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Jacobi's Last Multiplier and the Complete Symmetry Group of the Ermakov-Pinney Equation

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Abstract

The Ermakov-Pinney equation possesses three Lie point symmetries with the algebra $sl(2, R)$. This algebra does not provide a representation of the complete symmetry group of the Ermakov-Pinney equation. We show how the representation of the group can be obtained with the use of the method described in Nucci, *J. Nonlin. Math. Phys.* **12** (2005) (this issue), which is based on the properties of Jacobi's last multiplier (Bianchi L, Lezioni sulla teoria dei gruppi continui finiti di trasformazioni, Enrico Spoerri, Pisa, 1918), the method of reduction of order (Nucci,*J. Math. Phys* **37** (1996), 1772–1775) and an interactive code for calculating symmetries (Nucci, Interactive REDUCE programs for calcuating classical, non-classical and Lie-Bäcklund symmetries for differential equations (preprint: Georgia Institute of Technology, Math 062090-051, 1990, and CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends in Theoretical Developments and Computational Methods, Editor: Ibragimov N H, CRC Press, Boca Raton, 1996, 415–481).

1 Introduction

Given a (system of) differential equations(s) the desired outcome of an investigation is the demonstration of an explicit solution preferably in terms of functions analytic away from movable polelike singularities[1]. Too frequently the desired object is not obviously attainable and one must resort to indirect methods. A standard method, given a (system of) ordinary differential equation(s) of unknown integrability, is to seek a transformation

[1]A singularity is movable if its location is not determined by the ordinary differential equation itself but by specific initial conditions. One may include singularities in terms of rational powers – covered by the so-called 'weak' Painlevé Property – and any relevant remarks below may be regarded as applying to this case *mutatis mutandis*.

or series of transformations to bring one to a set of coordinates in which the system is manifestly integrable.

This immediately raises the question of the meaning of integrability. Naturally, if after a (series of) transformations(s) the system is obviously integrable, *ie* a specific solution can be demonstrated, the ultimate ambition has been achieved. In the field of differential equations the achievement of such an ambition has a set of measure zero. One can look to singularity analysis to provide the promise of integrability through the satisfaction of the requirement that a given system possess an analytic solution. In this paper we are not concerned with this aspect of integrability as our whole thrust is symmetry. Nevertheless we are well aware that symmetry and singularity are not necessarily disparate.

In this paper we are dealing with symmetry as evinced by the existence of differential operators, termed symmetries, which leave differential equations invariant under the infinitesimal transformations they generate[2]. The original idea of the symmetries as introduced by Lie concerned the infinitesimal version of a transformation relating coordinate systems to coordinate systems, *ie* the generator of the infinitesimal transformations (in the case of two variables) was

$$\Gamma = \xi(x,y)\partial_x + \eta(x,y)\partial_y. \tag{1.1}$$

This was later extended by Lie to include contact transformations generated by symmetries of the form

$$\Gamma = \xi(x,y,y')\partial_x + \eta(x,y,y')\partial_y + \zeta(x,y,y')\partial_{y'}, \tag{1.2}$$

where

$$\zeta = \frac{\partial\eta}{\partial x} + y'\left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right) - y'^2\frac{\partial\xi}{\partial y'} \tag{1.3}$$

and

$$\frac{\partial\eta}{\partial y'} = y'\frac{\partial\xi}{\partial y'}. \tag{1.4}$$

The attraction of the contact transformation was the ability to have transformations of the extended phase space to itself.

Such a constraint was not deemed necessary by Noether [24] who could allow the y' dependence in ξ and η to be general since derivatives higher than the second were not involved[3].

Towards the end of the twentieth century practical observation forced the introduction of a 'new' class of symmetries called 'hidden' symmetries on account of their origin [1, 2, 3]. These hidden symmetries arose as point symmetries on reduction or increase of order. Their origins could be contact or generalised symmetries, but often the source symmetries were nonlocal, *ie* symmetries which contain integrals the integrands of which cannot be

[2]The existence of symmetries is not limited to differential equations. Functions also possess symmetries and an area of interest comprises those functions which are integrals of differential equations.

