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Universal Solitonic Hierarchy

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Abstract

We describe recent results on the construction of hierarchies of nonlinear evolution equations associated to generalized second order spectral problems. The first results in this subject had been presented by Francesco Calogero.

Introduction

In order to introduce the G-model considered recently in joint with L. Martinez Alonso papers, one can exploit an invariance of this model under hodograph type transformations. Namely, let us begin with the hierarchy of times t_i, $i \in \mathbb{Z}$ and the corresponding set of commuting diferentiations $D_i = \partial_{t_i}$. Choose t_0 as a basic independent variable and change variables $\mathbf{t} = (\dots, t_{-1}, t_0, t_1, \dots)$ into $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ by

$$x_0 = q(\mathbf{t}), \quad x_j = t_j,\, j \neq 0 \;\Rightarrow\; b_0 dx_0 = dt_0 + \sum_1^\infty g_k dt_k - \sum_1^\infty b_k dt_{-k}. \tag{0.1}$$

Then

$$D_0(q) = \frac{1}{b_0}, \quad D_j(q) = \begin{cases} \dfrac{g_j}{b_0} & \text{for } j > 0, \\ -\dfrac{b_j}{b_0} & \text{for } j < 0 \end{cases} \tag{0.2}$$

and we see that

$$D_1\left(\frac{D_1 q}{D_0 q}\right) = D_0\left(\frac{D_2 q}{D_0 q}\right) \;\Leftrightarrow\; D_1(g_1) = D_0(g_2). \tag{0.3}$$

This three-dimensional nonlinear PDE for the function $q(\mathbf{t})$ represents one of multi-dimensional versions of equations of G-model considered in the last paper of ref. [3].

The full set of equations of the G-model can be introduced by

$$D_n(G) =< A_n, G >, \quad < A, B > \stackrel{\text{def}}{=} AD_0(B) - BD_0(A), \tag{0.4}$$

where G is an auxiliary function and, for any $n > 0$, we set

$$A_n \stackrel{\text{def}}{=} \lambda^n + g_1\lambda^{n-1} + \cdots + g_n, \quad A_{-n} \stackrel{\text{def}}{=} \frac{b_0}{\lambda^n} + \frac{b_1}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}}{\lambda}. \tag{0.5}$$

In §1 we reformulate some results obtained by [3].

In §§2 and 3 we discuss applications of the G-model to the generalized second order spectral problems

$$\psi_{xx} = U(x, \lambda)\psi, \quad D_x \equiv D_0$$

with potential

$$U = U(x, \lambda) = u_0(x)\lambda^m + u_1(x)\lambda^{m-1} + \cdots + u_m(x) + \sum_{k=1}^{\infty} \lambda^{-k} v_k(x), \quad x = t_0. \tag{0.6}$$

The main point is that this generalization allows us to prove (Theorem 3) that there exist well-defined mappings $U \leftrightarrow G$ of potentials (0.6) into solutions of the equations (0.4) and vice versa.

All solitonic hierarchies are defined by the equations (0.4) up to this mapping and do not depend from the particular spectral problem.

1 Integrability of G-models

Excluding an auxiliary function G from equations (0.4) by cross differentiation we get

$$D_n(A_m) - D_m(A_n) =< A_n, A_m >, \quad \text{for any} \quad n, m. \tag{1.1}$$

Replacing the wronskian in the right hand side by the commuatator,

$$[AD_0, BD_0] =< A, B > D_0, \quad < A, B >= AD_0(B) - BD_0(A),$$

we get the zero-curvature representation of the G-model:

$$[D_n - A_nD_0, D_m - A_mD_0] = 0, \quad \text{for any} \quad n, m. \tag{1.2}$$

Due formulae (0.5) one obtains for the infinite *set of dynamical variables*

$$\ldots, b_j, b_{j-1}, \ldots, b_0, g_1, g_2, \ldots, g_j, \ldots . \tag{1.3}$$

the infinite set of equations which we take here for a definition of the G-model. It is important to notice that the change of variables (0.1) gives rise to an analogous to (1.2) set of commuting operators $\hat{D}_j = \partial_{x_j}$:

$$\hat{D}_{-n} = D_{-n} + b_nD_0, \quad \hat{D}_0 = b_0D_0, \quad \hat{D}_n = D_n - g_nD_0. \tag{1.4}$$

