x as ξ, the vector NLS system [27]

$$U_t = U_{xx} + \langle U, V\rangle U, \quad V_t = -V_{xx} - \langle U, V\rangle V$$

and its higher symmetry hierarchy is λ-homogeneous with weights $\mu, \xi, \lambda_U, \lambda_V$ equal to $2, 1, 1, 1$, as well as $0, 0, 1, -1$. The latter weights have been used as an extra filter in our investigation of PDEs in the complex-valued case. To accommodate multiple weightings, the program SVSym allows the specification of a list of weight sets $\{\mu, \xi, \nu, \lambda_i\}$.

The program places two a priori restrictions on possible weights to ensure that the number of possible homogeneous terms allowed to appear in the PDEs and symmetries is necessarily finite. A first restriction is that x cannot have zero weight in all weight sets as this would permit an unlimited differential order of x- derivatives (note the differential order of t- derivatives is always finite due to the ordering $t >_T x$ combined with the fixed number p of t- derivatives on the l.h.s.). A second restriction is that there must be a weight set $\{\mu^*, \xi^*, \nu^*, \lambda^*_i\}$ for which the total weight of the l.h.s. of the PDEs $\phi^i_{pt\,qx}$ and symmetries ϕ^i_τ is non-negative, namely $w^{\text{PDE}}_i := \lambda^*_i + p\mu^* + q\xi^* \geq 0$ (with $\xi^* \neq 0$), and

[2] SVSym is a completely new program, more general than the earlier program used for the classification of evolutionary PDEs in [27, 29], which was limited to positive weights and a strictly polynomial ansatz.

$w_i^{\rm sym} := \lambda_i^* + \nu^* \geq 0$, as then the r.h.s. of the PDEs and symmetries can be generated by the following finite ansatz:

$$u^i_{pt\,qx} = f^i, \quad U^i_{pt\,qx} = \sum_{k,J} F^{ikJ} U^k_J, \qquad \text{and} \qquad u^i_\tau = g^i, \quad U^i_\tau = \sum_{k,J} G^{ikJ} U^k_J. \tag{3.1}$$

Here J is a jet index, denoting partial derivatives including those of zeroth order (only t- and x-derivatives of u^k, U^k are used whose total ordering with respect to $t >_T x$ is lower than that of the l.h.s. of the PDEs). In (3.1), U^k_J has *-weight ≥ 0; $f^i, g^i, F^{ikJ}, G^{ikJ}$ are the most general polynomials (of appropriate weight) that can be built from the following scalar polynomial variables, all of which are constructed to have positive *-weight: u^k_J, products $u^{k_1}_{J_1} u^{k_2}_{J_2}$ where one of the two factors has a negative *-weight, and dot products $\langle U^{k_1}_{J_1}, U^{k_2}_{J_2}\rangle$ where at most one of the two factors may have a negative *-weight. The undetermined coefficients in all polynomials are arbitrary functions of similarly constructed scalar variables u^k_J, $u^{k_1}_{J_1} u^{k_2}_{J_2}$, $\langle U^{k_1}_{J_1}, U^{k_2}_{J_2}\rangle$ whose weight is zero in all weight sets. All weight sets other than the *-weight set are used as an extra filter on the generated terms. (The restriction of using polynomial variables built from at most two scalar or vector factors could obviously be generalized to allow more factors such that the total weight of the product still satisfies the previous conditions.)

It is important to emphasize that the ansatz (3.1) for the PDEs and symmetries need not be strictly polynomial if $\mu^* < 0$ or $\xi^* < 0$ or any $\lambda_i^* \leq 0$.

A few flags give convenient means to cut down the generality of the ansatz, for example by discarding t-derivatives or imposing constant coefficients in the PDEs or symmetries or both. Extra conditions or inequalities on coefficients can be added easily.

The following examples show the ansatz generated for the classifications in theorems 2 and 3. The vector SG ansatz with $\mu = -1, \nu = 3, \lambda_U = 0$ is:

$$\begin{aligned} U_{tx} &= a_1 U, \\ U_\tau &= b_1 U_{3x} + b_2 U_{2x}\langle U_x, U\rangle + b_3 U_x \langle U_{2x}, U\rangle + b_4 U_x \langle U_x, U_x\rangle + b_5 U_x \langle U, U_x\rangle^2 + b_6 U \langle U_{3x}, U\rangle \\ &\quad + b_7 U \langle U_{2x}, U_x\rangle + b_8 U \langle U_x, U_x\rangle \langle U, U_x\rangle + b_9 U \langle U_{2x}, U\rangle\langle U, U_x\rangle + b_{10} U \langle U_x, U\rangle^3 \end{aligned}$$

where all coefficients a_i, b_j are undetermined functions of $\langle U, U\rangle, \langle U_x, U_t\rangle$. The ansatz for the vector mKdV inverse flow with $\mu = -1, \nu = 3, \lambda_U = 1$ is:

$$\begin{aligned} U_{tx} &= a_1 U_t \langle U, U_t\rangle + a_2 U, \\ U_\tau &= b_1 U_{3x} + b_2 U_{2x} \langle U, U_t\rangle + b_3 U_x \langle U_x, U_t\rangle + b_4 U_x \langle U, U_t\rangle^2 + b_5 U_x \langle U, U\rangle + b_6 U_t \langle U_x, U_t\rangle^2 \\ &\quad + b_7 U_t \langle U_x, U_t\rangle\langle U, U_t\rangle^2 + b_8 U_t \langle U_x, U_t\rangle\langle U, U\rangle + b_9 U_t \langle U, U_t\rangle^2 \langle U, U\rangle + b_{10} U_t \langle U, U_t\rangle^4 \\ &\quad + b_{11} U_t \langle U, U_t\rangle\langle U, U_x\rangle + b_{12} U_t \langle U, U_{2x}\rangle + b_{13} U_t \langle U_x, U_x\rangle + b_{14} U_t \langle U, U\rangle^2 \\ &\quad + b_{15} U \langle U_x, U_t\rangle\langle U, U_t\rangle + b_{16} U \langle U, U_t\rangle\langle U, U\rangle + b_{17} U \langle U, U_t\rangle^3 + b_{18} U \langle U, U_x\rangle \end{aligned}$$

where all coefficients a_i, b_j are undetermined functions of $\langle U_t, U_t\rangle$. The ansatz for the vector NLS inverse flow is similar, using a pair of vectors U, V, with $\mu = -2, \nu = 6, \lambda_U = \lambda_V = 1$, and imposing the additional the filter $\mu = \nu = 0, \lambda_U = 1, \lambda_V = -1$.