[3]This is in the context of the invariance of the Action Integral of a Lagrangian of the form $L(x,y,y')$. Naturally an higher-order Lagrangian would permit higher-order generalised symmetries with $\xi = \xi(x,y,y',y'',\ldots)$ and $\eta = \eta(x,y,y',y'',\ldots)$.

evaluated since they depend upon the dependent variable and its derivatives. As is often the case with new ideas, nonlocal symmetries were originally associated with esoteric origins. In fact they can be found in the simplest of situations. Consider the second-order differential equation

$$y'' = 0 \tag{1.5}$$

which has a symmetry $\Gamma = \xi\partial_x + \eta\partial_y$ if

$$\Gamma^{[2]} y''_{|y''=0} = 0, \tag{1.6}$$

where

$$\Gamma^{[2]} = \xi\partial_x + \partial_y + \left(\eta' - y'\xi'\right)\partial_{y'} + \left(\eta'' - 2y''\xi' - y'\xi''\right)\partial_{y''}. \tag{1.7}$$

Obviously the action of (1.7) on (1.6) when (1.6) is taken into account gives

$$\eta'' = y'\xi'' \tag{1.8}$$

so that

$$\begin{aligned} \eta' &= A + y'\xi \\ \eta &= Ax + B + y'\xi \end{aligned} \tag{1.9}$$

and for any ξ of one's choice there is an η. The natural consequence is that (1.5) possesses an infinite number of Lie symmetries[4].

Given a (system of) ordinary differential equation(s) the determination of the Lie symmetries has importance as they can lead to the reduction of the order of the system, the determination of first integrals and the identification of more appropriate variables in which the properties of the system become more apparent.

For some equations (systems) certain symmetries are obvious by inspection, for example an ignorable coordinate has its accompanying symmetry. For others invariance under rescaling, *ie* the possession of some form of homogeneity can also be obvious. The list of obvious symmetries, invariably point, tends not to go past those mentioned. One then looks to algorithmic methods. They exist and have a variety of implementations in terms of symbolic manipulation packages[5]. Unfortunately they tend to be limited in the extent of the nature of the symmetry able to be elucidated. This does depend upon the order of the subject differential equation(s). In the case of first-order ordinary differential equations no algorithm exists. In the case of second-order equations the restriction to point symmetries is effective, often too so. For nth-order equations, $n \geq 3$, contact symmetries may be sought. Apart from these specific instances the only way to find a symmetry is to abandon the algorithmic route. One can do this by making *a priori* assumptions on the structure of the coefficient functions of the symmetry. Unfortunately such assumptions

[4]For the present we do not enter into the discussion of the utility of these symmetries. Indeed the very question of the utility [9] of a symmetry becomes part of the central theme of this paper as developed below.

[5]See the review by Hereman [10] for an assessment of the packages available at that time.

generally reflect the prejudices of the assumer rather than the properties of the equation under investigation.

One recognises in the foregoing that something of an impasse exists. Either one contents oneself with the algorithmic determination of point (contact as appropriate[6]) symmetries or makes assumptions about the possible wider variety of symmetries to be admitted. The former contentment gives a sense of completeness within limits. The second accommodation is fraught with the uncertainty of correctness of choice. Neither route is satisfactory and the existence of a further option can only be welcomed. This option is to be found in the last multiplier of Jacobi. Naturally this is not the ideal answer. Nothing in Mathematics is perfect for the explanation of reality. Nevertheless Jacobi's last multiplier combines nicely with the symmetries of Lie to enable some advance in the resolution of the properties of ordinary differential equations.

The exploitation of Jacobi's last multiplier to the purpose of finding Lie symmetries has been presented in [28] and need not be repeated here.

In this paper we wish to address the matter of the use of the last multiplier of Jacobi in the identification of the complete symmetry group of an ordinary differential equation. We recall that the complete symmetry group of an ordinary differential equation[7] is the minimal set of symmetries required to specify completely the differential equation. The concept was introduced by Krause [11, 12] in the context of the Kepler Problem and involved the introduction of nonlocal symmetries to provide a sufficient set of symmetries to specify completely the system of equations of the Kepler Problem. The question of the necessity of nonlocal symmetries to complete the specification has been a matter of debate since the work of Krause. Nucci showed [27] that Krause's nonlocal symmetries came from the point symmetries of an equivalent nonautonomous system of first-order equations in which one of the dependent variables has been taken as the new independent variable. This result was extended to include a whole class of systems possessing the distinguishing feature of the Kepler Problem, *videlicet* the Laplace-Runge-Lenz vector [30, 15].

In several papers of Andriopoulos *et al* [4, 5] the more exoteric origins of complete symmetry groups and their properties were explored.

