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Jacobi's Three-Body System Moves like a Free Particle

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La competizione che conta, non è quella con gli uomini, bensì quella con le cose[1]
Guido Calogero

Abstract

The problem of three bodies which attract each other with forces proportional to the cube of the inverse of their distance and move on a line was reduced to one quadrature by Jacobi [23]. Here we show that the equations of motions admit a five-dimensional Lie symmetry algebra and can be reduced to a single second-order linear equation, i.e. the equation of motion of a single free particle on the line.

1 Introduction

In January 2001 the first Whiteman prize for notable exposition on the history of mathematics was awarded to Thomas Hawkins by the American Mathematical Society. In the citation, published in the Notices of AMS **48** 416-417 (2001), one reads that Thomas Hawkins "... has written extensively on the history of Lie groups. In particular he has traced their origins to [Lie's] work in the 1870s on differential equations ... the *idée fixe* guiding Lie's work was the development of a Galois theory of differential equations ... [Hawkins's book [13]] highlights the fascinating interaction of geometry, analysis, mathematical physics, algebra and topology ...". Also Hawkins had established "the nature and extent of Jacobi's influence upon Lie" [14]. This is particularly noteworthy since 2004 marks two hundred years since Jacobi's birth. "Given the fact that the Jacobi Identity is fundamental to the theory of Lie groups, Jacobi's influence upon Lie will come as no surprise. But the bald fact that he inherited the Identity from Jacobi fails to convey fully or accurately the historical dimension of the impact of Jacobi's work on partial differential equations" [14].
In the Introduction of his book [40] Olver wrote that "it is impossible to overestimate the importance of Lie's contribution to modern science and mathematics. Nevertheless anyone who is already familiar with [it] ... is perhaps surprised to know that its original inspirational source was the field of differential equations".

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[1]The competition which is worthy of consideration is not with men, rather with matters.

Lie's monumental work on transformation groups, [27], [28] and [29], and in particular contact transformations [30], led him to achieve his goal [31]. Many books have been dedicated to this subject and its generalizations ([1], [4], [41], [40], [5], [42], [43], [15], [19], [20], [21], [16]).
Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations. Any known integration technique can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, i.e. the Lie symmetry algebra. In particular Bianchi's theorem ([3], [40]) states that, if an admitted n-dimensional solvable Lie symmetry algebra is found, then the general solution of the corresponding n^{th}-order system of ordinary differential equations can be obtained by quadratures. The admitted Lie symmetry algebra can easily be derived by a straightforward although lengthy procedure. As computer algebra software becomes widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is becoming easier to perform.
In this paper we apply Lie group analysis to the following system of three second-order ordinary differential equations:

$$\begin{aligned} \ddot{u}_1 &= \frac{a_{12}}{(u_1-u_2)^3}+\frac{a_{13}}{(u_1-u_3)^3}, \\ \ddot{u}_2 &= \frac{a_{21}}{(u_2-u_1)^3}+\frac{a_{23}}{(u_2-u_3)^3}, \\ \ddot{u}_3 &= \frac{a_{31}}{(u_3-u_1)^3}+\frac{a_{32}}{(u_3-u_2)^3}, \end{aligned} \tag{1.1}$$

in which $a_{ij}, (i,j=1,2,3; i\neq j)$ are not null arbitrary constants, different from each other, and there is no relationship among them. This system is described in Calogero's book [8], although it was studied by Calogero as early as 1969 [6] in a paper where he found the solution of the corresponding Schrödinger equation for two particular cases. System (1.1) is a generalization of the equation of motion of three bodies attracting each other with forces proportional to the cube of the inverse of their distance and moving on a line. It was studied by Jacobi in [23] which is an incomplete paper found after his death [24]. In that paper Jacobi was able to reduce the problem to a single quadrature. Actually as early as 1845 in his seminal paper on the last multiplier [22] at §28[2] Jacobi wrote "Observo tamen, casu quo trium corporum quae in eadem recta moventur mutuae attractiones *cubis* distantiarum inverse proportionales sint, motum totum tantum ab *unica Quadratura* pendere"[3].
Using Lie group analysis we show that system (1.1) admits a five-dimensional Lie symmetry algebra and can be reduced to a Riccati equation, i.e. a single second-order linear equation, and therefore:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2}=0, \tag{1.2}$$

which may be interpreted as the equation of one-dimensional motion of a single free particle. In fact Lie showed that any second-order linear (or linearizable through a point transformation) equation can be transformed into (1.2) [31].

[2]This is mentioned in [24].
[3]However, I remark that in the case of three bodies moving on the same line and attracting each other with forces proportional to the *cube* of the inverse of their distance the entire motion depends on a *single quadrature* only.