3.2 Computing symmetry conditions

In the special case of evolutionary PDEs, all substitutions of ϕ^i_t and ϕ^i_τ in the symmetry conditions $\phi^i_{[t,\tau]} := D_\tau \phi^i_t - D_t \phi^i_\tau = 0$ are done only once because the r.h.s. of the PDEs and symmetries do not contain t- or τ- derivatives. As a consequence, the symmetry conditions are separately linear in the undetermined coefficients of the PDEs and of the symmetries.

The situation is different when the l.h.s. of the PDEs contain an x-derivative of ϕ^i, such as in the hyperbolic case ϕ^i_{tx}. Then any t- or x-derivatives that appear in the r.h.s. of the PDEs and symmetries lead to (repeated) substitutions of ϕ^i_{tx} through the PDEs. The symmetry conditions become polynomially non-linear in the coefficients of the PDEs but stay linear in the coefficients of the symmetries. Due to this non-linearity, which is further amplified whenever the coefficients are functions of t-derivatives of ϕ^i, the symmetry conditions can become extremely large. The size of intermediate expressions can become exceedingly high during the repeated substitution and simplification process. To get over this memory hurdle, the linearity in the coefficients of the symmetry is exploited. If the r.h.s. of the symmetry consists of s terms with undetermined coefficients $b_1, \ldots, b_s$ then the full symmetry condition is computed via

$$0 = \phi^i_{[t,\tau]} = \sum_{k=1}^{s} \phi^i_{[t,\tau]}\Big|_{\phi^j_\tau = g^j_{(k)}} \tag{3.2}$$

where $g^j_{(k)}$ denotes the kth term in the r.h.s. of $\phi^j_\tau = g^j$, i.e. all coefficients except b_k are replaced by zero. This trade-off of computing memory for computing time is also justified by the fact that the time to simplify large expressions grows non-linearly with their size, at least in the computer algebra system Reduce that is used.

3.3 Solving the symmetry conditions

Compared to the case of evolutionary PDEs, the symmetry conditions for the general case of PDEs with l.h.s. $\phi^i_{pt\,qx}$ are not purely algebraic but may involve ordinary and partial differential equations and may be inhomogeneous and non-linear in the undetermined coefficients of the PDEs. The system of symmetry conditions is fed into the computer program Crack, which was used to solve the algebraic symmetry conditions in the classification of evolutionary vector equations [27] and coupled evolutionary scalar-vector systems [29]. Originally, Crack has been developed as a solver for overdetermined PDE systems, and this functionality is essentially required for solving the polynomially nonlinear differential system of symmetry conditions that occur in the classifications of vector hyperbolic equations in Sec. 2. Solutions of non-polynomial form, like square roots in the equations found in theorem 3, may appear and often are in reach of the Reduce ODE solver that is employed by Crack when appropriate.

The strength of Crack compared with other computer algebra packages for solving overdetermined systems lies in the flexibility of its approach, the possibility to run it in a fully automatic or interactive mode, and its support in handling especially large systems (safety, process control, heuristic algorithms to cut down the size of equations, ability to take advantage of the linearity of coefficient functions of the symmetries). In addition, Crack provides a verbose mode reporting all computational steps, which allows in principle the solutions to be checked by (human) inspection (in practise this is hard due to the high number of steps).

4 Sigma Model interpretations

The wave equation $u_{tx} = 0$ and SG equation $u_{tx} = \sin u$ are well-known to be intimately related to two of the simplest sigma models [20], which are described by a geometrical wave equation for maps on 2-dimensional Minkowski space $\mathbb{R}^{1,1}$ into target spaces S^1 and S^2, respectively. In general a sigma model on $\mathbb{R}^{1,1}$ is given by the nonlinear wave equation

$$0 = {}^g\nabla_t \partial_x u^A = \partial_t \partial_x u^A + \Gamma^A{}_{BC}(u)\partial_t u^B \partial_x u^C \tag{4.1}$$

for scalar functions $u^A(t,x)$ representing a map into some Riemannian target space (M, g_{AB}), $A = 1,\dots,\dim M$. Here ${}^g\nabla_t = \partial_t + \Gamma^A{}_{BC}(u)\partial_t u^B$ is the pullback to $\mathbb{R}^{1,1}$ of the (torsion-free) covariant derivative determined by the metric g_{AB} in local coordinates u^A on the manifold M, given in terms of the Christoffel symbol $\Gamma^A{}_{BC}$ of the metric. This wave equation has the geometrical action principle $S[u] = \int \langle \partial_t u, \partial_x u\rangle_g \mathrm{d}t\mathrm{d}x$ where $\langle\cdot,\cdot\rangle_g$ denotes the inner product in $T(M)$ given by the metric $g_{AB}(u)$. Another geometrical meaning for this wave equation arises if we view $u^A(t,x)$ for fixed t as a curve embedded into the manifold M. The sigma model thereby describes a flow of this curve, whose arclength is preserved due to the conservation law

$$\partial_t(\partial_x u^A \partial_x u^B g_{AB}(u)) = 0. \tag{4.2}$$

Because of the reflection symmetry $t \leftrightarrow x$ in the model, there is also a conservation law

$$\partial_x(\partial_t u^A \partial_t u^B g_{AB}(u)) = 0. \tag{4.3}$$

These conservation laws are connected with invariance of the model under conformal scalings on t, x, namely $t \to t' := \alpha(t)$, $x \to x' := \beta(x)$. This freedom can be fixed in a natural manner by putting $\alpha(t) = \langle \partial_t u, \partial_t u\rangle_g$ and $\beta(x) = \langle \partial_x u, \partial_x u\rangle_g$. Finally, note that in general the sigma model equation (4.1) is λ-homogeneous with scaling weight $\lambda = 0$ on u^A while t and x have opposite weights of $\mu = -1$ and $+1$, in the conventions of Sec. 2.

