# Triangular Newton Equations with Maximal Number of Integrals of Motion 

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#### Abstract

We study two-dimensional triangular systems of Newton equations (acceleration $=$ velocity-independent force) admitting three functionally independent quadratic integrals of motion. The main idea is to exploit the fact that the first component $M_{1}\left(q_{1}\right)$ of a triangular force depends on one variable only. By using the existence of extra integrals of motion we reduce the problem to solving a simultaneous system of three linear ordinary differential equations with nonconstant coefficients for $M_{1}\left(q_{1}\right)$. With the help of computer algebra we have found and solved these ordinary differential equations in all cases. A complete list of superintegrable triangular equations in two dimensions is been given. Most of these equations were not known before.


## 1 Introduction

In the last few years we have witnessed $[3,4,8,5]$ considerable interest in Hamiltonian systems that admit a maximal number of $2 n-1$ functionally independent integrals of motion. Their trajectories are completely determined as the intersection of surfaces of constant value of these integrals of motion. In the case in which one of these surfaces is compact the trajectories are closed and solutions are periodic. Such systems are very exceptional, but they are interesting in classical and quantum mechanics because their solutions can be described analytically and in the quantum case their spectrum can be often characterized in an algebraic way.

Remarkably some very important systems in physics, such as the homogeneous harmonic oscillator and the Kepler problem are superintegrable and they admit $2 n-1$ integrals of motion depending at most quadratically on momenta (velocities). This is the feature of Hamiltonian systems solvable through separation of variables in the HamiltonJacobi equation in more then one system of coordinates. Majority of presently known superintegrable systems are separable and their all integrals of motion depend at most quadratically on momenta.

A first example of a superintegrable system with integrals depending on higher powers of the momenta can be ascribed to C. G. J. Jacobi [2]. It is the system of $n$ particles in one dimension interacting with each other by an inverse square potential. This system is called now the Calogero system since it has been rediscovered in the quantum case by F . Calogero and then subsequently solved in the classical case too [1]. For $n>3$ this system has integrals that essentially depend on higher (than two) powers of the momenta and this system is not separable through the classical version of the Hamilton-Jacobi method. In [8] it has been shown that the Calogero system is superintegrable as well as the Euler-Calogero system [5] where the particle variables are coupled to internal spin variables.

For superintegrable systems of equations that are not potential little is known. In this paper we consider new a type of superintegrable systems that naturally arise in the framework of theory of quasipotential Newton equations [6]

$$
\begin{equation*}
\ddot{q}=M(q)=-A^{-1} \nabla k(q), \quad q=\left(q_{1}, q_{2}\right)^{t} \tag{1.1}
\end{equation*}
$$

with a triangular force $M=\left(M_{1}, M_{2}\right)^{t}$ so that the Newton equation has the coordinate form

$$
\begin{equation*}
\ddot{q}_{1}=M_{1}\left(q_{1}\right), \quad \ddot{q}_{2}=M_{2}\left(q_{1}, q_{2}\right) . \tag{1.2}
\end{equation*}
$$

We assume here that equation (1.1) admits an integral of motion, $E(q, \dot{q})=\frac{1}{2} \dot{q}^{t} A(q) \dot{q}+$ $k(q)$, that is a quadratic function of velocities $\dot{q}$ and is functionally independent of the energy integral, $G=\frac{1}{2} \dot{q}_{1}^{2}-\int M_{1}\left(q_{1}\right) d q_{1}$, admitted by the first equation $\ddot{q}_{1}=M_{1}\left(q_{1}\right)$.

The objective of this work is to find forces $M$ for which equations (1.2) admit three functionally independent integrals of motion which are all quadratic in the velocities. Such systems are often called superintegrable or completely degenerate since in the compact energy case all frequencies in action-angle variables are commensurable.

The main idea is to use the assumption about existence of three independent integrals, together with the property that $M_{1}\left(q_{1}\right)$ depends on one variable only, to derive ordinary differential equations (ODEs) to be satisfied by $M_{1}\left(q_{1}\right)$. We do this by observing that to each quadratic integral of motion there is related the quasipotential form (1.1) of the force $M$ so that $\nabla k=-A M$. Cross-differentiation of $\nabla k$ leads to a first order linear partial differential equation ( PDE ) for $M_{1}, M_{2}$ and a similar equation is related to the second integral of motion $F=\frac{1}{2} \dot{q}^{t} B(q) \dot{q}+l(q)$. These equations and their differential consequences lead to ODEs for $M_{1}\left(q_{1}\right)$.

In order to execute this idea in practice it has been necessary firstly to classify a linear pencil of matrices $A(q)+\mu B(q)$ with respect to affine triangular transformations $q \mapsto S q+h, S=\left(\begin{array}{ll}a & 0 \\ c & d\end{array}\right)$ preserving the triangular form of equation (1.2).

In Section 2 we recall [6] relevant facts from theory of quasipotential Newton equations: the notion of biquasipotential systems and the notion of the fundamental equation. Our method of determining all completely degenerate triangular Newton equations is presented in Section 3. In Section 4 we classify linear pencils $A(q)+\mu B(q)$ with respect to affine triangular transformations. The result is summarized by Figure 1. These pencils, $A+\mu B$, are the starting point for implementation of the method of Section 3. The results of all cases are presented in Section 5 which ends with Table 1 of all superintegrable triangular systems.

## 2 Biquasipotential Newton equations

A Newton equation, $\ddot{q}=M(q)$, is called quasipotential if the force $M(q)$ can be written as

$$
\begin{equation*}
M(q)=-A(q)^{-1} \nabla k(q) \tag{2.1}
\end{equation*}
$$

where $A$ is a symmetric $n \times n$ matrix satisfying the so called cyclic condition,

$$
\begin{equation*}
\partial_{i} A_{j k}+\partial_{j} A_{k i}+\partial_{k} A_{i j}=0 \quad \text { for all } i, j, k=1,2 \tag{2.2}
\end{equation*}
$$

and $k(q)$ is a scalar function. We use here the notation $\partial_{i}=\partial / \partial x_{i}$ and below we also use the notation $\partial_{i j}=\partial^{2} / \partial x_{i} \partial x_{j}$ and $\partial_{i j k}=\partial^{3} / \partial x_{i} \partial x_{j} \partial x_{k}$. A quasipotential system is a generalization of a potential system when $A(q)=I d$. A quasipotential system always admits an integral of motion depending quadratically on velocities of the form

$$
\begin{equation*}
E(q, \dot{q})=\frac{1}{2} \sum_{i, j=1}^{n} \dot{q}_{i} A_{i j}(q) \dot{q}_{j}+k(q)=\frac{1}{2} \dot{q}^{t} A \dot{q}+k(q) \tag{2.3}
\end{equation*}
$$

Proposition 1. If $A$ is nonsingular, the following statements are equivalent:
(i) The system $\ddot{q}=M(q)$ is quasipotential.
(ii) The system $\ddot{q}=M(q)$ has an integral of motion quadratic in velocities given by $E=\frac{1}{2} \dot{q}^{t} A(q) \dot{q}+k(q)$.