One may ask 'How can it happen that the last multiplier of Jacobi can help in the determination of a complete symmetry group?' Firstly there exists an interchangeability between integrals and symmetries in the method of the last multiplier of Jacobi. Secondly the ability to identify integrals of equations is not in one-to-one correspondence with the representation of its complete symmetry group as we see below. In addition there is not necessarily a one-to-one correspondence between symmetries and integrals. This is a familiar feature from the results of Noether's Theorem applied to, say, the Kepler Problem. In terms of Noether's Theorem the integral associated with invariance under time-translation is the energy and yet the angular momentum and the Laplace-Runge-Lenz vectors are also autonomous. In terms of the Lie theory these vectors follow also from invariance under time-translation [13]. In [31] we have applied the known relationship between Jacobi's last multiplier, Lie symmetries and first integrals to the Kepler Problem.

In this paper we investigate a variation of a well-known equation, *videlicet* the Ermakov-Pinney equation [8, 32] in its most elementary form. The simplest Ermakov-Pinney equa-

[6]In this we are neglecting a possible foreprocessing such as one finds in the method of reduction of order [27, 29].

[7]Equally this applies to a system of ordinary differential equations.

tion is

$$\ddot{z} = \frac{1}{z^3}, \tag{1.10}$$

but we make some transformation of it via $z^2 \longrightarrow 1/x$ to give

$$2x\ddot{x} - 3\dot{x}^2 + 4x^4 = 0 \tag{1.11}$$

which, like many equations, is more complicated in its mathematically correct form than in its naturally ordained form. However, (1.11) has the advantage over (1.10) of possessing the Painlevé Property and having an analytic solution. The equation (1.10), equally (1.11) since they are related by means of a point transformation, possess the algebra $sl(2,R)$ of Lie point symmetries. This particular algebra is insufficient to specify completely the equations (1.10/1.11). Our interest in this paper is to find a suitable representation of the complete symmetry group of (1.11) and in the process demonstrate the utility of the last multiplier of Jacobi for the purpose.

2 The inadequacy of $sl(2,R)$

The Lie point symmetries of the the Ermakov-Pinney equation, (1.11), are

$$\begin{aligned}\Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t - x\partial_x \\ \Gamma_3 &= t^2\partial_t - 2tx\partial_x.\end{aligned} \tag{2.1}$$

Although the complete symmetry group of a scalar second-order ordinary differential equation is three-dimensional [4], this particular algebra is not suitable for the purpose. If we take a general scalar second-order equation, *videlicet*

$$\ddot{x} = f\left(t, x, \dot{x}\right), \tag{2.2}$$

the action of $\Gamma_1^{[2]}$, the second extension of Γ_1, is to remove the variable t from f. The second extension of Γ_2,

$$\Gamma_2^{[2]} = t\partial_t - x\partial_x - 2\dot{x}\partial_{\dot{x}} - 3\ddot{x}\partial_{\ddot{x}},$$

on the now autonomous (2.2) leads to the associated Lagrange's system for the characteristics,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\dot{x}}{2\dot{x}} = \frac{\mathrm{d}f}{3f},$$

and they are easily found to be $\dot{x}/x^2$ and f/x^3. Now (2.2) has the form

$$\ddot{x} = x^3 f_1\left(\frac{\dot{x}}{x^2}\right). \tag{2.3}$$

The second extension of Γ_3 is

$$\Gamma_3^{[2]} = t^2\partial_t - 2tx\partial_x - \left(2x + 4t\dot{x}\right)\partial_{\dot{x}} - \left(6\dot{x} + 6t\ddot{x}\right)\partial_{\ddot{x}}$$

which we may write as

$$\Gamma^{[2]}_{3\mathrm{eff}} = 2x\partial_{\dot{x}} + 6\dot{x}\partial_{\ddot{x}}$$

after taking Γ_1 and Γ_2 into account. The action of $\Gamma^{[2]}_{3\mathrm{eff}}$ on (2.3) gives

$$6\dot{x} = 2x^2 f_1', \quad ie \quad f_1 = \tfrac{3}{2}\frac{\dot{x}^2}{x^4} + \tfrac{1}{2}K, \tag{2.4}$$

where K is a constant of integration. Equation (2.3) is now

$$2x\ddot{x} - 3\dot{x}^2 + Kx^4 = 0. \tag{2.5}$$

Equation (2.5) is not precisely (1.11) since the arbitrary constant is present. A further symmetry is required to fix the value of K. That makes four symmetries and a scalar second-order ordinary differential equation needs only three symmetries for its complete specification.

Since we have used our supply of Lie point symmetries, we must of necessity look towards nonlocal symmetries to provide the necessary symmetries. The calculation of nonlocal symmetries is not easy since there does not exist a general algorithm such as does exist for calculating the Lie point symmetries of, say, second-order equations. We see below that the Jacobi's last multiplier provides us with a systematic route to find symmetries.