$O(N)$-invariant vector wave equations arise from sigma models in various ways and for certain types of target spaces. One direct way is by considering target spaces embedded as an $O(N)$ symmetric hypersurface in Euclidean space $\mathbb{R}^{N+1}$ or Minkowski space $\mathbb{R}^{N,1}$. Vector coordinates for the hypersurface given by its embedding lead to natural vector generalizations of the variant scalar wave equation $u_{tx} = -u(1-u^2)^{-1}u_t u_x$ which is the sigma model for $S^1 \subset \mathbb{R}^2$. A second way involves introducing a Cartan connection and frame bundle on flat homogeneous target spaces $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^{N+1}$ based on pairs of Lie algebras $\mathfrak{g} \supset \mathfrak{h}$. With a suitable choice of frame, the flat connection components yield natural vector generalizations of the variant SG equation $u_{tx} = u\sqrt{1-u_t^2}$ which comes from the linear sigma model for $\mathfrak{so}(3)/\mathfrak{so}(2) \simeq \mathbb{R}^2$. The SG equation itself is related in a similar way to a moving frame formulation of the nonlinear sigma model $S^2 = SO(3)/SO(2)$ shown in [2].

In Sec. 4.1 we show that the vector wave equation (2.13) in theorem 2 arises from the nonlinear sigma model for an $O(N)$ symmetric paraboloid hypersurface in $\mathbb{R}^{N+1}$. In Sec. 4.2 we derive the first two hyperbolic vector equations in theorem 3 from the Cartan connection of linear sigma models viewed as a flow of curves on flat homogeneous spaces associated with the Lie algebras $\mathfrak{so}(N+1) \subset \mathfrak{so}(N+2), \mathfrak{su}(N+1)$.

4.1 N-sphere and N-dimensional paraboloid sigma models

Consider firstly the N-sphere target space S^N as an embedded hypersurface in $\mathbb{R}^{N+1}$. We describe S^N by the $N+1$-component unit vector u^i, $i = 1, \ldots, N+1$, constrained to satisfy $1 = u^i u^j \delta_{ij}$. If we resolve the constraint by expressing $u^{N+1} = \pm\sqrt{1 - u^A u^B \delta_{AB}}$ with $u^i = (u^A, u^{N+1})$, $A = 1, \ldots, N$, then u^A represents a set of coordinates on S^N. In particular the upper or lower hemisphere of S^N is mapped by u^A to the open unit ball in $\mathbb{R}^N$. In these coordinates, the Riemannian metric on S^N as induced by the Euclidean metric δ_{ij} on $\mathbb{R}^{N+1}$ is given by

$$g_{AB}(u) = \delta_{AB} + (1 - u^C u_C)^{-1} u_A u_B \tag{4.4}$$

with the inverse metric $g^{AB} = \delta^{AB} - u^A u^B$, where $u_A := \delta_{AB} u^B$. The Christoffel symbol and covariant derivative on S^N are readily found from the condition ${}^g\nabla_C g^{AB}(u) = 0$, giving

$${}^g\nabla_C = \frac{\partial}{\partial u^C} + u^A g_{BC}(u), \quad \Gamma^A{}_{BC}(u) = u^A(\delta_{BC} + (1 - u^D u_D)^{-1} u_B u_C). \tag{4.5}$$

Hence the curvature is given by

$$[{}^g\nabla_A, {}^g\nabla_B] := R_{ABC}{}^D = 2 g_{C[A}(u) \delta_{B]}{}^D \chi, \quad \chi = 1 \tag{4.6}$$

where χ denotes the scalar curvature. Note this indicates S^N is an $O(N)$ symmetric space of (positive) constant curvature.

The sigma model equation (4.1) for the target space S^N thus has the form

$$0 = \partial_t \partial_x u^A + u^A(\partial_t u^B \partial_x u_B + (1 - u^B u_B)^{-1} u_C u_D \partial_t u^C \partial_x u^D) \tag{4.7}$$

which is an $O(N)$-invariant hyperbolic vector equation for $u^A(t, x)$. The scalar case $N = 1$ is seen to reduce to $0 = u_{tx} + u(1 - u^2)^{-1} u_t u_x$. Integrability of this sigma model for all $N \geq 1$ is a well-established result via a Lax pair associated with $S^N = SO(N+1)/SO(N)$ as a symmetric space [32, 20], and also more recently via a bi-Hamiltonian structure (higher-symmetry recursion operator) derived from a moving frame formulation of S^N as a constant curvature space [23, 2]. Its higher symmetries were first found in [20] using a Bäcklund transformation technique which relies on conformally rescaling $t \to t'$, $x \to x'$ through the conservation laws (4.2) and (4.3). The hierarchy of higher symmetries is homogeneous and has positive scaling weight $\nu = 3, 5, \ldots$, with respect to the scaling group $(x', t', u^A) \to (e^\epsilon x', e^{-\epsilon} t', u^A)$, $\epsilon \in \mathbb{R}$. But when expressed in the original variables t, x, the higher symmetries are all homogeneous of weight $\nu = 0$. For this reason the vector wave equation (4.7) does not appear in our classifications.