It can be proved (Cf [3]) that equations (1.1) are equivalent to equations (0.4) if we insert in the latter the asimptotic expansions

$$G = \begin{cases} 1 + \frac{g_1}{\lambda^1} + \cdots + \frac{g_n}{\lambda^n} + \ldots & \text{as } \lambda \longrightarrow \infty, \\ b_0 + \lambda b_1 + \cdots + \lambda^n b_n + \ldots & \text{as } \lambda \longrightarrow 0. \end{cases} \tag{1.5}$$

Thus, the function G can be used as the *generating function* of the hierarchy (1.2) and the infinite set of equations of the G-model can be rewritten in very compact form

$$D_n(G) =< (\lambda^n G)_+, G >, \quad n \in \mathbb{Z},$$

where, for any $n > 0$,

$$(\lambda^n G)_+ = \lambda^n + g_1\lambda^{n-1} + \cdots + g_n = A_n, \quad (\lambda^{-n}G)_+ = \frac{b_0}{\lambda^n} + \frac{b_1}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}}{\lambda} = A_{-n}.$$

Summing up we formulate for future references:

Theorem 1 ([3]) The nonlinear partial differential equations (1.2), (0.5) for the infinite set of dynamical variables (1.3) are equivalent to either equation of the following triad

$$D_n(G) =< A_n, G >, \quad D_n(H) = D_0(A_nH), \quad \hat{D}_n(b_0H) = \lambda b_0 D_0(A_{n-1}H), \qquad (1.6)$$

where the operators $\hat{D}_n$ are defined by (1.4), $H = G^{-1}$ and G is the generating function (1.5).

The substituting formal expansions (1.5) into equation $D_n(G) =< A_n, G >$ define the action of D_n on the dynamical variables g_n and b_n, respectively. For example, for $n = 1$, we have $A_1 = \lambda + g_1$ and obtain in this case an infinite system of $1+1$-dimensional equations

$$D_1(g_n) = D_0(g_{n+1}) + < g_1, g_n >, \qquad D_1(b_n) = D_0(b_{n-1}) + < g_1, b_n > . \qquad (1.7)$$

Analogous formal expansions of $H = G^{-1}$ as $\lambda \longrightarrow \infty$ and $\lambda \longrightarrow 0$ yield the two series of conservation laws for the equations $D_n(G) =< A_n, G >$. The coefficients of these formal series for H are defined by inverting the series (1.5) as functions of dynamical variables (1.3).

Coming back to the change of variables (0.1), which was our starting point and will be used later on, we can now demonstrate the *invariance* of equations (1.2) by proving that

$$\hat{D}_n(H) = \hat{D}_1(A_{n-1}H), \quad \forall\, n \neq 0. \qquad (1.8)$$

Thanks to (1.6) this equation means that the transformation $x_0 = q(\mathbf{t})$ does not change the form of equations yet results in the shift of the numeration $n \mapsto n-1$ of independent variables. Since

$$D_1(A_j) = D_0(A_{j+1}) + < g_1, A_j >,$$

the equation (1.8) is a corollary of the last equation in (1.6):

$$\hat{D}_n(H) - \hat{D}_1(A_{n-1}H) = D_0(A_n)H + \lambda A_{n-1}D_0(H) - (D_1 - g_1D_0)(A_{n-1}H) = 0.$$

Integration

It is most natural to ask what kind of restrictions will arise if the dynamics in the G-model is defind by a finite number of independent variables t_n.