Proof. By differentiating (2.3) with respect to time we find

$$
\begin{align*}
\dot{E} & =\sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{i j} \ddot{q}_{j}+\partial_{i} k\right) \dot{q}_{i}+\frac{1}{2} \sum_{i, j, k=1}^{n} \partial_{k} A_{i j} \dot{q}_{i} \dot{q}_{j} \dot{q}_{k}  \tag{2.4}\\
& =\sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{i j} \ddot{q}_{j}+\partial_{i} k\right) \dot{q}_{i}+\frac{1}{6} \sum_{i, j, k=1}^{n}\left(\partial_{i} A_{j k}+\partial_{j} A_{k i}+\partial_{k} A_{i j}\right) \dot{q}_{i} \dot{q}_{j} \dot{q}_{k} .
\end{align*}
$$

At different powers of $\dot{q}_{i}$ in (2.4) we get equations (2.1) and (2.2): $A \ddot{q}+\nabla k=0, \partial_{i} A_{j k}+$ $\partial_{j} A_{k i}+\partial_{k} A_{i j}=0$ for all $i, j, k=1,2, \ldots, n$.

Remark 1. In two dimensions the general (symmetric) solution to the cyclic condition is:

$$
A=\left[\begin{array}{cc}
a q_{2}^{2}+b q_{2}+\alpha_{1} & -a q_{1} q_{2}-\frac{1}{2} b q_{1}-\frac{1}{2} c q_{2}+\frac{1}{2} \beta  \tag{2.5}\\
-a q_{1} q_{2}-\frac{1}{2} b q_{1}-\frac{1}{2} c q_{2}+\frac{1}{2} \beta & a q_{1}^{2}+c q_{1}+\gamma
\end{array}\right]
$$

where $a, b, c, \alpha, \beta$ and $\gamma$ are real constants.
Suppose a Newton equation, $\ddot{q}=M(q)$, admits two functionally independent integrals of motion, $E(q, \dot{q})=\frac{1}{2} \dot{q}^{t} A(q) \dot{q}+k(q)$ and $F(q, \dot{q})=\frac{1}{2} \dot{q}^{t} B(q) \dot{q}+l(q)$, where $A$ and $B$ are $n \times n$ symmetric nonsingular matrices satisfying the cyclic condition. Then we must have

$$
\begin{equation*}
\ddot{q}=M=-A^{-1} \nabla k=-B^{-1} \nabla l \tag{2.6}
\end{equation*}
$$

and the system is called biquasipotential. We have assumed both $A$ and $B$ to be nonsingular. However, if $A$ is nonsingular and $B$ not, we can always take a new integral, $\tilde{F}=E+\lambda F$, with the nonsingular matrix $\tilde{B}=A+\lambda B$.

From (2.6) we have $A^{-1} \nabla k=B^{-1} \nabla l$. The compatibility conditions for derivatives of $l, \partial_{12} l=\partial_{21} l$, yields

$$
\begin{equation*}
\partial_{1}\left(B A^{-1} \nabla k\right)_{2}-\partial_{2}\left(B A^{-1} \nabla k\right)_{1}=0, \tag{2.7}
\end{equation*}
$$

where $(\cdot)_{i}$ denotes the $i$ th component of the vector. If we substitute $k=K \operatorname{det} A$ in (2.7) and use the cyclic condition for $A$, we obtain a second-order linear PDE for $K(q)$ which is satisfied by $K=k / \operatorname{det} A$ :

$$
\begin{align*}
0= & 2\left(A_{12} B_{22}-A_{22} B_{12}\right) \partial_{11} K-2\left(A_{11} B_{22}-A_{22} B_{11}\right) \partial_{12} K \\
& +2\left(A_{11} B_{12}-A_{12} B_{11}\right) \partial_{22} K \\
& +3\left(A_{12} \partial_{1} B_{22}-B_{12} \partial_{1} A_{22}+A_{22} \partial_{2} B_{11}-B_{22} \partial_{2} A_{11}\right) \partial_{1} K  \tag{2.8}\\
& -3\left(A_{11} \partial_{1} B_{22}-B_{11} \partial_{1} A_{22}+A_{12} \partial_{2} B_{11}-B_{12} \partial_{2} A_{11}\right) \partial_{2} K \\
& +3\left(\partial_{1} A_{22} \partial_{2} B_{11}-\partial_{1} B_{22} \partial_{2} A_{11}\right) K .
\end{align*}
$$

This equation is called the fundamental equation associated with the pair of matrices $(A, B)$. If we substitute $A=I d$ into the fundamental equation, it reduces to the BertrandDarboux equation [7] with $K=V$.

By interchanging $A$ and $B$ in the fundamental equation (2.8), we see that it only changes sign. Thus $l / \operatorname{det} B$ must also be a solution to (2.8) if (2.6) holds. The fundamental equation is invariant under the transformation $A \rightarrow \lambda A+\mu B, B \rightarrow \lambda^{\prime} A+\mu^{\prime} B$, where $\lambda, \lambda^{\prime}, \mu, \mu^{\prime} \in \mathbf{R}$. The exact relationship between pairs $(A, B)$ and the fundamental equation is given by the following theorem, [6].

Theorem 1. Let $(A, B)$ be a pair of linearly independent matrices satisfying the cyclic conditions (2.2). Then there is a 1-1 relationship between the linear span $\{\lambda A+\mu B$ : $\lambda, \mu \in \mathbf{R}\}$ of $A$ and $B$ and the fundamental equation (2.8), i.e.,

1. any two linearly independent matrices $A^{\prime}=\alpha A+\beta B^{\prime}, B^{\prime}=\gamma A+\delta B$ determine the same fundamental equation as $(A, B)$ does and
2. if the pair $\left(A^{\prime}, B^{\prime}\right)$ determines the same fundamental equation as $(A, B)$ does, then the matrices $A^{\prime}$ and $B^{\prime}$ belong to the linear span $\{\lambda A+\mu B\}$ of $A$ and $B$.

## 3 Completely degenerate triangular Newton equations

In this Section we study triangular Newton equations in two dimensions,

$$
\begin{equation*}
\ddot{q}_{1}=M_{1}\left(q_{1}\right), \quad \ddot{q}_{2}=M_{2}\left(q_{1}, q_{2}\right), \tag{3.1}
\end{equation*}
$$

admitting three functionally independent integrals of motion which are quadratic functions of the velocities. Any triangular equation always has

$$
G=\frac{1}{2} \dot{q}_{1}^{2}-\int M_{1}\left(q_{1}\right) d q_{1}=\dot{q}^{t} \underbrace{\left(\begin{array}{ll}
1 & 0  \tag{3.2}\\
0 & 0
\end{array}\right)}_{=: C} \dot{q}-\int M_{1}\left(q_{1}\right) d q_{1}
$$

as an integral of motion. Moreover we assume that there exist two extra integrals, $E=$ $\frac{1}{2} \dot{q}^{t} A(q) \dot{q}+k(q)$ and $F=\frac{1}{2} \dot{q}^{t} B(q) \dot{q}+l(q)$, which are functionally independent.