3 The last multipliers for the Ermakov-Pinney equation

We write the Ermakov-Pinney equation, (1.11), as a system of first-order differential equations, *videlicet*

$$\dot{w}_1 = w_2 \tag{3.1}$$

$$\dot{w}_2 = \tfrac{3}{2}\frac{w_2^2}{w_1} - 2w_1^3, \tag{3.2}$$

where we we define $w_1 = x$ and $w_2 = \dot{x}$ as the new variables, in the first step of the method of reduction of order [27, 29]. The symmetries (2.1) are now symmetries of (3.1,3.2), *ie*

$$\begin{aligned} \Lambda_1 &= \partial_t \\ \Lambda_2 &= t\partial_t - w_1\partial_{w_1} - 2w_2\partial_{w_2} \\ \Lambda_3 &= t^2\partial_t - 2tw_1\partial_{w_1} - (2w_1 + 4tw_2)\partial_{w_2}. \end{aligned} \tag{3.3}$$

We use the vector field of the two-dimensional system and two of the symmetries at a time to calculate the corresponding Jacobi's last multiplier ([17], [7]) M_{12}, M_{13} and M_{23}. The subscripts refer to the symmetries used.

$$C_{12} = \begin{bmatrix} 1 & w_2 & \tfrac{3}{2}\dfrac{w_2^2}{w_1} - 2w_1^3 \\ 1 & 0 & 0 \\ t & -w_1 & -2w_2 \end{bmatrix}$$

$$\begin{aligned}\frac{1}{M_{12}} &= \det C_{12}\\ &= \tfrac{1}{2}\left(w_2^2+4w_1^4\right),\end{aligned} \tag{3.4}$$

$$C_{13}=\begin{bmatrix} 1 & w_2 & \frac{3}{2}\dfrac{w_2^2}{w_1}-2w_1^3\\ 1 & 0 & 0\\ t^2 & -2tw_1 & -(2w_1+4tw_2)\end{bmatrix}$$

$$\begin{aligned}\frac{1}{M_{13}} &= \det C_{13}\\ &= t\left(w_2^2+4w_1^4\right)+2w_1w_2 \quad\text{and}\end{aligned} \tag{3.5}$$

$$C_{23}=\begin{bmatrix} 1 & w_2 & \frac{3}{2}\dfrac{w_2^2}{w_1}-2w_1^3\\ t & -w_1 & -2w_2\\ t^2 & -2tw_1 & -(2w_1+4tw_2)\end{bmatrix}$$

$$\begin{aligned}\frac{1}{M_{23}} &= \det C_{23}\\ &= \tfrac{1}{2}t^2\left(w_2^2+4w_1^4\right)+2tw_1w_2+2w_1^2.\end{aligned} \tag{3.6}$$

The first integrals are obtained by taking the quotients of the Jacobi's last multipliers ([17],[7]). Thus we have

$$\begin{aligned}I_{13} &= \frac{M_{12}}{M_{23}}\\ &= t^2+4\frac{tw_1w_2+w_1^2}{w_2^2+4w_1^4},\end{aligned} \tag{3.7}$$

$$\begin{aligned}I_{23} &= \frac{M_{12}}{M_{13}}\\ &= 2t+4\frac{w_1w_2}{w_2^2+4w_1^4} \quad\text{and}\end{aligned} \tag{3.8}$$

$$\begin{aligned}I_{12} &= \frac{M_{13}}{M_{23}}\\ &= \tfrac{1}{2}\frac{t^2\left(w_2^2+4w_1^4\right)+4tw_1w_2+2w_1^2}{t\left(w_2^2+4w_1^4\right)+2w_1w_2}.\end{aligned} \tag{3.9}$$

We note that

$$\tfrac{1}{4}I_{23}^2 = I_{13}-\frac{16}{J^2},$$

where

$$J=\frac{w_2^2}{w_1^3}+4w_1=\frac{\dot{x}^2}{x^3}+4x \tag{3.10}$$

is an autonomous integral[8]. The explicitly time-dependent integral can be taken as

$$I=t+\frac{2w_1w_2}{w_2^2+4w_1^4}=t+\frac{2x\dot{x}}{\dot{x}^2+4x^4}. \tag{3.11}$$

[8]This integral may be obtained directly from the second-order equation, (1.11), by use of the integrating factor $\dot{x}/x^4$.

