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A Chorin-Type Formula for Solutions to a Closure Model for the von Kármán–Howarth Equation [1]

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Abstract

The article is devoted to studying the Millionshtchikov closure model (a particular case of a model introduced by Oberlack [14]) for isotropic turbulence dynamics which appears in the context of the theory of the von Kármán-Howarth equation. We write the model in an abstract form that enables us to apply the theory of contractive semigroups and then to present a solution to the initial-boundary value problem by Chorin-type formula.

1 Introduction

For decaying isotropic turbulence von Kármán and Howarth [8] found that the governing equation for the two-point velocity correlation functions is given by

$$\frac{\partial B_{LL}}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \left(B_{LL,L} + 2\nu \frac{\partial B_{LL}}{\partial r} \right). \tag{1.1}$$

Here ν is the kinematic viscosity, B_{LL} is the longitudinal two-point correlation function of velocity fluctuations and $B_{LL,L}$ is the corresponding two-point moment of the third-order. Equation (1.1) is not closed since it contains two unknowns B_{LL} and $B_{LL,L}$ which cannot be defined from (1.1) without the use of additional hypotheses. The simplest assumption is the Kármán–Howarth's hypothesis on the similarity of the correlation functions B_{LL} and $B_{LL,L}$ which are (see [12])

$$B_{LL}(r,t) = V^2(t) f(\eta), \quad B_{LL,L}(r,t) = V^3(t) h(\eta), \quad \eta = r/L(t), \tag{1.2}$$

where $V^2(t)$ is the scale for the turbulent kinetic energy, $V^3(t)$ is the scale for the turbulent transfer and $L(t)$ is the global length scale of the turbulence. Substituting these hypothesized expressions into equation (1.1), we straightforwardly demonstrate that this equation

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admits a complete similarity solution of type (1.2) only when $Re_l = VL/\nu = const.$ It is known that this condition is normally not satisfied in experimental measurements of decaying isotropic turbulence at finite Reynolds number. In [3] Batchelor et al. carried out a similarity analysis of this problem in Fourier space and shown that a similarity solution could be found during the final period of decay when the nonlinear terms become negligible. Millionshtchikov outlined in [13] a more general hypotheses which produces parametric models of isotropic turbulence based on a closure procedure for von the Kármán–Howarth equation. The essence of these hypotheses is that $B_{LL,L}$ is given by the following relation of gradient-type

$$B_{LL,L} = 2K\frac{\partial B_{LL}}{\partial r}, \tag{1.3}$$

where K is the turbulent kinematic viscosity which is characterized by a single length and velocity scale. Millionshtchikov's hypotheses assumes [13] that

$$K = \kappa_1 ur, \qquad u^2 = B_{LL}(0,t), \tag{1.4}$$

where κ_1 denotes an empirical constant.

An original way of closing the von Kármán–Howarth equation was suggested by Oberlack in [14] which connects the two-point correlation functions of the third-order $B_{LL,L}$ and the second order B_{LL} by using the gradient type hypothesis that according to [14] takes the form

$$K = \kappa_2 r\sqrt{D_{LL}}, \quad D_{LL} = 2[u^2 - \varepsilon B_{LL}(r,t)], \quad \varepsilon = 1. \tag{1.5}$$

The Millionshtchikov hypotheses is a consequence of the above formula in the case when $\varepsilon = 0$. The model (1.5) holds for a wide range of well accepted turbulence theories for homogeneous isotropic turbulence such as Kolmogorov's first and second similarity hypothesis and the integral invariant theory, which is a generalization of Loitsiansky's and Birkhoff's integrals.

We note that Hasselmann (see, [7]) was the first to hypothesize a connection between the correlation functions of the second- and third-order. His model for isotropic turbulence contains one empirical constant and a rather complicated expression for the turbulent viscosity coefficient. Onufriev [16] obtained the closed mathematical model of isotropic turbulence dynamics based on a system of two partial differential equations for the longitudinal two-point double and triple correlation functions. He used the finite-dimensional probability density equation and the Millionshtchikov approach for closing this system.

Besides the models based on solving the von Kármán–Howarth equation, there are some other approaches to investigate theoretically isotropic turbulence dynamics, which include: analysis of possible self-similar solutions of the unclosed von Kármán–Howarth equation or its spectral analog, attempts to close the equation for the energy spectrum, direct numerical simulation of isotropic turbulence decay based on solving the Navier–Stokes equations. These approaches and the results obtained therein have been derived by Barenblatt et al. [2], Monin [12], Korneev et al. [9], Schumann et al. [17], George [6], Speciale et al. [18] and others.