Theorem 2. Let $G(\lambda)$ be generating function from Theorem 1 and, for fixed $N > 0$, there exist constants $\varepsilon_1, \ldots, \varepsilon_N$ such that

$$D_N + \varepsilon_1 D_{N-1} + \varepsilon_2 D_{N-2} + \cdots + \varepsilon_N D_0 = 0.$$

Then there exist formal series with constant coefficients

$$\varepsilon(\lambda) = 1 + \frac{\varepsilon_1}{\lambda} + \frac{\varepsilon_2}{\lambda^2} + \frac{\varepsilon_3}{\lambda^3} + \ldots$$

such that the product $\lambda^N \varepsilon(\lambda) G(\lambda)$ is a polynomial in λ:

$$\hat{G}(\lambda) = \varepsilon(\lambda) G(\lambda) = 1 + \frac{\hat{g}_1}{\lambda} + \frac{\hat{g}_2}{\lambda^2} + \cdots + \frac{\hat{g}_n}{\lambda^n}.$$

Proof. We have

$$D_0(g_{n+1}) = D_n(g_1) = -[\varepsilon_1 D_{n-1} + \cdots + \varepsilon_n D_0](g_1) = -D_0[\varepsilon_1 g_n + \cdots + \varepsilon_n g_1]$$

since $D_n g_1 = D_0 g_{n+1}$ (see Theorem 1). We use now that

$$D_0(g) = D_0(h) \;\Leftrightarrow\; g = h + \text{const}$$

which means that there exist a constant ε_{n+1} such that

$$g_{n+1} + \varepsilon_1 g_n + \cdots + \varepsilon_n g_1 + \varepsilon_{n+1} = 0.$$

It remains to notice that, for any m,

$$-g_{m+1} = \varepsilon_1 g_m + \cdots + \varepsilon_m g_1 + \varepsilon_{m+1} \;\Rightarrow\; -g_{m+2} = \varepsilon_1 g_{m+1} + \cdots + \varepsilon_m g_2 + \varepsilon_{m+1} g_1 + \varepsilon_{m+2}$$

since

$$D_0(g_{m+2}) = D_1(g_{m+1}) - < g_1,\, g_{m+1} > .$$

Indeed, insert $g_{m+1} = -\varepsilon_1 g_m - \cdots - \varepsilon_m g_1$; we obtain

$$- < g_1,\, g_{m+1} > = \varepsilon_1 < g_1,\, g_m > + \cdots + \varepsilon_{m-1} < g_1, g_2 > - \varepsilon_{m+1} D_0 g_1,$$

$$-D_1(g_{m+1}) = D_1[\varepsilon_1 g_m + \cdots + \varepsilon_m g_1] =$$

$$D_0[\varepsilon_1 g_{m+1} + \cdots + \varepsilon_m g_2] + \varepsilon_1 < g_1, g_m > + \cdots + \varepsilon_{m-1} < g_1, g_2 > . \blacksquare$$

Corollary. The roots $\gamma_1, \ldots, \gamma_N$ of the polynomial $\lambda^N \varepsilon(\lambda) G(\lambda)$ satisfy the hyperbolic quasilinear system as follows

$$(D_1 + \varepsilon_1 D_0)\gamma_j = (\sum_{k \neq j} \gamma_k) D_0 \gamma_j, \quad j = 1,\, 2,\, \ldots, N. \tag{1.9}$$

Indeed: the modifyed generating function $\hat{G} = \varepsilon(\lambda) G(\lambda)$ satisfies the same equations

$$D_n \hat{G} = < A_n, \hat{G} >$$

as the original generating function and

$$\hat{G}(\lambda) = \prod_1^N (1 - \frac{\gamma_k}{\lambda}) \;\Rightarrow\; D_n \hat{G}|_{\lambda=\gamma_j} = -\frac{D_n \gamma_j}{\gamma_j} \prod_{k \neq j} (1 - \frac{\gamma_k}{\gamma_j}). \tag{1.10}$$

If $n = 1$, this yields (1.9) since

$$A_1 = \lambda + g_1 = \lambda - \sum \gamma_k - \varepsilon_1 \;\Rightarrow\; A_1|_{\lambda=\gamma_j} = -\sum_{k \neq j} \gamma_k - \varepsilon_1.$$

The hyperbolic system (1.9) has many applications including the gas dynamics [4] as well as finite-gap potentials in spectral theory [6].