2 Lie group analysis

If we apply Lie group analysis[4] to system (1.1), then we obtain a five-dimensional Lie symmetry algebra generated by the following five operators[5]:

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= 2t\partial_t + \sum_{k=1}^{3} u_k \partial_{u_k}, \\ X_3 &= t^2\partial_t + t\sum_{k=1}^{3} u_k \partial_{u_k}, \\ X_4 &= t\sum_{k=1}^{3} \partial_{u_k}, \\ X_5 &= \sum_{k=1}^{3} \partial_{u_k}. \end{aligned} \tag{2.1}$$

This algebra is isomorphic to $sl(2,I\!R) \ltimes 2A_1$ and is not solvable, but contains a two-dimensional abelian subalgebra $2A_1$ generated by X_4 and X_5. This means that we can reduce the dimension of system (1.1) by two, namely obtaining two second-order equations which admit the Lie symmetry algebra $sl(2,I\!R)$ generated by X_1, X_2, X_3. In fact a basis of differential invariants of L_2 of order ≤ 1 is given by:

$$\xi_1 = u_1 - u_2, \qquad \xi_2 = u_1 - u_3, \qquad t = t \tag{2.2}$$

and then system (1.1) can easily be reduced to the following:

$$\begin{aligned} \ddot{\xi}_1 &= \frac{\left(a_{13}\xi_1^3 + a_{21}\xi_2^3 + a_{12}\xi_2^3\right)(\xi_1-\xi_2)^3 + a_{23}\xi_1^3\xi_2^3}{(\xi_1-\xi_2)^3\xi_1^3\xi_2^3}, \\ \ddot{\xi}_2 &= \frac{\left((a_{13}+a_{31})\,\xi_1^3 + a_{12}\xi_2^3\right)(\xi_1-\xi_2)^3 - a_{32}\xi_1^3\xi_2^3}{(\xi_1-\xi_2)^3\xi_1^3\xi_2^3} \end{aligned} \tag{2.3}$$

which admits the Lie symmetry algebra generated by X_1, X_2, X_3. In [18] it was shown how to integrate third-order differential equations which admit a three-dimensional Lie symmetry algebra. In particular, if the algebra is not solvable, a reduction to a Riccati equation can always be attained by using a two-dimensional subalgebra. Clarkson and Olver [10] have given the theoretical explanation of the appearance of a Riccati equation through prolongation of Lie algebras. In this case we can use the two-dimensional subalgebra of $sl(2,I\!R)$ generated by X_1, and X_2 in order to reduce system (2.3) to two first-order equations of which one is a Riccati equation and the other is one that can easily be integrated with one quadrature. In fact, when we use the following basis of differential invariants of order ≤ 1:

$$z = \frac{\xi_2}{\xi_1}, \qquad w_1 = \dot{\xi}_1\xi_1, \qquad w_2 = \dot{\xi}_2\xi_1, \tag{2.4}$$

[4]Through this paper we have used our own interactive REDUCE programs for calculating Lie symmetries [35], [36].

[5]It is interesting to recollect that the two-body Kepler problem also admits a five-dimensional Lie symmetry algebra although isomorphic to $so(3) \oplus A_2$.

system (2.3) reduces to:

$$\frac{\mathrm{d}w_1}{\mathrm{d}z} = -\frac{\left(\left(a_{21}+w_1^2\right)z^3+a_{13}+a_{12}z^3\right)(z-1)^3-a_{23}z^3}{(w_1z-w_2)(z-1)^3z^3}, \tag{2.5}$$

$$\frac{\mathrm{d}w_2}{\mathrm{d}z} = -\frac{(a_{31}+w_1w_2z^3+a_{13}+a_{12}z^3)(z-1)^3+a_{32}z^3}{(w_1z-w_2)(z-1)^3z^3} \tag{2.6}$$

which suggests the simplifying transformation $w_2 = w_1z - r_2$. Then equation (2.5) becomes

$$\frac{\mathrm{d}w_1}{\mathrm{d}z} = -\frac{\left(\left(a_{21}+w_1^2\right)z^3+a_{13}+a_{12}z^3\right)(z-1)^3-a_{23}z^3}{(z-1)^3r_2z^3}, \tag{2.7}$$

which is a Riccati equation for $w_1(z)$, and equation (2.6) becomes

$$\frac{\mathrm{d}r_2}{\mathrm{d}z} = -\frac{(a_{21}z^4-a_{31})(z-1)^3-(a_{23}z+a_{32})z^3+(a_{12}z^3+a_{13})(z-1)^4}{(z-1)^3r_2z^3} \tag{2.8}$$