Analysis of the literature has shown that there is a relatively small number of publications devoted to numerical experiments of the von Kármán-Howarth equation, and there are very few results devoted to the mathematical study, known to the authors, that are based on closed models of the von Kármán-Howarth equation. Exceptions are selfsimilar solutions to the model (1.1)-(1.4) which were obtained in [13]; selfsimilar and numerical solutions to model (1.1), (1.3), (1.5) which were presented in [14].

Our intention is to study some open mathematical problems of the dynamics of isotropic turbulence using two-point turbulence models. We begin the study of isotropic turbulence dynamics with an investigation of the Millionshtchikov model which occupies an intermediate location according to complexity in the hierarchy of closing the von Kármán–Howarth equation. Firstly we write the model in an abstract form on the product of $R^+ \times X \times Y$, where we can rewrite equation (1.1) in a form suitable to employ the theory of contractive semigroups. Then using Chorin's formula [5] (or a Trotter–Kato formula for pairs of contractive continuous semigroup) we find a solution to our model. This formula was discovered by Chorin in an attempt to find solutions of the Navier–Stokes equations.

Remark 1. Millionshtchikov's [13] model may be considered as a particular case of Oberlack's model [14], [15] by using formula (1.5) for $\varepsilon = 0$ and the present study of the Millionshtchikov model is the first step for investigation of the general model (1.1), (1.3), (1.5).

2 Problem Formulation

In order to study the dynamics of homogeneous isotropic turbulence we use the closed von Kármán–Howarth equation in dimensionless form

$$\frac{\partial \tilde{B}_{LL}}{\partial \tilde{t}} = H\tilde{B}_{LL} \equiv \frac{2}{\tilde{r}^4}\frac{\partial}{\partial \tilde{r}}\tilde{r}^4\left(\tilde{K} + \frac{1}{Re_M}\right)\frac{\partial \tilde{B}_{LL}}{\partial \tilde{r}}. \tag{2.1}$$

where $\tilde{B}_{LL} = B_{LL}/U_\infty^2$; $\tilde{t} = U_\infty t/M = x/M$, t is time, connected with x the distance from the grid in a wind tunnel. U_∞ is the velocity of the stream in the working section of the wind channel, $\tilde{r} = r/M$ (r is a distance between two points of space) and M is a size of a grid mesh; $Re_M = U_\infty M/\nu$, ν designates a kinematic viscosity coefficient; $\tilde{K} = \kappa_1 \tilde{u}\tilde{r}$, $\tilde{u}^2 = \tilde{B}_{LL}(0,t)$ and κ_1 is the empirical constant.

For the initial condition we set a positive decreasing function $\tilde{B}_{0LL}(\tilde{r})$, consistent with the experimental data:

$$\tilde{B}_{LL}(\tilde{r}, t_0) = \tilde{B}_{0LL}(\tilde{r}), \quad \tilde{t} = \tilde{t}_0, \quad \tilde{r} \geq 0. \tag{2.2}$$

Equation (1.1) is supplemented by the boundary conditions:

$$\tilde{B}_{LL,L} = 2K\frac{\partial \tilde{B}_{LL}}{\partial \tilde{r}} = 0, \quad \tilde{r} = 0 \quad \text{and} \quad \tilde{B}_{LL} = 0, \quad \tilde{r} \to \infty. \tag{2.3}$$

Bellow the tilde symbol for dimensionless variables is omitted for convenience.

3 Solution to the Problem

In this section we address an abstract form of equation (2.1) and construct approximate solutions to the above problem.