Using the *roots variables* $\vec{\gamma} = (\gamma_1, \ldots, \gamma_N)$ defined in (1.10) as dynamical ones we can write down the whole set of equations $D_n \hat{G} =< A_n, \hat{G} >$, where $n = 1, 2, \ldots, N-1$, in the form analogous to (1.9):

$$D_n(\gamma_j) = a_{jn}(\vec{\gamma}) D_0(\gamma_j), \quad n = 1, \ldots, N-1.$$

Introduce now an auxiliary vector field

$$D_0(\vec{\gamma}) = X(\vec{\gamma}), \quad X = (X^1, \ldots, X^N).$$

Then the above $N-1$ equations for the roots variables can be rewritten as $N-1$ dynamical systems

$$D_n(\vec{\gamma}) = \hat{A}_n(\vec{\gamma}), \quad \text{where } n = 1, 2, \ldots, N-1. \tag{1.11}$$

These dynamical systems can be integrated in quadratures with the help of

Ferapontov's Lemma ([5]) Let

$$D_0(\gamma_j) = z_j(\gamma_j) \left(\prod_{k \neq j} \gamma_{jk} \right)^{-1}, \quad \gamma_{jk} \stackrel{\text{def}}{=} \gamma_j - \gamma_k \tag{1.12}$$

where the N functions in one variable $z_j(\gamma_j)$ are arbitrary.

Then N dynamical systems (1.12) together with (1.11) for root variables $\vec{\gamma} = (\gamma_1, \ldots, \gamma_N)$ define N commuting vector fields.

Remark. Together with (1.12) the dynamical systems (1.11) can be rewritten as

$$d\vec{\gamma} = d\vec{t} A(\vec{\gamma}) \quad \text{or, equivalently, as} \quad d\vec{t} = d\vec{\tilde{\gamma}} \widetilde{A}(\vec{\gamma}) \tag{1.13}$$

where

$$d\vec{\tilde{\gamma}} \stackrel{\text{def}}{=} (z_1(\gamma_1) d\gamma_1, \ldots, z_N(\gamma_N) d\gamma_N)$$

and the differential form in the right hand side (1.13) is a closed one by commutativity of the corresponding vector fields.

2 Generalized spectral problems

Applications of the G-model to the second order spectral problems

$$\psi_{xx} = U(x,\lambda)\psi \tag{2.1}$$

are based on a mapping $G \mapsto U$ defined by the exact formula

$$U(x,\lambda) = \{D_x, \frac{1}{G(x,\lambda)}\} + \frac{z(\lambda)}{G(x,\lambda)^2}, \quad D_x \equiv D_0 \tag{2.2}$$

where $z(\lambda)$ is an x-independent normalizing coefficient and

$$\{D_x, F(x)\} \stackrel{\text{def}}{=} \frac{3}{4}\frac{F_x^2}{F^2} - \frac{1}{2}\frac{F_{xx}}{F} = \varphi_{xx} + \varphi_x^2, \quad \text{if} \quad F = e^{-2\varphi}. \tag{2.3}$$

Rewriting (2.2) in the bilinear form

$$4UG^2 + G_x^2 - 2GG_{xx} = z(\lambda) \tag{2.4}$$

and differentating it with respect to x we get the third order linear differential equation

$$G_{xxx} = 4UG_x + 2U_xG. \tag{2.5}$$

On the other hand, the deffirentiating (2.2) with respect the "time" τ and using equations (1.6), (2.5) one gets

$$D_\tau(G) = < A, G > \Rightarrow \; 2D_\tau(U) = 4UA_x + 2U_xA - A_{xxx}, \quad x \equiv t_0. \tag{2.6}$$