The two integrals, (3.10) and (3.11), provide the solution

$$x = \frac{4J}{16 + J^2(I-t)^2}. \tag{3.12}$$

4 Symmetries from the multipliers

Given the multipliers and the autonomy of the equation one can determine the symmetries from

$$\det \begin{bmatrix} w_2 & \dfrac{3w_2^2}{2w_1} - 2w_1^3 \\ \eta & \zeta \end{bmatrix} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_2^2 + 4w_1^4\right) \\ t\left(w_2^2 + 4w_1^4\right) + 2w_1w_2 \\ \frac{1}{2}t^2\left(w_2^2 + 4w_1^4\right) + 2tw_1w_2 + 2w_1^2 \end{array} \right. , \tag{4.1}$$

where we assume that the symmetry is $\Gamma = \xi\partial_t + \eta\partial_{w_1} + \zeta\partial_{w_2}$. The absence of ξ from (4.1) is covered by the relationship $\zeta = \dot{\eta} - w_2\dot{\xi}$.

Consider the case of the first multiplier, M_{12}. Equation (4.1) gives

$$\zeta w_2 - \eta\left(\frac{3w_2^2}{2w_1} - 2w_1^3\right) = \frac{1}{2}\left(w_2^2 + 4w_1^4\right) \tag{4.2}$$

which, of course, gives an infinite number of symmetries as

$$\begin{aligned} \eta = \exp\left[\int\left(\frac{3w_2}{2w_1} - \frac{2w_1^3}{w_2}\right)\mathrm{d}t\right]&\left\{A + \int\left[\frac{1}{2}\left(w_2 + \frac{4w_1^4}{w_2}\right) + w_2\dot{\xi}\right]\right. \\ &\left.\times \exp\left[-\int\left(\frac{3w_2}{2w_1} - \frac{2w_1^3}{w_2}\right)\mathrm{d}t\right]\right\}. \end{aligned} \tag{4.3}$$

Similar expressions can be obtained using the two other multipliers.

Alternatively one may look in (4.2) and the two related equations for point symmetries (in terms of the $x = w_1$ variable). The symmetries, not surprisingly, obtained are just the three symmetries already listed in (2.1/3.3).

One may wonder whether the nonlocal symmetries are of any use. It happens that there is no way to determine the invariants of the symmetry using the form in (4.3) (and its two companions) even if one puts $\xi = 0$ and $A = 0$. Alternatively one can put $\eta = 0$ and $A = 0$ to obtain an expression for ξ. In the case of the first multiplier, *ie* using (4.3), we find that

$$\xi_1 = -t - \frac{w_1}{w_2}.$$

However, when we apply the symmetry, we find that it is just the same as the effect of Λ_1 of (3.3). The same is true of the symmetries corresponding to the second and third multipliers.

Although in principle Jacobi's last multiplier can lead us to new symmetries in the manner of solving (4.1), there has been no success.

5 Implementation of the method of reduction of order

The method of reduction of order [27] consists of two parts. The system of differential equations is written in terms of first-order equations and ignorable variables [29] may be eliminated to reduce the order of the system. To enable an algorithmic calculation of symmetries the reduced system of the first-order equations is rewritten to include at least one second-order equation. There is always the question of an appropriate choice of variables. As it happens, the last multiplier of Jacobi provides a method to identify the useful variables [28]. We firstly use Strategy 3 and then use Strategy 2.

The Jacobi last multiplier of the system (3.1) and (3.2) may formally be written as

$$JLM = \exp\left[-\int\left(\frac{\partial\dot{w}_1}{\partial w_1}+\frac{\partial\dot{w}_2}{\partial w_2}\right)\mathrm{d}t\right] = \exp\left[-\int\frac{3w_2}{w_1}\mathrm{d}t\right]$$

and the integral may be formally evaluated by the introduction of a new variable[9]

$$\dot{z} = \frac{w_2}{w_1}. \tag{5.1}$$

The very definition of z introduces a second-order equation. This is Strategy 3 [28].