The functional independence of these integrals can inferred from the linear independence of the matrices, $A, B$ and $C$, as follows from Proposition 2 below.
Proposition 2. Let $E, F$ and $G$ be three integrals of motion to a given two-dimensional system of equations, where $E, F$ and $G$ are defined as

$$
\begin{equation*}
E=\frac{1}{2} \dot{q}^{t} A(q) \dot{q}+k(q), \quad F=\frac{1}{2} \dot{q}^{t} B(q) \dot{q}+l(q), \quad G=\frac{1}{2} \dot{q}^{t} C(q) \dot{q}+m(q) \tag{3.3}
\end{equation*}
$$

and $A, B$ and $C$ satisfy the cyclic condition (2.2). Then $E, F$ and $G$ are functionally independent if and only if their respective symmetric matrices $A, B$ and $C$ are linearly independent.
Proof. To show that functional independence implies linear independence of $A, B$ and $C$ assume that there exist $\lambda, \mu$ and $\nu \in \mathbf{R}$ such that $\lambda A+\mu B+\nu C=0$. Then $K=\lambda E+$ $\mu F+\nu G$ is an integral of motion not depending on velocities $\dot{q}$ and $0=\dot{K}=\left(\dot{q}_{1}, \dot{q}_{2}\right) \cdot \nabla_{q} K$ implies that $\nabla_{q} K=\lambda \nabla_{q} E+\mu \nabla_{q} F+\nu \nabla_{q} G=0$ since $\dot{q}_{1}$ and $\dot{q}_{2}$ are arbitrary. The linear dependence $\lambda A+\mu B+\nu C=0$ implies also that $\lambda \nabla_{\dot{q}}\left(\dot{q}^{t} A \dot{q}\right)+\mu \nabla_{\dot{q}}\left(\dot{q}^{t} B \dot{q}\right)+\nu \nabla_{\dot{q}}\left(\dot{q}^{t} C \dot{q}\right)=0$ and therefore the full gradients $\lambda \nabla E+\mu \nabla F+\nu \nabla G=0$, where $\nabla=\left(\nabla_{q}, \nabla_{\dot{q}}\right)^{t}$, are linearly dependent. To show implication in the other direction assume that $E, F$ and $G$ are functionally dependent so that there exist $\lambda, \mu$ and $\nu$ such that for the full gradients $\lambda \nabla E+\mu \nabla F+\nu \nabla G=0$. This implies that $\lambda \nabla_{q} k+\mu \nabla_{q} l+\nu \nabla_{q} m=0$ and that $0=$ $\nabla\left[\dot{q}^{t}(\lambda A+\mu B+\nu C) \dot{q}\right]=\nabla_{q}\left[\dot{q}^{t}(\lambda A+\mu B+\nu C) \dot{q}\right]+\nabla_{\dot{q}}\left[\dot{q}^{t}(\lambda A+\mu B+\nu C) \dot{q}\right]$. Since each term has to vanish on its own, we get $2(\lambda A+\mu B+\nu C) \dot{q}=0$ for arbitrary $\dot{q}$ and therefore $\lambda A+\mu B+\nu C=0$.

We know that by solving the fundamental equation we obtain biquasipotential systems. The integral $G=\frac{1}{2} \dot{q}_{1}^{2}-\int M_{1}\left(q_{1}\right) d q_{1}$ can be written as $G=\frac{1}{2} \dot{q}^{t} C(q) \dot{q}+m(q)$ with $C=$ $\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$ and $m(q)=-\int M_{1}\left(q_{1}\right) d q_{1}$.

The existence of three functionally independent quadratic integrals, $E, F$ and $G$, for the system, $\ddot{q}=M(q)$, results in two fundamental equations for $K=k \operatorname{det}(A)$, one for $A$ and $B$ and one for $A$ and $C$. Simultaneous solutions to these two fundamental equations give superintegrable triangular systems. However, the solution of two fundamental equations which are second-order linear PDEs for one unknown function is a nontrivial task.

The main idea of our approach is to study conditions for the components, $M_{1}\left(q_{1}\right)$ and $M_{2}\left(q_{1}, q_{2}\right)$, of a quasipotential superintegrable force and to derive ODEs which have to be satisfied by the component $M_{1}$. The advantage of this approach is that one works with a system of first-order linear PDEs instead of fundamental equations that are PDEs of second order.

An arbitrary triangular force $M$ is not quasipotential, i.e. $M \neq-A^{-1} \nabla k$, since $k$ may not exist. For a quasipotential system we have from (2.6) that

$$
\begin{equation*}
A M=\binom{A_{11} M_{1}+A_{12} M_{2}}{A_{21} M_{1}+A_{22} M_{2}}=-\binom{\partial_{1} k}{\partial_{2} k}, \tag{3.4}
\end{equation*}
$$

where $A$ satisfies the cyclic condition (2.2). A sufficient condition for the existence of $k$ is that $\partial_{2}(A M)_{1}=\partial_{1}(A M)_{2}$. By requiring this in (3.4) we get a first-order partial
differential equation for components of M which, after using triangularity, $\partial_{2} M_{1}=0$, and the cyclic condition (2.2), yields

$$
\begin{equation*}
-3\left(\partial_{1} A_{12}\right) M_{1}-A_{12} \partial_{1} M_{1}=-3\left(\partial_{2} A_{12}\right) M_{2}+A_{22} \partial_{1} M_{2}-A_{12} \partial_{2} M_{2} . \tag{3.5}
\end{equation*}
$$

If $A$ and $C$ are linearly independent, equation (3.5) is a sufficient and necessary condition for the system (3.1) to admit an extra integral, $E(q, \dot{q})$, that depends quadratically on the velocities.

If we assume that the system (3.1) has two functionally independent integrals of motion, $E$ and $F$, we get two linearly independent equations to be fulfilled by the components of the force $M$,

$$
\begin{align*}
& \alpha M_{1}+\beta \partial_{1} M_{1}=\gamma M_{2}+\delta \partial_{1} M_{2}+\beta \partial_{2} M_{2}  \tag{3.6}\\
& \alpha^{\prime} M_{1}+\beta^{\prime} \partial_{1} M_{1}=\gamma^{\prime} M_{2}+\delta^{\prime} \partial_{1} M_{2}+\beta^{\prime} \partial_{2} M_{2} \tag{3.7}
\end{align*}
$$

where the coefficient vectors $\left(\alpha=-3 \partial_{1} A_{12}, \beta=-A_{12}, \gamma=-3 \partial_{2} A_{12}, \delta=A_{22}\right)^{t}$ and $\left(\alpha^{\prime}=-3 \partial_{1} B_{12}, \beta^{\prime}=-B_{12}, \gamma^{\prime}=-3 \partial_{2} B_{12}, \delta^{\prime}=B_{22}\right)^{t}$ are linearly independent.