Consider the operator

$$H = \frac{2}{r^4}\frac{\partial}{\partial r} r^4 \left(\kappa_1 r u(t) + \frac{1}{Re_M}\right)\frac{\partial}{\partial r} = \\ 2\kappa_1 u(t)\left[r\frac{\partial^2}{\partial r^2} + 5\frac{\partial}{\partial r}\right] + 2\frac{1}{Re_M}\left[\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right]. \tag{3.1}$$

By introducing the new variable $s = 2\sqrt{r}$ and replacing r by s in the first operator

$$\left[r\frac{\partial^2}{\partial r^2} + 5\frac{\partial}{\partial r}\right] \tag{3.2}$$

we can rewrite (3.1) in the following form

$$H = \frac{2}{Re_M}\left[\kappa_1 Re_M u(t)\left(\frac{\partial^2}{\partial s^2} + \frac{9}{s}\frac{\partial}{\partial s}\right) + \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\right]. \tag{3.3}$$

The operator

$$B_k = \frac{\partial^2}{\partial q^2} + \frac{k}{q}\frac{\partial}{\partial q} \tag{3.4}$$

is often called the Bessel operator and represents the spherical Laplacian acting on a $(k+1)$-dimensional sphere S^{k+1}. The operator H admits a representation in the form of the spherical Laplacian on the product $S^5 \times S^{10}$ with appropriate metrics.

Let Z be a manifold isometric to the product $X \times Y$ (where X and Y are, respectively, manifolds of dimension n and m with the metrics dx^2 and dy^2) and endowed with the metric

$$dz^2 = dx^2 + \gamma^2 dy^2 \tag{3.5}$$

where γ is a smooth positive function. Calculating the Laplace operator Δ_Z on the manifold Z we arrive at the following (see [19])

$$\Delta_Z = \Delta_X + \gamma^{-2}\Delta_Y, \tag{3.6}$$

where Δ_X and Δ_Y are the Laplace operators on X and Y. We define $X = S^5$, $Y = S^{10}$ and dx^2, dy^2 which are, respectively, the metrics on S^5, S^{10}. Let $\gamma = 1/\sqrt{\kappa_1 \nu^{-1} u(t)}$. Then the operator H is calculated as follows:

$$H = \Delta_Z \equiv \frac{2}{Re_M}\left[\Delta_{S^5} + \gamma^{-2}\Delta_{S^{10}}\right], \tag{3.7}$$

where $Z = S^5 \times S^{10}$ with the above metric. Here the operator Δ_{S^i} is the corresponding Bessel operator. The representation obtained for the operator H gives a reason to assume that H generates a semigroup in some functional space. To prove this property we use Chorin's formula [5] (or the Trotter–Kato formula for pairs of continuous contractive semigroups in some Banach space). The basis of employing this approach uses the results of [1] for the Bessel operator which ensures that B_k generates an analytic contractive semigroup $e^{B_k t}$ in the functional space H_0^λ:

$$H_0^\lambda = \{u \in C_0 \,:\, u \in h^\lambda([0,\infty))\}, \tag{3.8}$$

where C_0 is a set of bounded, uniformly continuous functions $\{f\}$ on $[0,\infty)$ such that $f(q) \to 0$ as $q \to \infty$ and h^λ denotes the little Hölder space. With the Hölder norm on u, H_0 becomes a Banach space [1]. Therefore $u(t) = e^{B_k t}u(0)$ is a weak solution of the equation $u_t = B_k u$ for $u(0) \in H_0^\lambda$ [1]. Moreover the well-known result about regularity of solutions to parabolic equations [10] guarantees that in fact $u(t)$ is the classical solution to the above equation for $t > 0$.

It is useful to present this analytic semigroup in an explicit form. To this end we consider the equation

$$\frac{\partial v}{\partial t} = x\frac{\partial^2 v}{\partial x^2} + \alpha\frac{\partial v}{\partial x} \tag{3.9}$$

for $\alpha > 1$ and $x \in R^+$, $t > 0$ keeping in mind that there exists a correspondence between the Bessel operator B_k and the operator

$$D_\alpha = x\frac{\partial^2}{\partial x^2} + \alpha\frac{\partial}{\partial x} \tag{3.10}$$

by change of variables $q = 2\sqrt{x}$ at $k = 2\alpha - 1$. According to the theory for parabolic equations which are degenerate on a boundary (see, for example [20]), the correctly formulated problem to equation (3.9) with initial condition

$$v(x,0) = v_0(x), \quad x \in R^+ \tag{3.11}$$

requires to set free the boundary $\{x = 0\}$ from boundary conditions. The degeneracy of (3.9) at $\{x = 0\}$ may be observed since the first term on the right hand side becomes zero and formally a hyperbolic equation is obtained. The following formula [20],

$$v(x,t) = \int_0^\infty G(x,\xi,t)v_0(\xi)d\xi, \tag{3.12}$$

gives a (positive) solution to problem (3.9), (3.11) (for the positive function $v_0(x)$), where $G(x,\xi,t)$ is a Green's function for (3.9), (3.11) which has the form

$$G(x,\xi,t) = \frac{1}{t}\left(\frac{x}{\xi}\right)^{m/2}\exp\left(-\frac{x+\xi}{t}\right)I_m\left(2\frac{x^{1/2}\xi^{1/2}}{t}\right), \quad m = \alpha - 1. \tag{3.13}$$