To account for the vector wave equation (2.13) and its higher symmetry (2.15) with SG scaling weight in theorem 2, we look for a target space with an appropriate Riemannian metric. Working backward from the nonlinear terms in the equation

$$0 = \partial_t \partial_x u^A + u^A (1 + u^C u_C)^{-1} \partial_t u^B \partial_x u_B \tag{4.8}$$

we deduce the Christoffel symbol

$$\Gamma^A{}_{BC}(u) = u^A \delta_{BC} (1 + u^D u_D)^{-1} \tag{4.9}$$

and hence

$$ {}^{g}\nabla_C = \frac{\partial}{\partial u^C} + u^A \delta_{BC} (1 + u^D u_D)^{-1} \tag{4.10} $$

is the covariant derivative. From the condition ${}^{g}\nabla_C g^{AB}(u) = 0$, it is straightforward to derive $g^{AB} = \delta^{AB} + (1 + u^C u_C)^{-1} u^A u^B$ which then gives the metric

$$ g_{AB}(u) = \delta_{AB} - u_A u_B. \tag{4.11} $$

Finally we are able to identify this metric as the induced Riemannian metric on the N-dimensional target space given by the paraboloid hypersurface

$$ u^{N+1} = \frac{1}{2} u^A u^B \delta_{AB} \quad \text{in } \mathbb{R}^{N+1}. \tag{4.12} $$

This target space is an $O(N)$ symmetric Riemannian manifold, whose scalar curvature is non-constant

$$ \chi = (1 + 2u^{N+1})^{-1}, \quad R_{ABC}{}^{D} = 2\delta_{C[A} g_{B]}{}^{D}(u) \chi \tag{4.13} $$

where $g_A{}^B := \delta_{AC} g^{CB}$. (Curiously, the metric tensor $g_A{}^B(u)$ on the N-dimensional paraboloid is the same as the inverse metric tensor $g^B{}_A(u) := g^{BC}(u)\delta_{CA}$ on the N-sphere.)

We note there is a Lorentzian version of these sigma models (4.7) and (4.8) obtained via replacing the Euclidean space $\mathbb{R}^{N+1}$ by Minkowski space $\mathbb{R}^{N,1}$. Correspondingly, S^N is replaced by its Lorentzian counterpart given by the N-dimensional hyperboloid space $u^{N+1} = \pm\sqrt{1 + u^A u^B \delta_{AB}}$ which comes from the spacelike hypersurface $1 = u^i u^j \eta_{ij}$ in $\mathbb{R}^{N,1}$, where η_{ij} is the Minkowski metric and $u^i = (u^A, u^{N+1})$. Its induced Riemannian metric is $g_{AB}(u) = \delta_{AB} + (1 + u^C u_C)^{-1} u_A u_B$, with constant (negative) scalar curvature $\chi = -1$. In contrast, the N-dimensional paraboloid space $u^{N+1} = \frac{1}{2} u^A u^B \delta_{AB}$ is its own Lorentzian counterpart, but with the metric replaced by $g_{AB}(u) = \delta_{AB} + u_A u_B$ as induced by the Minkowski metric η_{ij}, and with non-constant scalar curvature $\chi = (1 - 2u^{N+1})^{-1}$. Note in the Lorentzian case this paraboloid hypersurface has a null tangent direction at points $2u^{N+1} = 1$ in $\mathbb{R}^{N,1}$ where both the metric and curvature become singular.

Thus the nonlinear sigma model (4.8) for the N-dimensional paraboloid target space in the Euclidean and Lorentzian cases reproduces the vector wave equation (2.13) with the vector U identified as the local coordinates u^A. Integrability of this model does not seem to be known in the literature, so its higher symmetry (2.15) stated in theorem 2 is a new result. Unlike the S^N model, the existence of a hierarchy of higher symmetries is an open question, apart from the scalar case $N = 1$ of this model (4.8) which gives the same integrable S^1 sigma model as the scalar reduction of the S^N model (4.7). As noted in the theorem, for $N \geq 1$ we checked that there are higher symmetries of all weights $\nu = 4, 5, 6, 7$. This strongly suggests the model (4.8) is indeed integrable. A proof will require exhibiting e.g. a Lax pair, a bi-Hamiltonian structure, or a symmetry-recursion operator.

4.2 Flat homogeneous sigma models

We now turn to the flat homogeneous target spaces $M = \mathfrak{g}/\mathfrak{h}$ based on Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ chosen to be semi-simple and compact, with the structure of a symmetric space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ such that $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$ where M is identified with the vector space $\mathfrak{p}$ in this decomposition. These target spaces $M = \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{p}$ are thus flat symmetric Riemannian manifolds whose metric is given by the Cartan-Killing inner product $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ on $\mathfrak{g}$ restricted to M. A few main examples are $\mathfrak{so}(k+1)/\mathfrak{so}(k)$, $\mathfrak{su}(k)/\mathfrak{so}(k)$, $\mathfrak{su}(k+1)/\mathfrak{u}(k)$ (see [12] for a complete list and properties). The sigma model (4.1) for all such flat target spaces is just the linear wave equation

$$0 = \partial_t \partial_x u^A \tag{4.14}$$

for scalar functions $u^A(t, x)$. Nevertheless, viewed as a flow of curves in $M \simeq \mathfrak{g}/\mathfrak{h}$, there is a natural frame bundle formulation for this model using a Cartan connection whose components will satisfy an $O(N)$-invariant nonlinear vector wave equation [7]. Details of Cartan connections and frame bundles for homogeneous spaces are presented in [24].