Equations (3.6) and (3.7) can also be written in a matrix form as

$$
\underbrace{\left(\begin{array}{cc}
\delta & \beta  \tag{3.8}\\
\delta^{\prime} & \beta^{\prime}
\end{array}\right)}_{=: D}\binom{\partial_{1} M_{2}}{\partial_{2} M_{2}}=\left(\begin{array}{cc}
\alpha & -\gamma \\
\alpha^{\prime} & -\gamma^{\prime}
\end{array}\right)\binom{M_{1}}{M_{2}}+\left(\begin{array}{cc}
\beta & 0 \\
\beta^{\prime} & 0
\end{array}\right)\binom{\partial_{1} M_{1}}{\partial_{2} M_{1}} .
$$

Without loosing generality we may assume that $\delta \beta^{\prime}-\beta \delta^{\prime} \neq 0$ since the condition $0=$ $\delta \beta^{\prime}-\beta \delta^{\prime}=A_{12} B_{22}-B_{12} A_{22}$ means that $A, B$ and $C$ have to be linearly dependent contrary to the assumption. We can express the derivatives $\partial_{1} M_{2}$ and $\partial_{2} M_{2}$ as

$$
\begin{aligned}
\binom{\partial_{1} M_{2}}{\partial_{2} M_{2}}= & \frac{1}{\delta \beta^{\prime}-\delta^{\prime} \beta}\left(\begin{array}{cc}
\beta^{\prime} & -\beta \\
-\delta^{\prime} & \delta
\end{array}\right)\left(\begin{array}{cc}
\alpha & -\gamma \\
\alpha^{\prime} & -\gamma^{\prime}
\end{array}\right)\binom{M_{1}}{M_{2}} \\
& +\frac{1}{\delta \beta^{\prime}-\delta^{\prime} \beta}\left(\begin{array}{cc}
\beta^{\prime} & -\beta \\
-\delta^{\prime} & \delta
\end{array}\right)\left(\begin{array}{cc}
\beta & 0 \\
\beta^{\prime} & 0
\end{array}\right)\binom{\partial_{1} M_{1}}{\partial_{2} M_{1}} \\
= & \frac{1}{\delta \beta^{\prime}-\delta^{\prime} \beta}\left(\begin{array}{cc}
\beta^{\prime} \alpha-\beta \alpha^{\prime} & -\beta^{\prime} \gamma+\beta \gamma^{\prime} \\
-\delta^{\prime} \alpha+\delta \alpha^{\prime} & \delta^{\prime} \gamma-\delta \gamma^{\prime}
\end{array}\right)\binom{M_{1}}{M_{2}}+\binom{0}{\partial_{1} M_{1}} .
\end{aligned}
$$

Equality of the mixed derivatives, $\partial_{2}\left(\partial_{1} M_{2}\right)=\partial_{1}\left(\partial_{2} M_{2}\right)$, leads to a third partial differential equation, similar to (3.6) and (3.7), viz.

$$
\begin{equation*}
\alpha^{\prime \prime} M_{1}+\beta^{\prime \prime} \partial_{1} M_{1}-\partial_{11} M_{1}=\gamma^{\prime \prime} M_{2}+\delta^{\prime \prime} \partial_{1} M_{2}+\varepsilon^{\prime \prime} \partial_{2} M_{2}, \tag{3.9}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\alpha^{\prime \prime}=\partial_{2}\left(\frac{\beta^{\prime} \alpha-\beta \alpha^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}\right)+\partial_{1}\left(\frac{\delta^{\prime} \alpha-\delta \alpha^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}\right), & \beta^{\prime \prime}=\frac{\delta^{\prime} \alpha-\delta \alpha^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}, \\
\gamma^{\prime \prime}=\partial_{1}\left(\frac{\delta^{\prime} \gamma-\delta \gamma^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}\right)+\partial_{2}\left(\frac{\beta^{\prime} \gamma-\beta \gamma^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}\right), \quad \delta^{\prime \prime}=\frac{\delta^{\prime} \gamma-\delta \gamma^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}}, \quad \varepsilon^{\prime \prime}=\frac{\beta^{\prime} \gamma-\beta \gamma^{\prime}}{\beta^{\prime} \delta-\beta \delta^{\prime}} .
\end{array}
$$

Equations (3.6), (3.7) and (3.9) can now be rewritten as the following linear system of equations for derivatives of $M_{1}$ and $M_{2}$,

$$
\underbrace{\left(\begin{array}{ccc}
\alpha & \beta & 0  \tag{3.10}\\
\alpha^{\prime} & \beta^{\prime} & 0 \\
\alpha^{\prime \prime} & \beta^{\prime \prime} & -1
\end{array}\right)}_{=: P}\left(\begin{array}{c}
M_{1} \\
\partial_{1} M_{1} \\
\partial_{11} M_{1}
\end{array}\right)=\underbrace{\left(\begin{array}{ccc}
\gamma & \delta & \beta \\
\gamma^{\prime} & \delta^{\prime} & \beta^{\prime} \\
\gamma^{\prime \prime} & \delta^{\prime \prime} & \varepsilon^{\prime \prime}
\end{array}\right)}_{=: Q}\left(\begin{array}{c}
M_{2} \\
\partial_{1} M_{2} \\
\partial_{2} M_{2}
\end{array}\right) .
$$

If $\operatorname{det}(Q)=0$, then rows in matrix $Q$ are linearly dependent and by linearly combining the three equations in (3.10) we can directly get a second-order ordinary differential equation for $M_{1}$.

On the other hand, if $\operatorname{det}(Q) \neq 0$, we can multiply both sides with $Q^{-1}$ and we then get

$$
\left(\begin{array}{c}
M_{2}  \tag{3.11}\\
\partial_{1} M_{2} \\
\partial_{2} M_{2}
\end{array}\right)=\underbrace{Q^{-1} P}_{=: R}\left(\begin{array}{c}
M_{1} \\
\partial_{1} M_{1} \\
\partial_{11} M_{1}
\end{array}\right)
$$

or in components

$$
\begin{aligned}
M_{2} & =R_{11} M_{1}+R_{12} \partial_{1} M_{1}+R_{13} \partial_{11} M_{1}, \\
\partial_{1} M_{2} & =R_{21} M_{1}+R_{22} \partial_{1} M_{1}+R_{23} \partial_{11} M_{1}, \\
\partial_{2} M_{2} & =R_{31} M_{1}+R_{32} \partial_{1} M_{1}+R_{33} \partial_{11} M_{1} .
\end{aligned}
$$

The compatibility conditions for the derivatives, $\partial_{2}\left(\partial_{1} M_{2}\right)=\partial_{1}\left(\partial_{2} M_{2}\right), \partial_{1}\left(M_{2}\right)=$ $\partial_{1} M_{2}$ and $\partial_{2}\left(M_{2}\right)=\partial_{2} M_{2}$, lead immediately to three ordinary differential equations for $M_{1}$.