Remark. Recall that if ψ_1, ψ_2 constitute a basis of the linear space of solutions (2.1) then $G = \{\psi_1^2, \psi_2^2, \psi_1\psi_2\}$ is a fundamental system of solutions of (2.5). Vice versa, in order to reconstruct ψ_1, ψ_2 from a given G and the wronskian $w =< \psi_1, \psi_2 >= \psi_1\psi_{2,x} - \psi_2\psi_{1,x}$ we have

$$G = \psi_1\psi_2 \;\Rightarrow\; \frac{G_x}{G} = \frac{\psi_{1,x}}{\psi_1} + \frac{\psi_{2,x}}{\psi_2}, \qquad \frac{< \psi_1, \psi_2 >}{G} = \frac{\psi_{2,x}}{\psi_2} - \frac{\psi_{1,x}}{\psi_1}.$$

Thus

$$\frac{\psi_{1,x}}{\psi_1} = \frac{1}{2}\left(\frac{G_x}{G} - \frac{w}{G}\right), \quad \frac{\psi_{2,x}}{\psi_2} = \frac{1}{2}\left(\frac{G_x}{G} + \frac{w}{G}\right).$$

Recall also that the second equation in (2.6) which describes the evolution of the potential U is the consistency condition for the equations

$$\psi_{xx} = U\psi, \quad \text{and} \quad \psi_\tau = A\psi_x - \frac{1}{2}A_x\psi.$$

Lastly, one can notice that latter equation for ψ together with $G = \psi_1\psi_2$ yield the first of equations (2.6) $D_\tau(G) =< A, G >$.

In the general case the mapping $G \mapsto U$ defined by (2.2), where

$$z(\lambda) = \lambda^m + \alpha_1 z^{m-1} + \cdots + \alpha_0 + \sum_{j>0} \beta_j \lambda^{-j},$$

and G represented by the asimptotic expansion, yields the formal Laurent series

$$U = U(x,\lambda) = \lambda^m + u_1(x)\lambda^{m-1} + \cdots + u_m(x) + \sum_{k=1}^{\infty} \lambda^{-k} v_k(x), \quad \text{where} \quad x = t_0. \tag{2.7}$$

Equation (2.1) with the potential U represented as $\lambda \longrightarrow \infty$ by its asimptotic expansion (2.7) is said to be *the generalized spectral problem.*

Theorem 3. Let the potential $U(x,\lambda)$ of the generalized spectral problem (2.1) be represented by the formal Laurent series (2.7) with $m > 0$. Then the third order equation (2.5) associated with the spectral problem possesses a unique normalized solution $G = Y = \sum y_k \lambda^{-k}$ such that

$$Y(x,\lambda) = 1 + \sum_{k=1}^{\infty} \lambda^{-k} y_k(x), \quad -2Y_{xx}Y + Y_x^2 + 4UY^2 = 4\lambda^m. \tag{2.8}$$

Proof. We need to prove that equation (2.8) uniquely defines the coefficients y_k, where $k = 1, 2, \ldots$ in terms of the coefficients of the formal powers series (2.7).

It suffices to verify that consequently equating in (2.8) the coefficients of equal powers of λ

$$\lambda^m,\ \lambda^{m-1}, \ldots, \lambda^0,\ \lambda^{-1}, \ldots$$

we get a triangular type system. Thus we find

$$y_1 = -\frac{1}{2}\widetilde{u}_1,\ \ldots, y_n = -\frac{1}{2}\widetilde{u}_n + \Phi_n(y_1, \ldots, y_{n-1}; \widetilde{u_1}, \ldots, \widetilde{u}_{n-1}), \ldots,$$

where

$$\widetilde{u}_j = \begin{cases} u_j & \text{for } j \le m \\ v_{j-m} & \text{for } j > m \end{cases}$$

and functions $\Phi_n(\cdot;\cdot)$ are differential polynomials in their arguments. The first m coefficients are obtained in a purely algebraic way since the equation (2.8) rewritten in the form (2.2) implies that the coefficients $u_1, \ldots, u_m$ of the potential coincide with the corresponding coefficients of the series $\lambda^m Y^{-2}$. ∎