Here $I_q(x)$ is the modified Bessel function. It is easy to check that the Green's function G satisfies the equality

$$\int_0^\infty G(x,\xi,t)d\xi = 1. \tag{3.14}$$

The formula (3.12) and the results obtained by Tersenov [20] for degenerate parabolic equations of the form (3.9) give the explicit representation for the semigroup $e^{B_k t}$ and arguments to prove that B_k generates an analytic contractive semigroup in H_0^λ. For example the strong continuity of $e^{B_k t}$ at $t = 0$ follows from the formula

$$v(x,t) = \int_0^\infty G(x,\xi,t)\{v_0(\xi) - v_0(x)\}d\xi + v_0(x) \tag{3.15}$$

and the equality

$$\lim_{t\to 0}\int_0^\infty G(x,\xi,t)\{v_0(\xi)-v_0(x)\}d\xi = 0. \tag{3.16}$$

We note that the convergence of $v(x,t)$ to zero as $x\to\infty$ is a consequence of the rapidly decreasing the Green's function $G(x,\xi,t)$ as $x\to\infty$.

Therefore $u(q,t)\equiv v(x,t)$ is a solution to equation $u_t = B_k u$ with initial data u_0. In fact $u(q,t)$ is the classical solution of the equation $u_t = B_k u$ in the domain $q\geq 0$, $t>0$ due to the result on regularity of weak solutions to parabolic equations [10] (or for degenerate parabolic equations of the form (3.9); see [20]). We have that $(\partial/\partial q)u(0,t)=0$ for $t>0$ due to the spherical symmetry of the solutions. The function $v(x,t)$, being the solution of (3.9), defines another semigroup $e^{D_\alpha t}: H_0^\mu \to H_0^\mu$, $\mu\leq\lambda$, $t>0$; $u(q,t)=v(x,t)=e^{D_\alpha t}v_0(x)$.

Following the ideas of [5] we reduce the study of problem (1.1)–(1.5) to the study of the two partial problems

$$\frac{\partial B_{1LL}}{\partial t} = 2\kappa_1 u(t)\left[r\frac{\partial^2}{\partial r^2}+5\frac{\partial}{\partial r}\right]B_{1LL} \tag{3.17}$$

and

$$\frac{\partial B_{2LL}}{\partial t} = 2\frac{1}{Re_M}\left[\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right]B_{2LL} \tag{3.18}$$

with initial-boundary conditions

$$B_{iLL}(r,0)=B_{i0LL}(r),\quad i=1,2, \tag{3.19}$$

$$2K\frac{\partial B_{iLL}}{\partial r}(r,t)=0,\quad r=0,\quad t>0,\quad B_{i0LL}=0,\quad r\to\infty. \tag{3.20}$$

Using Chorin's formula we can solve problem (2.1)–(2.3).

First we study the problem associated to the equations (3.17), (3.19), (3.20) emerging from equation (2.1). This problem is of special interest in view of studying the dynamics of isotropic turbulence in the case when the influence of the kinematic molecular viscosity coefficient ν is negligibly small in comparison with the turbulent viscosity coefficient K.

Introducing formally the change of time variable

$$d\tau = 2\kappa_1 u dt,\quad \tau(0)=0,\quad u^2(t)=B_{1LL}(0,t) \tag{3.21}$$

we can rewrite equation (3.17) and the initial-boundary conditions (3.19), (3.20) in the following form:

$$\frac{\partial v}{\partial\tau} = r\frac{\partial^2 v}{\partial r^2}+5\frac{\partial v}{\partial r}, \tag{3.22}$$

$$v(r,0)=v_0(r)\equiv B_{10LL}(r), \tag{3.23}$$

$$2K\frac{\partial v}{\partial r}=0,\quad r=0,\quad v=0,\quad r\to\infty, \tag{3.24}$$

where $v(r,\tau) = B_{1LL}(r,t)$. Therefore we can work with the new equation (3.22) and subsequently translate our results to (3.17). By using the formula

$$dt = \frac{d\tau}{2\kappa_1\sqrt{v(0,\tau)}} \tag{3.25}$$

we can recover the original time t for the positive function $v(0,\tau)$.