From $\partial_{2}\left(\partial_{1} M_{2}\right)=\partial_{1}\left(\partial_{2} M_{2}\right)$ we get

$$
\begin{equation*}
a M_{1}+b \partial_{1} M_{1}+c \partial_{11} M_{1}+d \partial_{111} M_{1}=0 \tag{3.12}
\end{equation*}
$$

where $a=\partial_{2} R_{21}-\partial_{1} R_{31}, b=\partial_{2} R_{22}-\partial_{1} R_{32}-R_{31}, c=\partial_{2} R_{23}-\partial_{1} R_{33}-R_{32}$ and $d=-R_{33}$. From $\partial_{1}\left(M_{2}\right)=\partial_{1} M_{2}$ we get

$$
\begin{equation*}
a^{\prime} M_{1}+b^{\prime} \partial_{1} M_{1}+c^{\prime} \partial_{11} M_{1}+d^{\prime} \partial_{111} M_{1}=0 \tag{3.13}
\end{equation*}
$$

where $a^{\prime}=\partial_{1} R_{11}-R_{21}, b^{\prime}=\partial_{1} R_{12}+R_{11}-R_{22}, c^{\prime}=\partial_{1} R_{13}+R_{12}-R_{23}$ and $d^{\prime}=R_{13}$. Finally from $\partial_{2}\left(M_{2}\right)=\partial_{2} M_{2}$ we get

$$
\begin{equation*}
a^{\prime \prime} M_{1}+b^{\prime \prime} \partial_{1} M_{1}+c^{\prime \prime} \partial_{11} M_{1}=0 \tag{3.14}
\end{equation*}
$$

where $a^{\prime \prime}=\partial_{2} R_{11}-R_{31}, b^{\prime \prime}=\partial_{2} R_{12}-R_{32}$ and $c^{\prime \prime}=\partial_{2} R_{13}-R_{33}$.
If equations (3.12-3.14) are linearly independent, the highest derivatives can be eliminated to reduce the system (3.12-3.14) to a first-order differential equation. The solution to this first-order differential equation is required to satisfy (3.12-3.14). It may happen that a system of linear ODEs does not have a common solution, although higher derivatives can be eliminated. A list of superintegrable triangular Newton equations and the corresponding integrals is given in Table 1 and Table 2.

## 4 Classification of linear pencils $A+\mu B$

The results of Table 1 and Table 2 are calculated for standard forms of linear pencils $A+\mu B$ determined in a similar way as in [6]. Since triangular equations are not invariant w.r.t. the full affine group of transformations, we have to rework the classification of [6]. In general a biquasipotential system is a system of the form

$$
\begin{equation*}
\ddot{q}=M=-A^{-1} \nabla k=-B^{-1} \nabla l, \tag{4.1}
\end{equation*}
$$

where $A$ and $B$ are two linearly independent solutions to the cyclic condition (2.2), i.e. matrices of the form:

$$
\begin{align*}
& A=\left[\begin{array}{cc}
a_{1} q_{2}^{2}+b_{1} q_{2}+\alpha_{1} & -a_{1} q_{1} q_{2}-\frac{1}{2} b_{1} q_{1}-\frac{1}{2} c_{1} q_{2}+\frac{1}{2} \beta_{1} \\
* & a_{1} q_{1}^{2}+c_{1} q_{1}+\gamma_{1}
\end{array}\right]  \tag{4.2}\\
& B=\left[\begin{array}{cc}
a_{2} q_{2}^{2}+b_{2} q_{2}+\alpha_{2} & -a_{2} q_{1} q_{2}-\frac{1}{2} b_{2} q_{1}-\frac{1}{2} c_{2} q_{2}+\frac{1}{2} \beta_{2} \\
* & a_{2} q_{1}^{2}+c_{2} q_{1}+\gamma_{2}
\end{array}\right], \tag{4.3}
\end{align*}
$$

where $*$ denotes elements determined by symmetry, $A=A^{t}$. Each matrix, $A$ and $B$, depends upon six independent parameters and affine coordinate transformations, $\left(q_{1}, q_{2}\right) \mapsto$ $\left(Q_{1}, Q_{2}\right)$,

$$
\begin{equation*}
q=S Q+h, \quad S \in \mathrm{GL}(2, \mathbf{R}), \quad h \in \mathbf{R}^{2} \tag{4.4}
\end{equation*}
$$

can be used to reduce the number of parameters.
The system (3.1) is transformed to a new triangular system, i.e. $M_{1}\left(q_{1}(Q)\right)=\tilde{M}_{1}\left(Q_{1}\right)$, if $S$ is a triangular matrix,

$$
S=\left(\begin{array}{ll}
a & 0  \tag{4.5}\\
c & d
\end{array}\right)=\left(\begin{array}{ll}
a & 0 \\
0 & d
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
c^{\prime} & 1
\end{array}\right)
$$

where the triangular matrix $S$ can be further factorized (commutatively) into a product of two simpler matrices, a diagonal matrix and a lower triangular matrix with ones on the diagonal.

Under the coordinate transformation (4.4) the integral of motion (2.3) transforms as

$$
\begin{equation*}
E(q(Q), \dot{q}(\dot{Q}))=\dot{Q}^{t} S^{t} A(q(Q)) S \dot{Q}+k(q(Q)) \tag{4.6}
\end{equation*}
$$

where $\dot{q}=S \dot{Q}$. The integral remains quadratic w.r.t. velocities and the matrix $A_{Q}(Q)=$ $S^{t} A(q(Q)) S$ also satisfies the cyclic condition (2.2).