Firstly, we consider $M \simeq \mathbb{R}^{N+1} = \mathfrak{p}$ as given by $\mathfrak{g} = \mathfrak{so}(N+2)$ and $\mathfrak{h} = \mathfrak{so}(N+1)$. Recall, $\mathfrak{so}(k)$ is a real vector space of dimension $\frac{k(k-1)}{2}$ isomorphic to the Lie algebra of $k \times k$ skew-symmetric matrices. So $M = \mathfrak{so}(N+2)/\mathfrak{so}(N+1)$ is isomorphic to $\mathfrak{p} = \mathbb{R}^{N+1}$, as described by the following canonical decomposition

$$\begin{pmatrix} 0 & p \\ -p^{\mathrm{T}} & \mathbf{h} \end{pmatrix} \in \mathfrak{so}(N+2), \quad \mathbf{h} \in \mathfrak{so}(N+1), \quad p \in \mathbb{R}^{N+1} \tag{4.15}$$

parameterized by the $N+1$-component vector p^A, $A = 1, \ldots, N+1$. Note the Cartan-Killing inner product on $\mathfrak{g}$ is (up to a normalization factor) the trace of the product of $\mathfrak{so}(N+2)$ matrices (4.15) and hence reduces for $h = 0$ to the ordinary dot product δ_{AB} on vectors p^A. On the symmetric space $\mathfrak{p} = \mathfrak{so}(N+2)/\mathfrak{so}(N+1)$ we introduce an $\mathfrak{so}(N+2)$-valued flat connection ${}^{\mathfrak{g}}\omega_A$ with the decomposition ${}^{\mathfrak{g}}\omega_A = {}^{\mathfrak{h}}\omega_A + {}^{\mathfrak{p}}\omega_A$. As a consequence of the Lie algebra structure (4.15), ${}^{\mathfrak{h}}\omega_A$ provides a Cartan connection to which is associated an $\mathbb{R}^{N+1}$-valued orthonormal frame e^A for the vector space $\mathfrak{so}(N+2)/\mathfrak{so}(N+1) \simeq \mathbb{R}^{N+1}$. More precisely, the underlying (gauge) structure group here is $SO(N+1)$, namely the matrix Lie group associated with $\mathfrak{h} = \mathfrak{so}(N+1)$. Up to the gauge freedom of $SO(N+1)$ rotations, there is a canonical relation ${}^{\mathfrak{p}}\omega_A = e_A$ between the frame and the connection, where e_A is the coframe satisfying $\langle e_A, e^B \rangle_{\mathfrak{p}} = \delta_A{}^B$.

To derive an $O(N)$-invariant vector wave equation from the linear sigma model (4.14) for this flat symmetric target space $M = \mathfrak{so}(N+2)/\mathfrak{so}(N+1)$, we first write ${}^{\mathfrak{h}}\omega_t := {}^{\mathfrak{h}}\omega_A \partial_t u^A$, ${}^{\mathfrak{h}}\omega_x := {}^{\mathfrak{h}}\omega_A \partial_x u^A$, $e_t := e_A \partial_t u^A$, $e_x := e_A \partial_x u^A$, in terms of $u^A(t, x)$. These obey

$$0 = \partial_t e_x + {}^{\mathfrak{h}}\omega_t e_x = \partial_x e_t + {}^{\mathfrak{h}}\omega_x e_t \tag{4.16}$$

as a consequence of the wave equation (4.14) and the structure equation of the frame ${}^{\mathfrak{p}}\omega_A = e_A$. Interpreted geometrically, the second equality is the Cartan structure equation for vanishing torsion of the connection ${}^{\mathfrak{g}}\omega_A$. The Cartan structure equation for the curvature of this flat connection gives

$$\partial_x {}^{\mathfrak{h}}\omega_t - \partial_t {}^{\mathfrak{h}}\omega_x + [{}^{\mathfrak{h}}\omega_x, {}^{\mathfrak{h}}\omega_t] = [{}^{\mathfrak{p}}\omega_t, {}^{\mathfrak{p}}\omega_x] = e_t \wedge e_x \tag{4.17}$$

where $\wedge$ denotes the antisymmetric tensor product. Note from equation (4.16) we recover the conservation laws (4.2) and (4.3) for this sigma model, $0 = \partial_t\langle e_x, e_x\rangle_{\mathfrak{p}} = \partial_x\langle e_t, e_t\rangle_{\mathfrak{p}}$. By conformally scaling t, x, we normalize $\langle e_x, e_x\rangle_{\mathfrak{p}} = \langle e_t, e_t\rangle_{\mathfrak{p}} = 1$. Then we use the inherent gauge freedom on e_A using the rotation group $SO(N+1)$ to put

$$e_x{}^{N+1} = 1, \quad e_x{}^i = 0, \tag{4.18}$$

without loss of generality. Thus one of the conservation laws is satisfied identically; the other one determines

$$e_t{}^{N+1} = \pm\sqrt{1 - e_t{}^i e_t{}^j \delta_{ij}}. \tag{4.19}$$

In the index notation here, $e_x = (e_x{}^i, e_x{}^{N+1})$ and $e_t = (e_t{}^i, e_t{}^{N+1})$, $i = 1, \ldots, N$, are $(N+1\times 1)$ column matrices comprising the frame components of, respectively, the tangent vector $\partial_x u^A$ along the curve $u^A(t,x)$ and the direction vector $\partial_t u^A$ of the flow on $M = \mathfrak{so}(N+2)/\mathfrak{so}(N+1)$. The condition (4.18) geometrically means e^A restricted to the curve $u^A(t,x)$ is an adapted moving frame, consisting of N normal vectors and one tangential vector (so x has the geometrical meaning of arclength). There now remains $SO(N)$ gauge freedom that preserves the adapted form (4.18) of e_x, which is described by rotations in the normal space along the curve. We use this freedom to fix the connection ${}^{\mathfrak{h}}\omega_x$ so it corresponds geometrically to specializing the adapted moving frame to be parallel (cf. [6]) with respect to the curve $u^A(t,x)$ as follows. Write ${}^{\mathfrak{h}}\omega_x = ({}^{\mathfrak{h}}\omega_x{}^{ij}, {}^{\mathfrak{h}}\omega_x{}^{iN+1})$ and ${}^{\mathfrak{h}}\omega_t = ({}^{\mathfrak{h}}\omega_t{}^{ij}, {}^{\mathfrak{h}}\omega_t{}^{iN+1})$ for the components of the $(N+1\times N+1)$ connection matrices. The curvature equation (4.17) shows that the pair of matrices ${}^{\mathfrak{h}}\omega_x{}^{ij}, {}^{\mathfrak{h}}\omega_t{}^{ij}$ constitute a flat $\mathfrak{so}(N)$-valued connection, and consequently along the curve we put ${}^{\mathfrak{h}}\omega_x{}^{ij} = 0$ by the available gauge freedom. This leads to ${}^{\mathfrak{h}}\omega_t{}^{ij} = 0$ without loss of generality. Geometrically speaking, ${}^{\mathfrak{h}}\omega_x$ then describes the derivative of e^A under infinitesimal displacement along the curve $u^A(t,x)$ such that the normal vectors in the frame have a purely tangential derivative.