In the analysis of the system to determine the Lie symmetries Nucci's interactive Reduce program for calculating Lie symmetries [25, 26] produces a parabolic differential equation the characteristic of which is a first integral of the system [19]. This provides a new variable satisfying a trivial first-order equation. The variable is

$$u_1 = w_1 \exp\left[-\frac{w_2}{w_1}\right]. \tag{5.2}$$

The system (3.1) and (3.2) is now

$$\dot{u}_1 = 0 \tag{5.3}$$

$$\ddot{z} - \tfrac{1}{2}\dot{z}^2 + 2u_1^2\mathrm{e}^{2z} = 0. \tag{5.4}$$

Equation (5.4) is rendered more transparent by the transformation $u_2 = \exp[-2z]$ and becomes

$$\ddot{u}_2 + \frac{u_1^2}{u_2^3} = 0. \tag{5.5}$$

The system (5.3) and (5.5) possesses the four Lie point symmetries

$$\begin{aligned}\Gamma_1 &= \partial_t\\ \Gamma_2 &= t\partial_t - u_1\partial_{u_1}\\ \Gamma_3 &= t^2\partial_t + tu_2\partial_{u_2}\\ \Gamma_4 &= 2u_1\partial_{u_1} + u_2\partial_{u_2}\end{aligned} \tag{5.6}$$

with the algebra $A_1 \oplus_s A_{3,8}$ in the Mubarakzyanov classification scheme [20, 21, 22, 23]. As the latter subalgebra is $sl(2,R)$, we are in the same situation that we encountered in

[9] Note that it is not necessary to include the constant 3 in the definition of the new variable.

§2. Although we have the correct number of symmetries for the complete specification of the system (5.3) and (5.5), the algebra is not correct.

Since the system (3.1) and (3.2) is autonomous, we may replace it by the single first-order equation

$$\frac{\mathrm{d}w_2}{\mathrm{d}w_1} = \frac{3w_2^2 - 4w_1^4}{2w_1 w_2}. \tag{5.7}$$

The last multiplier for (5.7) is given by

$$JLM = \exp\left[-\int \frac{\partial}{\partial w_2}\left(\frac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)\mathrm{d}w_1\right] = \exp\left[-\int \frac{3w_2^2 + 4w_1^4}{2w_1 w_2^2}\mathrm{d}w_1\right].$$

We write the independent variable w_1 as y and introduce a new independent variable through $\mathrm{d}u/\mathrm{d}y = \left(3w_2^2 + 4y^4\right) / \left(2yw_2^2\right)$ so that (5.7) becomes the second-order equation

$$y^2\left(\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - 2\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right) + 6y\frac{\mathrm{d}u}{\mathrm{d}y} - 3 = 0. \tag{5.8}$$

A further simplification is achieved by the change of variables $y = \exp[\rho]$ and $s(\rho) = \exp[-2u]$. Then (5.8) becomes

$$s'' + 5s' + 6s = 0, \tag{5.9}$$

where prime indicates differentiation with respect to the new independent variable, ρ.

Equation (5.9) is a linear second-order equation and possesses eight Lie point symmetries. They are

$$\begin{aligned}
\Gamma_1 &= \exp[-2\rho]\,\partial_s \\
\Gamma_2 &= \exp[-3\rho]\,\partial_s \\
\Gamma_3 &= s\partial_s \\
\Gamma_4 &= \partial_\rho \\
\Gamma_5 &= \exp[-\rho]\,(\partial_\rho - 3s\partial_s) \\
\Gamma_6 &= \exp[\rho]\,(\partial_\rho - 2s\partial_s) \\
\Gamma_7 &= \exp[2\rho]\,\left(s\partial_\rho - 3s^2\partial_s\right) \\
\Gamma_8 &= \exp[3\rho]\,\left(s\partial_\rho - 2s^2\partial_s\right)
\end{aligned} \tag{5.10}$$

which is a representation of the algebra $sl(3, R)$.

The symmetries in (5.10) were listed in one of the standard orderings for the elements of $sl(3, R)$. The first two symmetries come from the solutions of (5.9), Γ_3 is the homogeneity symmetry, the next three are the elements of the subalgebra $sl(2, R)$ and the last two the nonfibre-preserving symmetries.