Note that nontrivial linear combinations $\left(\lambda A+\mu B, \lambda^{\prime} A+\mu^{\prime} B\right)$ generate the same biquasipotential system since $\lambda E+\mu F$ and $\lambda^{\prime} E+\mu^{\prime} F$ are quadratic integrals of motion. Since affine transformations or linear combinations do not change the polynomial degree of the matrices $A$ and $B$, we can divide the matrix pairs $(A, B)$ into inequivalent classes according to different polynomial degrees of $A$ and $B$. Each class is represented by an algebraically simple pair of matrices $A$ and $B$ (with smaller number of parameters) obtained with the use of affine triangular transformations,

$$
\binom{q_{1}}{q_{2}}=\left(\begin{array}{ll}
a & 0  \tag{4.7}\\
c & d
\end{array}\right)\binom{Q_{1}}{Q_{2}}+\binom{e}{f},
$$

and of taking linear combinations of matrices $A$ and $B$.
We now adopt the notation $A^{(i)}(i=0,1,2)$ for all matrices satisfying the cyclic condition and having the highest polynomial entries equal $i$. By $\left[A^{(i)}, B^{(j)}\right](i=0,1,2)$ we denote the class of (nonordered) pairs $(A, B)$ of linearly independent matrices $A$ and $B$, where one of the matrices has degree $i$ and the other has degree $j$. Obviously $\left[A^{(i)}, B^{(j)}\right]=$ $\left[A^{(j)}, B^{(i)}\right]$ and we thus have six such classes. Note that, if $(A, B) \in\left[A^{(2)}, B^{(2)}\right]$, we can kill $\alpha_{2}$ in $B$ by taking $B \mapsto B-\frac{\alpha_{2}}{\alpha_{1}} A$ so that every element in this class belongs to $\left[A^{(2)}, B^{(1)}\right]$. We are thus left with five classes which are invariant w.r.t. affine triangular transformations and w.r.t. taking linear combinations of matrices. In Figure 1 we give a list of matrices $(A, B)$ having the minimal number of free parameters.


Figure 1. Triangular affine inequivalent pairs $\left[A^{(i)}, B^{(j)}\right]$.
For the purpose of finding superintegrable triangular systems we can use $\alpha_{1}=\alpha_{2}=0$ since they can be removed with the use of $C=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$ and they do not appear in Table 1.

## 5 Summary of results

We now summarize superintegrable systems in Table 1 and their integrals of motion in Table 2. Equations and integrals of motion in these tables have been calculated with the use of computer algebra for all standard forms of matrix pairs $(A, B)$ showed in Figure 1. Remarkably nearly all superintegrable triangular equations in Table 1 belong to the case $\operatorname{det} Q=0$. Note that there is no superintegrable system in case $\left[A^{(1)}, B^{(1)}\right]$. There can be distinguished three types of equations: a) uncoupled, b) linear second-order inhomogeneous equations for $q_{2}$, with $q_{1}(t)$ determined from the first equation and c) nonlinear equation for $q_{2}$ as in the case $\left[A^{(2)}, B^{(1)}\right](a)$. They all admit three autonomous time-independent integrals of motion that depend quadratically on $\dot{q}_{1}$ and $\dot{q}_{2}$, as listed in Table 1.

Table 1: Superintegrable systems

| $\left[A^{(0)}, B^{(0)}\right]$ |
| :---: |
| $\begin{aligned} & \gamma_{1} \neq 0, \beta_{2} \neq 0 \\ & \quad M_{1}=C_{1} q_{1}+C_{2}, \quad M_{2}=C_{1} q_{2}+C_{3} \end{aligned}$ |
| $\left[A^{(1)}, B^{(0)}\right]$ (a) |
| $\begin{aligned} & \beta_{2}=0, \gamma_{2} \neq 0 \\ & \quad M_{1}=C_{1} q_{1}+C_{2}, \quad M_{2}=\frac{C_{1}}{4} q_{2}+\frac{C_{3}}{q_{2}^{3}} \end{aligned}$ |
| $\begin{aligned} & \beta_{2} \neq 0, \gamma_{2}=0 \\ & \quad M_{1}=C_{1}+\frac{C_{2}}{\sqrt{q_{1}}}, \quad M_{2}=-\frac{C_{2}}{2} \frac{q_{2}}{q_{1}^{3 / 2}}+\frac{C_{3}}{q_{1}^{3 / 2}} \end{aligned}$ |
| $\left[A^{(1)}, B^{(0)}\right](\mathrm{b})$ |
| $\begin{aligned} & \gamma_{1}=0, \gamma_{2} \neq 0 \\ & \quad M_{1}=C_{1} q_{1}+\frac{C_{2}}{q_{1}^{3}}, \quad M_{2}=4 C_{1} q_{2}+\frac{3}{2} \frac{\beta_{2}}{\gamma_{2}}\left(C_{1} q_{1}-\frac{C_{2}}{3 q_{1}^{3}}\right)+C_{3} \end{aligned}$ |
| $\begin{aligned} & \gamma_{1} \neq 0, \beta_{2} \neq 0, \gamma_{2}=0 \\ & \quad M_{1}=C_{1} q_{1}+C_{2}, \quad M_{2}=\frac{3}{2 \gamma_{1}}\left(\frac{C_{1}}{2} q_{1}^{2}+C_{2} q_{1}+C_{3}\right)+C_{1} q_{2} \end{aligned}$ |
| $\left[A^{(2)}, B^{(0)}\right]$ |
| $\begin{aligned} & \gamma_{1}=0, \gamma_{2} \neq 0 \\ & \quad M_{1}=C_{1} q_{1}+\frac{C_{2}}{q_{1}^{3}}, \quad M_{2}=C_{1} q_{2}-C_{2} \frac{\beta_{2}}{2 \gamma_{2}} \frac{1}{q_{1}^{3}}+\frac{C_{3}}{\left(\frac{\beta_{2}}{2 \gamma_{2}} q_{1}+q_{2}\right)^{3}} \end{aligned}$ |