which can easily be integrated:

$$r_2 = \sqrt{(-a_{21}-a_{12})z^2+2za_{12}-\frac{a_{32}+a_{23}}{(z-1)^2}+2\frac{a_{13}}{z}-a_{13}+a_{31}z^2-2\frac{a_{23}}{z-1}+b_1} \tag{2.9}$$

with b_1 an arbitrary constant. Then the following transformation

$$w_1 = \frac{1}{r_2r_1}\frac{\mathrm{d}r_1}{\mathrm{d}z} \tag{2.10}$$

changes the Riccati equation (2.5) into a linear second-order equation for $r_1(z)$, i.e.:

$$\begin{aligned} z(z-1)[(a_{31}-b_1z^2+a_{21}z^4+(z-2)a_{12}z^3)(z-1)^2 \\ +a_{32}z^2-\left((z-1)^2a_{13}-a_{23}z^2\right)(2z-1)]\frac{\mathrm{d}^2r_1}{\mathrm{d}z^2} \\ +[(a_{21}z^4-a_{31})(z-1)^3-(a_{23}z+a_{32})z^3+(a_{12}z^3+a_{13})(z-1)^4]\frac{\mathrm{d}r_1}{\mathrm{d}z} \\ -[(a_{13}+a_{21}z^3+a_{12}z^3)(z-1)^3-a_{23}z^3]r_1 = 0. \end{aligned} \tag{2.11}$$

It is well-known that any second-order linear equation can be transformed into equation (1.2) by a point transformation [31]. Indeed Jacobi's three-body system (1.1) moves like a free particle (1.2). It should be noted that, if the following relationship among the coupling constants is verified, i.e.:

$$a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} = 0, \tag{2.12}$$

then a sixth Lie symmetry is admitted by system (1.1), namely:

$$X_6 = -2t\partial_t + (u_3+u_2)\partial_{u_1} + (u_3+u_1)\partial_{u_2} + (u_2+u_1)\partial_{u_3}. \tag{2.13}$$

Finally, we mention that the problem of N bodies which attract each other with forces proportional to the cube of the inverse of their distance and move on a line [8], or more generally the following system:

$$\ddot{u}_n = \sum_{m=1,\, m\neq n}^{N} \frac{a_{nm}}{(u_n-u_m)^3} \tag{2.14}$$

admits a five-dimensional Lie symmetry algebra isomorphic to $sl(2, I\!R) \ltimes 2A_1$ generated by the following five operators:

$$
\begin{aligned}
X_1 &= \partial_t, \\
X_2 &= 2t\partial_t + \sum_{n=1}^{N} u_n \partial_{u_n}, \\
X_3 &= t^2\partial_t + t\sum_{n=1}^{N} u_n \partial_{u_n}, \\
X_4 &= t\sum_{n=1}^{N} \partial_{u_n}, \\
X_5 &= \sum_{n=1}^{N} \partial_{u_n}.
\end{aligned}
\qquad (2.15)
$$

Then system (2.14) can be reduced to $2N - 5$ first-order equations plus one Riccati equation.

3 Some final remarks

Our purpose in this paper is to demonstrate once more the power of Lie group analysis. One could know nothing about the physical properties (e.g., its first integrals) of Jacobi's three-body problem and still be able to discover its hidden linearity thanks to Lie's method. In [39] another solvable many-body problem introduced by Calogero [7], [9] was shown to be intrinsically linear by means of Lie symmetries. The Kepler problem and MICZ-Kepler problem were also shown to be equivalent to an isotropic two-dimensional system of linear harmonic oscillators in [26] thanks to Lie symmetries. In [33] Lie group analysis – when applied to Euler-Poisson equations as obtained from the reduction method [37] – unveiled the Kowalevski top [25] and its peculiar integral without making use of either Noether's theorem [34] or the Painlevé method [25]. In [38] Lie group analysis related the famous Lorenz system [32] to the Euler equations of a rigid body moving about a fixed point and subjected to a torsion depending on time and angular velocity, namely Lie group analysis transformed the "butterfly" into a "tornado". In [44] Lie group analysis was applied to a seminal model by Anderson [2], which describes HIV transmission in male homosexual/bisexual cohorts, and thus shown to be intrinsically linear for a particular relationship among the involved parameters. In [11] Lie group analysis was applied to a core group model for sexually transmitted disease formulated by Hadeler and Castillo-Chavez [12] and several instances of integrability, even linearity, were found which led to the general solution of the model.
In conclusion, Lie group analysis was and should still be considered an essential tool for anyone who wants to "compete" with equations of relevance in physics and other scientific fields. As brilliantly stated by Ibragimov [17] "cherchez le groupe".

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