When we express these in terms of the original variables, we find that

$$
\begin{aligned}
\Gamma_1 &\quad\longrightarrow\quad \Lambda_1 = \exp\left(4\int \frac{y^3}{w_2^2}\,\mathrm{d}y\right)\frac{w_2}{4y^3}\,(w_2^2+4\,y^4)\,\partial_{w_2}\\
\Gamma_2 &\quad\longrightarrow\quad \Lambda_2 = \exp\left(4\int \frac{y^3}{\mathrm{w_2}^2}\,\mathrm{d}y\right) w_2\,\partial_{w_2}\\
\Gamma_3 &\quad\longrightarrow\quad \text{zero in terms of the new coordinates}\\
\Gamma_4 &\quad\longrightarrow\quad \Lambda_4 = y\,\partial_y + 2\,w_2\,\partial_{w_2}\\
\Gamma_5 &\quad\longrightarrow\quad \Lambda_5 = \partial_y + \frac{3\,w_2}{2y}\,\partial_{w_2}\\
\Gamma_6 &\quad\longrightarrow\quad \Lambda_6 = y^2\,\partial_y + \frac{w_2}{8y^3}(w_2^2+20\,y^4)\,\partial_{w_2}\\
\Gamma_7 &\quad\longrightarrow\quad \Lambda_7 = \exp\left(-4\int \frac{y^3}{w_2^2}\,\mathrm{d}y\right)\left[\partial_y + \frac{1}{2y\,w_2}\,\left(3\,w_2^2-4\,y^4\right)\,\partial_{w_2}\right]\\
\Gamma_8 &\quad\longrightarrow\quad \Lambda_8 = \exp\left(-4\int \frac{y^3}{w_2^2}\,\mathrm{d}y\right)\left[y\partial_y + \frac{1}{2\,w_2}\,\left(3\,w_2^2-4\,y^4\right)\,\partial_{w_2}\right],
\end{aligned}
\tag{5.11}
$$

where we see that Λ_1, Λ_2, Λ_7 and Λ_8 become exponential nonlocal due to the presence of the factor $\exp\left[\pm 4\int\left(y^3/w_2^2\right)\,\mathrm{d}y\right]$.

We note that only the elements of $sl(2,R)$ are inherited by (5.7) as point symmetries. The normal representation of the complete symmetry group of (5.9) would be in terms of Γ_i, $i = 1, 3$, [4]. The fact that the usual symmetries become nonlocal symmetries of the first-order equation is doubtless the explanation for the problems we have had in determining the complete symmetry group. However, we may use the transitivity property of complete symmetry groups [6] to establish a representation of the group. Given a subalgebra of three symmetries of (5.9) and a reduction of order, those symmetries are symmetries of the reduced equation. If the reduction of order is based upon one of the symmetries of the subalgebra, the remaining two symmetries constitute the complete symmetry group of the reduced equation. There are three representations of the complete symmetry group of a linear second-order equation [5, 14]. Since the symmetry, Γ_3, is the symmetry of the reduction of order, it must be an element of a representation of the complete symmetry group of (5.9) for us to obtain the desired result. There are two such representations, *videlicet* $\{\Gamma_1,\Gamma_2,\Gamma_3\}$ and $\{\Gamma_7,\Gamma_8,\Gamma_3\}$. One finds it more than a little ironic that the relevant symmetries are the nonlocal symmetries of the reduced equation[10].

We verify that Γ_1 and Γ_2 indeed completely specify (5.7) by demonstrating that the actions of $\Lambda_1^{[1]}$ and $\Lambda_2^{[1]}$ on the general first-order equation

$$
\frac{\mathrm{d}w_2}{\mathrm{d}y} = f(y,w_2) \tag{5.12}
$$

[10]The problem with the $sl(2,R)$ subalgebra becomes more obvious since it does not include Γ_3 and so the reduction is intransitive [6].

explicitly lead to (5.7). The demonstration for $\Lambda_7^{[1]}$ and $\Lambda_8^{[1]}$ is similar and the calculations are not repeated here. The first extension of Λ_2 is

$$\Lambda_2^{[1]} = \Lambda_2 + \exp\left[4\int \frac{y^3}{\mathrm{w_2}^2}\,\mathrm{d}y\right]\left\{w_2\,\partial_{w_2} + \left(w_2' + \frac{4y^3}{w_2}\right)\partial_{w_2'}\right\} \tag{5.13}$$

and its action on (5.12) leads to the first-order partial differential equation

$$f + \frac{4y^3}{w_2} = w_2\frac{\partial f}{\partial w_2} \tag{5.14}$$

which is easily solved to give

$$f = w_2 A(y) - \frac{2y^3}{w_2}, \tag{5.15}$$

where $A(y)$ is an arbitrary function of its argument, so that now (5.12) takes the form

$$\frac{\mathrm{d}w_2}{\mathrm{d}y} = w_2 A(y) - \frac{2y^3}{w_2}. \tag{5.16}$$

In the case of Λ_1 the the first extension is

$$\Lambda_1^{[1]} = \Lambda_1 + \exp\left[4\int \frac{y^3}{\mathrm{w_2}^2}\,\mathrm{d}y\right]\left\{2w_2 + 4\frac{y^4}{w_2} - \frac{3w_2^3}{4y^4} + \left(\frac{3w_2^2}{4y^3} + y\right)w_2'\right\}\partial_{w_2'}. \tag{5.17}$$

The action of $\Lambda_1^{[1]}$ on (5.16) produces an algebraic equation for the function $A(y)$. We obtain

$$A(y) = \frac{3}{2y} \tag{5.18}$$

and equation (5.7) is recovered.