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|  | $\begin{aligned} & \gamma_{1}=0, \beta_{2} \neq 0, \gamma_{2}=0 \\ & \quad M_{1}=C_{1} q_{1}+\frac{C_{2}}{q_{1}^{3}}, \quad M_{2}=\left(C_{1}-\frac{3 C_{2}}{q_{1}^{4}}\right) q_{2}+\frac{C_{3}}{q_{1}^{3}} \end{aligned}$ |
| :---: | :---: |
|  | $\begin{aligned} & \gamma_{1} \neq 0, \beta_{2} \neq 0, \gamma_{2}=0 \\ & \quad M_{1}=C_{1} q_{1}+C_{2} \frac{2 q_{1}^{2}+\gamma_{1}}{\sqrt{q_{1}^{2}+\gamma_{1}}}, \quad M_{2}=\left(C_{1}+C_{2} \frac{2 q_{1}^{3}+3 \gamma_{1} q_{1}}{\left(q_{1}^{2}+\gamma_{1}\right)^{3 / 2}}\right) q_{2}+\frac{C_{3}}{\left(q_{1}^{2}+\gamma_{1}\right)^{3 / 2}} \end{aligned}$ |
|  | $\begin{aligned} & \gamma_{1} \neq 0, \beta_{2} \neq 0 \gamma_{2} \neq 0 \\ & \quad M_{1}=C_{1} q_{1}, \quad M_{2}=C_{1} q_{2} \end{aligned}$ |
|  | $\left[A^{(2)}, B^{(1)}\right]$ (a) |
|  | $\begin{aligned} & \beta_{1}=0, \beta_{2}=0 \\ & M_{1}=C_{1} \frac{\left(q_{1}+\gamma_{2}\right)\left(q_{1}^{2}+2 \gamma_{2} q_{1}+2 \gamma_{2}^{2}+\gamma_{1}\right)}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{3}}+C_{2} \frac{q_{1}^{4}+6 \gamma_{1} q_{1}^{2}+8 \gamma_{1} \gamma_{2} q_{1}+4 \gamma_{1} \gamma_{2}^{2}+\gamma_{1}^{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{3}} \\ & M_{2}=\frac{C_{1}}{4} \frac{\left(3 q_{1}^{2}+6 \gamma_{2} q_{1}+4 \gamma_{2}^{2}+\gamma_{1}\right) q_{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{3}}+C_{2} \frac{\left(q_{1}^{3}+3 \gamma_{1} q_{1}+2 \gamma_{1} \gamma_{2}\right) q_{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{3}}+\frac{C_{3}}{q_{2}^{3}} \end{aligned}$ |
|  | $\begin{aligned} & \beta_{1}=0, \gamma_{1}=0, \gamma_{2}=0, \beta_{2} \neq 0 \\ & M_{1}=\frac{C_{1}}{q_{1}^{2}}, \quad M_{2}=C_{1} \frac{q_{2}-\beta_{2}}{q_{1}^{3}} \end{aligned}$ |
|  | $\left[A^{(2)}, B^{(1)}\right](\mathrm{b})$ |
|  | $\begin{aligned} & \beta_{1}=0, \gamma_{1} \neq 0, \gamma_{2}=0 \\ & M_{1}=C_{1} \frac{\beta_{2} q_{1}+\gamma_{1}}{\left(q_{1}-\beta_{2}\right)^{3}}+C_{2} \frac{\left(2 \beta_{2}^{2}+\gamma_{1}\right) q_{1}^{2}+2 \gamma_{1} \beta_{2} q_{1}+\gamma_{1} \beta_{2}^{2}+2 \gamma_{1}^{2}}{\sqrt{q_{1}^{2}+\gamma_{1}}\left(q_{1}-\beta_{2}\right)^{3}} \\ & M_{2}=C_{1} \frac{\beta_{2} q_{2}}{\left(q_{1}-\beta_{2}\right)^{3}}+C_{2} \frac{\left(\left(2 \beta_{2}^{2}+\gamma_{1}\right) q_{1}^{3}+3 \gamma_{1} \beta_{2}^{2} q_{1}+2 \gamma_{1}^{2} \beta_{2}\right) q_{2}}{\left(q_{1}^{2}+\gamma_{1}\right)^{3 / 2}\left(q_{1}-\beta_{2}\right)^{3}}+\frac{C_{3}}{\left(q_{1}^{2}+\gamma_{1}\right)^{3 / 2}} \end{aligned}$ |
|  | $\begin{aligned} & \gamma_{1}=0, \gamma_{2}=0 \\ & M_{1}=\frac{C_{1} q_{1}}{\left(q_{1}-\beta_{2}\right)^{3}}+C_{2} \frac{4 q_{1}^{4}-6 \beta_{2} q_{1}^{2}+4 \beta_{2}^{2} q_{1}-\beta_{2}^{3}}{q_{1}^{3}\left(q_{1}-\beta_{2}\right)^{3}} \\ & M_{2}=\frac{C_{1}}{2} \frac{2 q_{1}^{3} q_{2}-3 \beta_{1} q_{1}^{2}+3 \beta_{1} \beta_{2} q_{1}-\beta_{1} \beta_{2}^{2}}{q_{1}^{3}\left(q_{1}-\beta_{2}\right)^{3}}+\frac{C_{2}}{2} \frac{\left(6 q_{1}^{2}-8 \beta_{2} q_{1}+3 \beta_{2}^{2}\right)\left(2 \beta_{2} q_{2}-\beta_{1}\right)}{q_{1}^{4}\left(q_{1}-\beta_{2}\right)^{3}}+\frac{C_{3}}{q_{1}^{3}} \end{aligned}$ |