Now, the torsion equation (4.16) can be used to find

$$ {}^{\mathfrak{h}}\omega_t{}^{iN+1} = 0, \tag{4.20}$$

$$ {}^{\mathfrak{h}}\omega_x{}^{iN+1} = (e_t{}^{N+1})^{-1}\partial_x e_t{}^i, \tag{4.21}$$

while from the curvature equation (4.17) we have

$$e_t{}^i = -\partial_t {}^{\mathfrak{h}}\omega_x{}^{iN+1}. \tag{4.22}$$

Thus the reduced Cartan equations (4.21) and (4.22) together with the relation (4.19) yield a system of coupled vector equations $\partial_x e_t{}^i = \pm(1 - e_t{}^i e_t{}^j \delta_{ij})^{1/2}\,{}^{\mathfrak{h}}\omega_x{}^{iN+1}$ and $\partial_t {}^{\mathfrak{h}}\omega_x{}^{iN+1} = -e_t{}^i$. Viewing ${}^{\mathfrak{h}}\omega_x{}^{iN+1}$ as an N-component vector $U(t,x)$, and eliminating $e_t{}^i$ from these equations, we obtain precisely the $O(N)$-invariant nonlinear vector wave equation (2.16) in theorem 3. As shown by the theorem, this vector wave equation possesses a higher symmetry given by the $+1$ flow in the vector mKdV hierarchy (2.6).

Integrability of the nonlinear vector wave equation (2.16) was first shown via a Lax pair [8, 3]. Related derivations of it using both a zero-curvature and a Lie-algebraic formulation of the Lax pair were obtained in [21, 7, 30]. Our derivation here is more intrinsically geometrical, based on the approach in [16, 23] relying on use of the Cartan connection of a parallel moving frame

$$e_x = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}, \quad {}^{\mathfrak{h}}\omega_x = \begin{pmatrix} 0 & U \\ -U^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N+1) \tag{4.23}$$

for a curve $u^A(t,x)$ associated to the sigma model (4.14) interpreted as a flow of curves on the flat symmetric space $\mathfrak{so}(N+2)/\mathfrak{so}(N+1) \simeq \mathbb{R}^{N+1}$. In particular, in this frame the connection components U geometrically represent differential invariants of the curve.

The second $O(N)$-invariant vector wave equation (2.18) in theorem 3 has a similar derivation, starting from the target space $M = \mathfrak{su}(N{+}1)/\mathfrak{so}(N{+}1)$. One notable difference in this sigma model is that the target space has dimension greater than $N+1$ and hence a reduction condition must be imposed on the Cartan connection and its associated frame, allowing them to be expressed in terms of a $N{+}1$-component vector. The higher symmetry structure for this vector wave equation obtained by theorem 3, identified as the $+1$ flow in the vector mKdV hierarchy (2.7), is a new result. Of course, a proof of its integrability will require e.g. checking that it admits the symmetry-recursion operator of the hierarchy. The same remarks apply to the other two vector wave equations (2.21) and (2.24) in theorem 3.

To outline the derivation of equation (2.18), recall $\mathfrak{su}(k)$ is a complex vector space isomorphic to the Lie algebra of $k \times k$ skew-hermitian matrices. The real and imaginary parts of these matrices belong to the real vector space $\mathfrak{so}(k)$ of skew-symmetric matrices and the real vector space $\mathfrak{s}(k) \simeq \mathfrak{su}(k)/\mathfrak{so}(k)$ defined by $k{\times}k$ symmetric trace-free matrices. Hence $\mathfrak{g} = \mathfrak{su}(N{+}1)$ has the decomposition $\mathfrak{g} = \mathfrak{h}{+}\mathrm{i}\mathfrak{p}$ where $\mathfrak{h} = \mathfrak{so}(N{+}1)$ and $\mathfrak{p} = \mathfrak{s}(N{+}1)$. There is a natural subset of matrices in $\mathfrak{su}(N+1)$ given by the special form

$$\frac{1}{n+1} p\cdot p\,\mathbb{I} - p^{\mathrm{T}}p \in \mathfrak{s}(N+1), \quad p \in \mathbb{R}^{N+1} \tag{4.24}$$

which is parametrized by the arbitrary $N+1$-component vector p^A. This set of symmetric trace-free matrices (4.24), hereafter denoted $\mathfrak{s}_p(N+1)$, provides a canonical reduction of $\mathfrak{su}(N+1)/\mathfrak{so}(N+1)$ to $\mathbb{R}^{N+1}$ as a $N+1$-dimensional manifold. The restriction of the Cartan-Killing metric of $\mathfrak{su}(N+1)$ to $\mathfrak{s}_p(N+1)$ yields a Riemannian metric $\langle\cdot,\cdot\rangle_{\mathfrak{s}_p}$ proportional to the trace of the product of matrices (4.24). In particular, the norm of a matrix (4.24) in this metric (with a suitable normalization) is simply $p\cdot p$. (Note the nonlinearity of the matrix norm in terms of p is due to the lack of a vector space structure for $\mathfrak{s}_p(N+1)$.) On this space $\mathfrak{s}_p(N+1)$ we introduce an $\mathfrak{su}(N+1)$-valued flat connection ${}^{\mathfrak{g}}\omega_A = {}^{\mathfrak{h}}\omega_A + \mathrm{i}{}^{\mathfrak{s}}\omega_A$, with ${}^{\mathfrak{h}}\omega_A$ defining a Cartan connection to which is associated an $\mathfrak{s}_p(N+1)$-valued frame e^A and a coframe $e_A = {}^{\mathfrak{s}}\omega_A$ such that $\langle e_A, e^B\rangle_{\mathfrak{s}_p} = \delta_A{}^B$. There is $SO(N+1)$ gauge freedom inherent in e_A and ${}^{\mathfrak{h}}\omega_A$.