Now we can incorporate G-model into the theory of spectral problems. Namely, choose $Y(x,\lambda)$ defined in Theorem 3 as the initial data at $t_n = 0$ for the dynamical equations (1.6). Then, thanks to eq. (2.6), the mapping (2.2) transforms solutions of this initial data problem into solutions of the hierarchy of evolutionary equations

$$2D_{t_n}(U) = 4UA_{n,x} + 2U_x A_n - A_{n,xxx}, \quad A_n = (\lambda^n Y)_+, \quad n = 1, 2, 3, \ldots \tag{2.9}$$

for the potential $U = U(\mathbf{t},\lambda)$, where $t_0 = x$, of the generalized spectral problem. In particular, for the generalized Schrödinger spectral problem with

$$U = \lambda + u_1 + \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \ldots$$

we find, by setting $A_1 = \lambda + y_1$ and $D_t = 2D_{t_1}$, the following generalization of Korteveg-de Vries equation:

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x = 2v_{1,x}, \quad v_{1,t} + 2v_1 u_x + uv_{1,x} = 2v_{2,x}, \; \ldots \tag{2.10}$$

which is equivalent to eq. (1.7).

Proof of Theorem 3 describes the inversion of the mapping introduced by the basic eq. (2.2) with $z(\lambda) = \lambda^m$. This mapping $Y = G \mapsto U$ defines a differential substitution $\widetilde{u}_j = \Psi_j(\vec{y})$ transforming equations (1.6) into (2.9). For example, expressing U in terms of Y for $m = 1$ and $m = 2$ with

$$U = \lambda + u_1 + \sum_{k\geq 1} \lambda^{-k} v_k, \quad \text{and} \quad U = \lambda^2 + \lambda u_1 + u_2 + \sum_{k\geq 1} \lambda^{-k} v_k,$$

respectively, we find, in addition to eq. $u_1 + 2y_1 = 0$ above, that

$$v_1 = -2y_2 + 3y_1^2 + \frac{1}{2}y_{1,xx}, \quad 2v_2 = -4y_3 + 12y_1y_2 - 8y_1^3 + y_{2,xx} - y_1y_{1,x} - \frac{1}{2}y_{1,x}^2 \tag{2.11}$$

if $m = 1$ and

$$u_2 = -2y_2 + 3y_1^2, \quad 2v_1 = -4y_3 + 12y_1y_2 - 8y_1^3 + y_{1,xx} \tag{2.12}$$

if $m = 2$.

The polynomial case

Under the conditions of Theorem 2 we have, with a slightly misused notations,

$$G(x, \lambda) = \lambda^N \varepsilon(\lambda) Y(\lambda) = \prod_1^N (\lambda - \gamma_k(x)). \tag{2.13}$$

The differential equations for the roots $\gamma_1, \ldots, \gamma_N$ are now obtained (Cf Ferapontov's Lemma) by substitution $\lambda = \gamma_j$, where $j = 1, \ldots, N$ in (2.4). This yields

$$\frac{d}{dx}\gamma_j = \left(\prod_{k\neq j} \gamma_{jk}\right)^{-1} \hat{z}(\gamma_j), \quad \hat{z}(\lambda) = \frac{1}{2}\sqrt{z(\lambda)} \tag{2.14}$$

since the potential $U(x, \lambda)$ is assumed to be regular without movable singularities in the $\lambda-$plane.