Table 2: Integrals of motion

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| $\begin{aligned} \gamma_{1} & \neq 0, \beta_{2} \neq 0, \gamma_{2} \neq 0 \\ E & =\frac{1}{2} q_{2}^{2} \dot{q}_{1}^{2}-q_{1} q_{2} \dot{q}_{1} \dot{q}_{2}+\frac{1}{2}\left(q_{1}^{2}+\gamma_{1}\right) \dot{q}_{2}^{2}-\frac{C_{1}}{2} \gamma_{1} q_{2}^{2} \\ F & =\frac{\beta_{2}}{2} \dot{q}_{1} \dot{q}_{2}+\frac{\gamma_{2}}{2} \dot{q}_{2}^{2}-\frac{C_{1}}{2} \beta_{2} q_{1} q_{2}-\frac{C_{1}}{2} \gamma_{2} q_{2}^{2} \quad G=\frac{1}{2} \dot{q}_{1}^{2}-\frac{C_{1}}{2} q_{1}^{2} \end{aligned}$ |
| :---: |
| $\left[A^{(2)}, B^{(1)}\right](a)$ |
| $\begin{aligned} \beta_{1}= & 0, \beta_{2}=0 \\ E= & \frac{1}{2} q_{2}^{2} \dot{q}_{1}^{2}-q_{1} q_{2} \dot{q}_{1} \dot{q}_{2}+\frac{1}{2}\left(q_{1}^{2}+\gamma_{1}\right) \dot{q}_{2}^{2}+\frac{C_{1}}{8} \frac{\left(q_{1}^{2}+4 \gamma_{2} q_{1}+\gamma_{1}+4 \gamma_{2}^{2}\right) q_{2}^{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{2}} \\ & +C_{2} \frac{\gamma_{1}\left(q_{1}+\gamma_{2}\right) q_{2}^{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{2}}+\frac{C_{3}}{2} \frac{q_{1}^{2}+\gamma_{1}}{q_{2}^{2}} \\ F= & -\frac{1}{2} q_{2} \dot{q}_{1} \dot{q}_{2}+\frac{1}{2}\left(q_{1}+\gamma_{2}\right) \dot{q}_{2}^{2}-\frac{C_{1}}{8} \frac{\left(q_{1}+\gamma_{2}\right) q_{2}^{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{2}}-\frac{C_{2}}{4} \frac{\left(q_{1}^{2}+\gamma_{1}\right) q_{2}^{2}}{\left(q_{1}^{2}+2 \gamma_{1} q_{1}-\gamma_{1}\right)^{2}} \\ & +\frac{C_{3}}{2} \frac{q_{1}+\gamma_{2}}{q_{2}^{2}} \\ G= & \frac{1}{2} \dot{q}_{1}^{2}+\frac{C_{1}}{2} \frac{\left(q_{1}+\gamma_{2}\right)^{2}}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{2}}+C_{2} \frac{\left(q_{1}^{2}+\gamma_{1}\right)\left(q_{1}+\gamma_{2}\right)}{\left(q_{1}^{2}+2 \gamma_{2} q_{1}-\gamma_{1}\right)^{2}} \end{aligned}$ |
| $\begin{aligned} & \beta_{1}=0, \gamma_{1}=0, \gamma_{2}=0, \beta_{2} \neq 0 \\ & \quad E=\frac{1}{2} q_{2}^{2} \dot{q}_{1}^{2}-q_{1} q_{2} \dot{q}_{1} \dot{q}_{2}+\frac{1}{2} q_{1}^{2} \dot{q}_{2}^{2}+C_{1} \frac{\beta_{2} q_{2}}{q_{1}} \\ & F=-\frac{1}{2}\left(q_{2}-\beta_{2}\right) \dot{q}_{1} \dot{q}_{2}+\frac{1}{2} q_{1} \dot{q}_{2}^{2}-\frac{C_{1}}{4} \frac{\left(q_{2}-\beta_{2}\right)^{2}}{q_{1}^{2}} \quad G=\frac{1}{2} \dot{q}_{1}^{2}+\frac{C_{1}}{q_{1}} \end{aligned}$ |
| $\left[A^{(2)}, B^{(1)}\right](b)$ |
| $\begin{aligned} \beta_{1}= & 0, \gamma_{1} \neq 0, \gamma_{2}=0 \\ E= & \frac{1}{2} q_{2}^{2} \dot{q}_{1}^{2}-q_{1} q_{2} \dot{q}_{1} \dot{q}_{2}+\frac{1}{2}\left(q_{1}^{2}+\gamma_{1}\right) \dot{q}_{2}^{2}+\frac{C_{1}}{2} \frac{\gamma_{1} q_{2}^{2}}{\left(q_{1}-\beta_{2}\right)^{2}} \\ & +C_{2} \frac{\gamma_{1}\left(\beta_{2} q_{1}+\gamma_{1}\right) q_{2}^{2}}{\left(q_{1}-\beta_{2}\right)^{2} \sqrt{q_{1}^{2}+\gamma_{1}}}-C_{3} \frac{q_{2}}{\sqrt{q_{1}^{2}+\gamma_{1}}} \\ F= & \frac{1}{2} q_{2} \dot{q}_{1}^{2}-\frac{1}{2}\left(q_{1}-\beta_{2}\right) \dot{q}_{1} \dot{q}_{2}+\frac{C_{1}}{2} \frac{\left(\beta_{2} q_{1}+\gamma_{1}\right) q_{2}}{\left(q_{1}-\beta_{2}\right)^{2}} \\ & +\frac{C_{2}}{2} \frac{\left(\left(2 \beta_{2}^{2}+\gamma_{1}\right) q_{1}^{2}+2 \gamma_{1} \beta_{2} q_{1}+2 \gamma_{1}^{2}+\gamma_{1} \beta_{2}^{2}\right) q_{2}}{\left(q_{1}-\beta_{2}\right)^{2} \sqrt{q_{1}^{2}+\gamma_{1}}-\frac{C_{3}}{2}} \frac{\beta_{2} q_{1}+\gamma_{1}}{\gamma_{1} \sqrt{q_{1}^{2}+\gamma_{1}}} \\ G= & \frac{1}{2} \dot{q}_{1}^{2}+\frac{C_{1}}{2} \frac{2 \beta_{2} q_{1}+\gamma_{1}-\beta_{2}^{2}}{\left(q_{1}-\beta_{2}\right)^{2}}+C_{2} \frac{\left(\beta_{2} q_{1}+\gamma_{1}\right) \sqrt{q_{1}^{2}+\gamma_{1}}}{\left(q_{1}-\beta_{2}\right)^{2}} \end{aligned}$ |

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$$
\begin{aligned}
\gamma_{1}= & 0, \gamma_{2}=0 \\
E= & \frac{1}{2} q_{2}^{2} \dot{q}_{1}^{2}-\left(q_{1} q_{2}-\frac{\beta_{2}}{2}\right) \dot{q}_{1} \dot{q}_{2}+\frac{1}{2} q_{1}^{2} \dot{q}_{2}^{2}+\frac{C_{1} \beta_{1}}{8} \frac{8 q_{1}^{2} q_{2}-4 \beta_{2} q_{1} q_{2}-2 \beta_{1} q_{1}+\beta_{1} \beta_{2}}{q_{1}^{2}\left(q_{1}-\beta_{2}\right)^{2}} \\
& +\frac{C_{2}}{8} \frac{16 q_{1}^{3} q_{2}^{2}-32 \beta_{2} q_{1}^{2} q_{2}^{2}+8 \beta_{1} q_{1}^{2} q_{2}+16 \beta_{2}^{2} q_{1} q_{2}^{2}-3 \beta_{1}^{2} q_{1}-4 \beta_{1} \beta_{2}^{2} q_{2}+2 \beta_{1}^{2} \beta_{2}}{q_{1}^{3}\left(q_{1}-\beta_{2}\right)^{2}} \\
& -\frac{C_{3}}{4} \frac{4 q_{1} q_{2}-\beta_{1}}{q_{1}^{2}} \\
F= & \frac{1}{2} q_{2} \dot{q}_{1}^{2}-\frac{1}{2}\left(q_{1}-\beta_{2}\right) \dot{q}_{1} \dot{q}_{2}+\frac{C_{1}}{8} \frac{4 q_{1}^{3} q_{2}+3 \beta_{1} q_{1}^{2}-4 \beta_{1} \beta_{2} q_{1}+\beta_{1} \beta_{2}^{2}}{q_{1}^{2}\left(q_{1}-\beta_{2}\right)^{2}} \\
& +\frac{C_{2}}{4} \frac{8 q_{1}^{3} q_{2}-12 \beta_{2} q_{1}^{2} q_{2}+2 \beta_{1} q_{1}^{2}-8 \beta_{2}^{2} q_{1} q_{2}-3 \beta_{1} \beta_{2} q_{1}-2 \beta_{2}^{3} q_{2}+\beta_{1} \beta_{2}^{2}}{q_{1}^{3}\left(q_{1}-\beta_{2}\right)^{2}} \\
& -\frac{C_{3}}{4} \frac{2 q_{1}-\beta_{2}}{q_{1}^{2}} \\
G= & \frac{1}{2} \dot{q}_{1}^{2}+\frac{C_{1}}{2} \frac{2 q_{1}-\beta_{2}}{\left(q_{1}-\beta_{2}\right)^{2}}+\frac{C_{2}}{2} \frac{\left(2 q_{1}-\beta_{2}\right)^{2}}{q_{1}^{2}\left(q_{1}-\beta_{2}\right)^{2}}
\end{aligned}
$$

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