To proceed we use the Riemannian space $M \simeq \mathfrak{s}_p(N+1) \subset \mathfrak{su}(N+1)/\mathfrak{so}(N+1)$ as the reduced target space in the sigma model for $u^A(t,x)$. Along the tangent vector $\partial_x u^A$ and the flow direction vector $\partial_t u^A$ for the curve $u^A(t,x)$, we have the frame columns e_x, e_t and the connection matrices ${}^{\mathfrak{h}}\omega_x, {}^{\mathfrak{h}}\omega_t$. These obey the torsion and curvature equations

$$0 = \partial_t e_x + {}^{\mathfrak{h}}\omega_t e_x = \partial_x e_t + {}^{\mathfrak{h}}\omega_x e_t, \tag{4.25}$$

$$\partial_x {}^{\mathfrak{h}}\omega_t - \partial_t {}^{\mathfrak{h}}\omega_x + [{}^{\mathfrak{h}}\omega_x, {}^{\mathfrak{h}}\omega_t] = [{}^{\mathfrak{s}}\omega_x, {}^{\mathfrak{s}}\omega_t] = [e_x, e_t]. \tag{4.26}$$

Using the sigma model conservation laws (4.2) and (4.3), we put $\langle e_x, e_x\rangle_{\mathfrak{s}p} = \langle e_t, e_t\rangle_{\mathfrak{s}p} = 1$. We further fix $e_x, {}^{\mathfrak{h}}\omega_x$ so they represent a parallel moving frame and connection (via the $SO(N+1)$ gauge freedom) adapted to the curve $u^A(t,x)$. Namely, in matrix notation,

$$e_x = \frac{1}{n+1}\mathbb{I} - \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{0} \end{pmatrix} \in \mathfrak{s}p(N+1), \quad {}^{\mathfrak{h}}\omega_x = \begin{pmatrix} 0 & U \\ -U^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \in \mathfrak{so}(N+1), \quad U \in \mathbb{R}^N \tag{4.27}$$

where $\mathbf{0} \in \mathfrak{so}(N)$, and also

$$e_t = \frac{1}{n+1}\mathbb{I} - \begin{pmatrix} v & V \\ V^{\mathrm{T}} & v^{-1}V^{\mathrm{T}}V \end{pmatrix} \in \mathfrak{s}p(N+1), \quad (V, v) \in \mathbb{R}^{N+1} \tag{4.28}$$

where

$$v^2 + V \cdot V = v. \tag{4.29}$$

Note e_x, e_t are of the form (4.24) with respective parametrizations chosen in vector index notation as $1 = p^{N+1}$, $0 = p^i$, and $\sqrt{v} = p^{N+1}$, $(\sqrt{v})^{-1}V^i = p^i$; the norm of e_t is $1 = (p^{N+1})^2 + p^i p^j \delta_{ij} = v + v^{-1}V^iV^j\delta_{ij}$. Now, from the $\mathfrak{so}(N)$ part of the curvature equation (4.26) together with the first half of the torsion equation (4.25), we find ${}^{\mathfrak{h}}\omega_t = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{0} \end{pmatrix}$. Then the remaining parts of equations (4.25) and (4.26) reduce to

$$\partial_x V = vU - v^{-1}U \cdot V\ V, \quad \partial_t U = -V, \tag{4.30}$$

$$\partial_x v = -2U \cdot V, \quad \partial_x(v^{-1}V^TV) = V^{\mathrm{T}}U + U^{\mathrm{T}}V. \tag{4.31}$$

We see equations (4.30) and (4.29) are equivalent to the $O(N)$-invariant nonlinear vector wave equation (2.18), while the last equation (4.31) is found to hold as an identity.

4.3 Other models and concluding remarks

The sigma models connected with the two $O(N)$-invariant nonlinear vector wave equations (2.16) and (2.18) in theorem 3 exhaust all the types of real symmetric spaces $\mathfrak{g}/\mathfrak{so}(N+1)$. The other two vector wave equations (2.21) and (2.24), which in contrast are complex-valued and $U(N)$-invariant, might then be expected to arise from hermitian symmetric spaces $\mathfrak{g}/\mathfrak{u}(N+1)$. However, sigma models based on that type of target space naturally take the form of coupled scalar-vector wave equations, and hence degenerate cases would have to be investigated in order to account for these purely vectorial wave equations (2.21) and (2.24).

Interestingly, the last vector wave equation (2.24) given in theorem 3 stands out from the others in one respect; it is neither well-defined in the scalar case $N = 1$ nor in the real-valued vector case (precisely when the expression (2.26) vanishes). Thus it exhibits an intrinsic complex-valued vectorial nature. Some insight on this situation comes from the NLS higher symmetry (2.25) admitted by this vector wave equation. In [15] the corresponding vector NLS hierarchy (2.10) is shown to originate from the matrix NLS equation $\mathrm{i}\partial_t\mathcal{U} = \partial_x^2\mathcal{U} + \mathcal{U}\mathcal{U}^\dagger\mathcal{U}$ where the matrix $\mathcal{U} = u^A\gamma_A$ is chosen to belong to the span of the gamma matrices γ_A, $A = 1, \ldots, N$, defined to satisfy the Clifford algebra relations $\gamma_A\gamma_B + \gamma_B\gamma_A = 2\delta_{AB}\mathbb{I}$. This observation suggests the vector wave equation (2.24) may arise in a similar fashion from a suitable matrix type of sigma model.

Finally, in connection with these remarks, we mention the deep work [26, 11, 28] of Snivolupov and Sokolov on the construction of integrable vector and matrix equations of hyperbolic and evolutionary type, related to special kinds of nonassociative algebras.