To establish solvability of problem (3.22)–(3.24) we use that equation (3.22) is of the form (3.9). Thus the function defined by the formula $v(r,\tau) = e^{D_\alpha \tau} v_0(r)$, $\alpha = 5$ is a positive solution to the above problem. The boundary condition $2Kv_r(\tau,r)(\equiv 2\kappa_1 urv_r(\tau,r)) = 0$ at $r = 0$ for $\tau > 0$ is satisfied in view of the equality $\sqrt{r}v_r(r,\tau) = u_q(q,\tau)$. Expressed in terms of (r,t) the function $v(r,\tau)$ takes the form

$$v(r,\tau) \equiv B_{1LL}(r,t) = e^{D_\alpha \tau(t)} B_{10LL}(r) \tag{3.26}$$

which solves problem (3.17), (3.19), (3.20).

It is easily verified that equation (3.22) admits the following exact solutions: $v(r,\tau) = e^{-r/\tau}\tau^{-\alpha}$ and $v(r,\tau) = r^{1-\alpha}e^{-r/\tau}\tau^{-\alpha-2}$. Using (3.25) the first solution corresponds to the well-known Millionshtchikov solution

$$B_{1LL}(r,t) = c_1 t^n e^{-rc_2 t^\beta}, \tag{3.27}$$

where c_1, c_2 are some constants and $n = -10/7$, $\beta = -2/7$ which agrees with Kolmogorov's law of decaying isotropic turbulent flow.

In the same way the solution to problem (3.18)–(3.20) is defined by the formula

$$B_{2LL}(r,t) = e^{B_k \theta(t)} B_{20LL}(r), \tag{3.28}$$

where $k = 4$ and θ is determined by

$$d\theta = 2\frac{1}{Re_M}dt. \tag{3.29}$$

Then the formula

$$S(t) = \lim_{n\to\infty} (e^{D_\alpha \tau(t/n)} e^{B_k \theta(t/n)})^n, \tag{3.30}$$

known as Chorin formula, defines a continuous semigroup $S(t)$ according to [4, 11] to the full equation (2.1). In this formula the power n means iteration. For example:

$$(e^{D_\alpha \tau(t/2)} e^{B_k \theta(t/2)})^2 = e^{D_\alpha \tau(t/2)} e^{B_k \theta(t/2)} e^{D_\alpha \tau(t/2)} e^{B_k \theta(t/2)}. \tag{3.31}$$

As a result we obtain that the function

$$B_{LL}(t) = S(t)B_{LL}(0) = \lim_{n\to\infty} (e^{B_k \theta(t/n)} e^{D_\alpha \tau(t/n)})^n B_{LL}(0), \quad B_{LL}(t) = B_{LL}(\cdot,t), \tag{3.32}$$

defines a solution to the full equation (2.1) of class $C([0,\infty); H_0^\mu)$.

We note that the important feature of formulas of this type is that the error for large n is $O(1/n)$ independent of Re_M (see, [11]). Therefore the functions

$$B^n_{LL}(t) = (e^{D_\alpha \tau(t/n)} e^{B_k \theta(t/n)})^n B_{LL}(0) \tag{3.33}$$

represent approximate smooth solutions to the closed von Kármán–Howarth equation (2.1) which satisfy the boundary conditions (2.3) exactly due to the existence result for problem (3.17), (3.19), (3.20).

4 Conclusion

Formula (3.33) is of special interest in view of its application to find the so-called infinitesimally close asymptotic solutions to the original problem. We divide the time scale into n parts and then iterates the following procedure: at first solve equation (3.17), then solve equation (3.18) etc. Using this formula we can obtain that B^n_{LL} is independent of Re_M as $Re_M \to \infty$. The latter enables us to take the limit in the original model as $Re_M \to \infty$ and to study the asymptotic behavior of solutions to the problem under consideration.

The solution obtained (3.32) may be used as the initial approximation to a solution of Oberlack's model [14] by using the so-called method of prolongation with respect to the parameter ε.

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