The same result is found if Λ_7 and Λ_8 are used.

In both cases the algebra is abelian.

The single first-order differential equation, (5.7), was obtained from the system of two first-order differential equations, (3.1) and (3.2), by means of eliminating the time which is an ignorable coordinate in the system. This corresponds to the symmetry $\Delta_3 = \partial_t$. Consequently we may invoke Theorem 1 of [6] and conjoin this symmetry to Λ_1 and Λ_2 (equally to Λ_7 and Λ_8) to obtain a representation of the complete symmetry group of the system. Note that we have demonstrated the existence of two representations. The variables in Λ_1 and Λ_2 must be adjusted for the presence of the new variable t. The representation of the complete symmetry group of the system (3.1) and (3.2) is given by

$$\begin{aligned}\Delta_1 &= \left\{\int\left(\frac{w_2^2}{4w_1^3} + w_1\right)\exp\left[4\int\frac{w_1^3}{w_2}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_t \\ &\quad - \left\{\left(\frac{w_2^3}{4w_1^3} + w_1w_2\right)\exp\left[4\int\frac{w_1^3}{w_2}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_{w_2} \\ \Delta_2 &= \left\{\int\exp\left[4\int\frac{w_1^3}{w_2}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_t - \left\{w_2\exp\left[4\int\frac{w_1^3}{w_2}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_{w_2}\end{aligned} \tag{5.19}$$

$$\Delta_3 = \partial_t.$$

Trivially the complete symmetry group of the Ermakov-Pinney equation is now

$$\begin{aligned}
\Sigma_1 &= \left\{\int\left(\frac{\dot{x}^2}{4x^3}+x\right)\exp\left[4\int\frac{x^3}{\dot{x}}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_t \\
\Sigma_2 &= \left\{\int\exp\left[4\int\frac{x^3}{\dot{x}}\mathrm{d}t\right]\mathrm{d}t\right\}\partial_t \\
\Sigma_3 &= \partial_t.
\end{aligned} \tag{5.20}$$

We note that this representation of the complete symmetry group is expressed in terms of generators of translations in time. Two of the elements of the algebra are quite nonlocal. The appearance of the symmetries can be rendered simpler by the introduction of a nonlocal variable, *videlicet*

$$W = \int\exp\left[4\int\frac{x^3}{\dot{x}}\mathrm{d}t\right]\mathrm{d}t \tag{5.21}$$

for then the symmetries listed in (5.20) become

$$\begin{aligned}
\Sigma_1 &= IW\partial_t \\
\Sigma_2 &= W\partial_t \\
\Sigma_3 &= \partial_t,
\end{aligned} \tag{5.22}$$

where $I = x + \dot{x}^2/(4x^3)$ is the first integral of (1.11) noted above. This type of structure has been previously observed in the case of contact symmetries of second-order equations [18].

6 Conclusion

In this paper we have determined representations of the complete symmetry group of the Ermakov-Pinney equation. This proved to be a nontrivial task. We employed the properties of the last multiplier of Jacobi in a variety of ways. In particular we used the Jacobi last multiplier of the reduced system to find a suitable new dependent variable for the reduced system, *ie* Strategy 2 in [28]. In the process of determining the symmetries we also made use of the ability of the code of Nucci to provide a first integral [19] so that the second equation of the system became a trivial first-order equation. Fortunately the culmination of the method of reduction of order was a linear second-order equation for which there are eight Lie point symmetries. One of those symmetries was lost in the reduction of the second-order equation to the first-order equation and three of them constituted the $sl(2,R)$ subalgebra which, although it is characteristic of equations of maximal symmetry, is very unsuitable for the construction of a representation of a complete symmetry group of a differential equation. Fortunately the remaining symmetries had the correct properties and so we were led to two representations of the complete symmetry group. They are equivalent.

In the case of a (system of) differential equation(s) the determination of a representation of the complete symmetry group becomes quite difficult if the number of point (contact for equations of order greater than two) symmetries is small for then one has to look for nonlocal symmetries and generally their determination is difficult. The methodology we have adapted in this paper has enabled us to determine these nonlocal symmetries in a manner which is quite algorithmic by means of a combination of several tools, Jacobi's last multiplier, the method of reduction of order and Nucci's interactive code.

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