Acknowledgements

One author (TW) wants to thank Winfried Neun for discussions, and the Konrad Zuse Institute Berlin for support through its fellowship program. Takayuki Tsuchida is thanked for comments on derivative-NLS −1 flows. Computations were performed partially on computer clusters of the SHARCNET consortium (www.sharcnet.com).

References

[1] Ablowitz M J and Clarkson P A, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge 1991.

[2] Anco S C, Wang J P, Integrable flows of curves in Riemannian and conformal manifolds, in preparation (2004).

[3] Bakas I, Park Q-H, Shin H-J, Lagrangian formulation of symmetric space sine-Gordon models, *Phys. Lett.* B **372** (1996), 45–52.

[4] Beukers F, Sanders J A, and Wang J P, One symmetry does not imply integrability, *J. Differential Equations* **146** (1998), 251-260; van der Kamp P H and Sanders J A, On testing integrability, *J. Nonl. Math. Phys.* **8** (2001), 561–574.

[5] Beukers F, Sanders J A, and Wang J P, On integrability of systems of evolution equations, *J. Differential Equations* **172** (2001), 396-408.

[6] Bishop R, There is more than one way to frame a curve, *Amer. Math. Monthly* **82** (1975), 246–251.

[7] D'Auria R, Regge T, and Sciuto S, A general scheme for bidimensional models with associated linear set, *Phys. Lett.* B **89** (1980) 363–366; ibid, Group-theoretical construction of two-dimensional models with infinite sets of conservation laws, *Nuclear Phys.* B **171** (1980) 167–188.

[8] Eichenherr H and Pohlmeyer K, Lax pairs for certain generalizations of the sine-Gordon equation, *Phys. Lett.* B **89** (1979), 76–78.

[9] Fokas A S, A symmetry approach to exactly solvable evolution equations, *J. Math. Phys.* **21** (1980), 1318–1325.

[10] Fokas A S, Symmetries and integrability, *Stud. Appl. Math.* **77** (1987) 253–299.

[11] Habibullin I T, Sokolov V V, and Yamilov R I, Multi-component integrable systems and nonassociative structures, in Nonlinear Physics: theory and experiment World Scientific Publishing, 1996.

[12] Helgason S, Differential Geometry, Lie Groups, and Symmetric Spaces, Amer. Math. Soc., Providence, 2001.

[13] Ibragimov N H (editor), CRC Handbook of Lie Group Analysis of Differential Equations, CRC Press, Boca Raton, 1996.

[14] Ibragimov N and Shabat A B, Evolutionary equations with nontrivial Lie-Backlund algebra, *Funct. Anal. Appl.* **14** (1980) 19–30.

[15] Kulish P P and Sklyanin E K, O(N)-invariant nonlinear Schrödinger equation — a new completely integrable system, *Phys. Lett.* A **84** (1981), 349–352.

[16] Mari Befa G, Sanders J A, and Wang J P, Integrable systems in three-dimensional Riemannian geometry, *J. Nonlinear Sci.* **12** (2002), 143–167.

[17] Mikhailov A V, Shabat A B, and Yamilov R I, The symmetry approach to classification of integrable equations: complete lists of integrable systems, *Russ. Math. Surv.* **42** (1987), 1–51.

[18] Mikhailov A V, Shabat A B, and Sokolov V V, Symmetry approach to classification of integrable equations, in What is Integrability?, Springer-Verlag, 1999.

[19] Olver P J and Sokolov V V, Integrable evolution equations on associate algebras, *Comm. Math. Phys.* **193** (1998), 245–269.

[20] Pohlmeyer K, Integrable Hamiltonian systems and interactions through quadratic constraints, *Comm. Math. Phys.* **46** (1976), 207–221.

[21] Pohlmeyer K and Rehren K-H, Reduction of the two-dimensional O(n) nonlinear σ-model, *J. Math. Phys.* **20** (1979), 2628–2632.

[22] Sanders J A and Wang J P, On the integrability of homogeneous scalar evolution equations, *J. Differential Equations* **147(2)** (1998), 410-434; ibid, On the integrability of non-polynomial scalar evolution equations, *J. Differential Equations* **166** (2000), 132-150.

[23] Sanders J A and Wang J P, Integrable systems in n-dimensional Riemannian geometry, *Moscow Mathematical Journal* **3** (2003), 1369–1393.

[24] Sharpe R W, Differential Geometry, Springer-Verlag, New York, 1997.

[25] Sokolov V V and Shabat A B, Classification of integrable evolution equations, *Sov. Sci. Rev. C* **4** (1984), 221–280.

[26] Sokolov V V and Svinolupov S I, Deformation of nonassociative algebras and integrable differential equations, Acta Appl. Math. **41** (1995), 323–339.

[27] Sokolov V V and Wolf T, Classification of integrable polynomial vector evolution equations, *J. Phys. A: Math. Gen.* **34** (2001), 11139-11148.

[28] Svinolupov S I and Sokolov V V, Deformations of Jordan triple systems and integrable equations, Theoret. and Math. Phys. **108** (1997), 1160 - 1163.

[29] Tsuchida T and Wolf T, Classification of integrable coupled systems with one scalar and one vector unknown, *RIMS Kokyuroku* **No. 1302** (2003), 68-90 (in Japanese); ibid, Classification of polynomial integrable systems of mixed scalar and vector evolution equations. I (preprint), nlin.SI/0412003.

[30] Wang J P, Generalized Hasimoto transformation and vector sine-Gordon equation, in Symmetry and Perturbation Theory, World Scientific, 2003.

[31] Wolf T, Applications of CRACK in the Classification of Integrable Systems, nlin.SI/0301032, to appear in CRM Proceedings.

[32] Zacharov V E and Mikhailov A V, Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method, Sov. Phys. JEPT **47** (1978), 1017–1027.

[33] Zhiber A V and Shabat A B, Klein Gordon equations with a nontrivial group, Sov. Phys. Dokl. **24** (1979), 607-609; ibid, Systems of equations $u_x = p(u, v)$, $v_y = q(u, v)$ possessing symmetries, Sov. Math. Dokl. **30** (1984), 23-26.