Dubrovin's equations (2.14) are completely defined by the function $z(\lambda)$. In its turn, $z(\lambda)$ is defined by the right hand side of (2.4) and has to be an $(m + 2N)$th degree polynomial in λ for spectral problems with mth degree polynomial potentials. The other way round, the formula (2.2) defines the potential as a degree m polynomial in λ for any polynomial $z(\lambda)$ of degree $m + 2N$ thanks to the following

Dubrovin's Lemma Let $z(\lambda)$ be meromorphic and the roots γ_j of an $N-$th degree unitary polynomial $G(x, \lambda)$ satisfy equations (2.14). Then

$$\lambda = \gamma_j \;\Rightarrow\; G(x, \lambda) = 0 \;\Rightarrow\; 2G_{xx}(x, \lambda) + \frac{z(\lambda) - G_x^2(x, \lambda)}{G(x, \lambda)} = 0.$$

Corollary Let us, under conditions of Dubrovin's Lemma, denote by $\hat{z}(\lambda, \vec{\gamma})$ an $N-$th degree polynomial in λ such that

$$\hat{z}(\lambda;\, \vec{\gamma})|_{\lambda=\gamma_j} = z(\lambda)|_{\lambda=\gamma_j}, \quad \hat{z}'(\lambda;\, \vec{\gamma})|_{\lambda=\gamma_j} = z'(\lambda)|_{\lambda=\gamma_j}.$$

Then eq. (2.2) for potential U of the generalized spectral problem can be expressed as follows

$$4U \prod (\lambda - \gamma_j)^2 = z(\lambda) - \hat{z}(\lambda;\, \vec{\gamma}).$$

3 The problem of constraints

The change of variables in the G-model defined by (2.2) allows us to reformulate the problem of reductions (of an infinite set of dynamical variables to a finite one) in terms of equations (2.9) for coefficients of the formal series (2.7). Thus the truncation $v_j = 0$ for $j \geq 1$ in (2.10) yields the classical KdV equation while the next possibility $v_j = 0$ for $j \geq 2$ leads to the system of equations for $u = u_1$ and $v = 4v_1$:

$$2u_t = u_{xxx} - 6uu_x + v_x, \quad v_t + 2vu_x + uv_x = 0.$$

The polynomial constraints have been considered in [3]. They correspond to the case where $U(\mathbf{t}, \lambda)$ and $z(\lambda)$ in (2.2) are polynomials of the same degree in λ and give rise to some interesting modifications of the classical solitonic hierarchies (see next subsection). These polynomial constraints are in good agreement with Dubrovin's Lemma but one has to keep in mind that this lemma and its corrolary do not forbid pole singularities in λ.

In order to highlight the possibility of more direct approach to the problem of constraints let us consider a simple example related with the Bürgers hierarchy. The infinite system of equations

$$D_1 G =< \lambda + g_1, G > \tag{3.1}$$

for the coefficients g_j can be reduced, apparently, to a single evolutionary equation for the function $u = g_1$ in a very elementary way:

$$g_2 = f(g_1, g_{1,x}) \;\Rightarrow\; u_t = Df(u, u_x), \quad \text{where} \quad u \equiv g_1, \quad D_t \equiv D_1. \tag{3.2}$$

Lemma Let an evolutionary equation of the second order $u_t = D_0 f(u, u_x)$ possess the conservation law with density $g = g(u, u_x)$. Then $g = a(u)u_x + b(u)$.

Proof. Denoting by $\sim$ equivalence modulo $\operatorname{Im} D_0$ and setting $g_0 = \partial_u g(u, u_1)$ and $g_1 = \partial_{u_1} g(u, u_1)$ we obtain by integration by parts

$$D_t(g) = g_0 D_0 f + g_1 D_0^2 f \sim D_0 f (g_0 - D_0 g_1) = f_1 g_{11} u_{xx}^2 + \dots$$

By assumption $D_t g \in \operatorname{Im} D_0$, so the right hand side should be represented by $D_0 h(u, u_x)$ and thus $f_1 g_{11} = 0$. ∎

The constraint (3.2) should be compatible with the original system if equations (3.1) allow one to express all coefficients g_j as differential functions of u (i.e., functions of u and derivatives of u with respect to x). Using Theorem 1 and the above Lemma one can readily prove the following

Proposition The constraint (3.2) is compatible with the system of equations (3.1) if and only if $g_2 = g_1^2 + \varepsilon_1 g_{1,x} + \varepsilon_2 g_1 + \varepsilon_3$, where ε_j are arbitrary constants.

Camassa-Holm equation

Straightforward application of Theorem 3 to the polynomial potentials (2.7) gives rise to the constraints $v_1 = \Psi(\vec{y}) = 0$. Using formulae (2.11), (2.12) it yields, for example,

$$m = 1 \;\Rightarrow\; 2y_2 = 3y_1^2 + \frac{1}{2}y_{1,xx}, \qquad m = 2 \;\Rightarrow\; 4y_3 = 12y_1y_2 - 8y_1^3 + D_0^2 y_1.$$

These constraints correspond to KdV and NLS hierarchies, respectively.

Invoking Theorem 1 and hodograph type transformations one can substantially enlarge the set of differential costraints defined by (2.2) in the polynomial case (see [3]). For simplicity we confine ourselves to the case $m = 1$ and, accordingly, set $z(\lambda) = \lambda + \varepsilon$ in (2.2). In terms of $H = G^{-1}$ this yields (Cf (2.3)):

$$U(\mathbf{t}, \lambda) = \{D_0, H\} + (\lambda + \varepsilon)H^2. \tag{3.3}$$

The main point is that (3.3) allows us now to relate the Taylor series expansion of H at $\lambda = 0$ (Cf (1.5)) with the asimptotic expansion $H = 1 + h_1\lambda^{-1} + \ldots$ at the $\lambda-$infinity used in Theorem 3 and which implies that $U = \lambda + \varepsilon + 2h_1$ in (3.3). Thus, at $\lambda = 0$ eq. (3.3) implies a new form of the constraint:

$$H = a_0 + \lambda a_1 + \ldots \;\Rightarrow\; 2h_1 = \{D_0, a_0\} + \varepsilon a_0^2 - \varepsilon$$

and a modified KdV equation (Cf [2]) as follows

$$(4D_1 - 2\varepsilon D_0)\varphi = D_0^3\varphi - \frac{1}{2}(D_0\varphi)^3 - 6\varepsilon e^{2\varphi} D_0\varphi, \quad a_0 = e^{\varphi}.$$

Considering equations with negative numbers in the hierarchy (1.6) it is natural to introduce *renormalization* as follows

$$\hat{H} = b_0 H, \quad b_0 a_0 = 1 \;\Rightarrow\; H \longrightarrow 1\,(\lambda \longrightarrow \infty), \quad \hat{H} \longrightarrow 1\,(\lambda \longrightarrow 0). \tag{3.4}$$

Combined with the "hodograph" (0.1) change of variables $\mathbf{t} \mapsto \mathbf{x}$ and $\hat{D}_0 = b_0 D_0$ it transforms (3.3) into

$$\hat{U}(\mathbf{x}, \lambda) = \lambda b_0^2 + \varepsilon = \{\hat{D}_0, \hat{H}\} + (\lambda + \varepsilon)\hat{H}^2, \quad a_0(\mathbf{t})b_0(\mathbf{x}) = 1 \tag{3.5}$$

since

$$\{\hat{D}_0, \hat{H}\}) = b_0^2(\{D_0, H)\} - \{D_0, a_0)\}), \quad \hat{D}_0 = b_0 D_0, \quad \hat{H} = b_0 H.$$

In the notations

$$D_x = \hat{D}_0, \quad D_\tau = \hat{D}_{-1}, \quad \hat{H} = 1 + \lambda u + \ldots, \quad \lambda \longrightarrow 0$$

we obtain, with Theorem 1,

$$\lambda D_\tau(\hat{H}) = D_x[(1 - \lambda u)\hat{H}] \;\Rightarrow\; b_{0,\tau} + (ub_0)_x = 0. \tag{3.6}$$

Eq. (3.5) yields the constraint

$$b_0^2 = 1 + 2\varepsilon u - \frac{1}{2}u_{xx}$$

which together with the last equation in (3.6) is equivalent to the Camassa-Holm equation [1].

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