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Compactly Supported Solutions of the Camassa-Holm Equation

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Abstract

We give a simple proof that for any non-zero initial data, the solution of the Camassa-Holm equation loses instantly the property of being compactly supported.

1 Introduction

In this paper, we consider the *Camassa-Holm* equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad t \geq 0,\ x \in \mathbb{R}. \tag{1.1}$$

This equation is a model for the unidirectional propagation of shallow water waves, with $u(t,x)$ representing the water's free surface above a flat bed in nondimensional variables *cf.* [3] (see also [18] for an alternative derivation). It also models axially symmetric waves in hyperelastic rods [15] and was first derived as an abstract bi-Hamiltonian equation [16]. Moreover, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group of the line [6, 20]. Some solutions exist for all times, while others have a finite life-span, modelling wave breaking [8]. The solitary waves of the Camassa-Holm equation are stable solitons [2, 14] with a peak at their crest (and thus have to be interpreted as weak solutions cf. [13]. Equation (1.1) is an integrable infinite-dimensional Hamiltonian system (see [5, 7, 12, 19]). There is in fact a whole hierarchy of integrable equations associated to the Camassa-Holm equation, cf. [17].

In this paper we will show that the classical solutions of the Camassa-Holm equation have infinite propagation speed: the only initial data $u(0,\cdot)$ of compact support for which the solution stays compactly supported for some time $T > 0$ is $u(0,x) \equiv 0$. In so doing we refine the results obtained in [4], by lessening the restriction that the classical solution u of (1.1) must be smooth. Our approach is also simpler than the one pursued in [4].

2 Main Results

Assume that $u_0 \in \mathbb{H}^4(\mathbb{R})$. Then [9] there is a maximal time $T = T(u_0) > 0$ such that (1.1) has a unique solution with

$$u \in C([0,T), \mathbb{H}^4(\mathbb{R})) \cap C^1([0,T), \mathbb{H}^3(\mathbb{R})) \cap C^2([0,T), \mathbb{H}^2(\mathbb{R})) \tag{2.1}$$

In view of the Sobolev embedding $\mathbb{H}^{k+1}(\mathbb{R}) \subset C^k(\mathbb{R})$ for $k \geq 0$, we have, in particular, that $u \in C^2([0,T) \times \mathbb{R}, \mathbb{R})$.

We begin by proving the following result for the function $m = u - u_{xx}$. It guarantees that if a classical solution u of (1.1) starts out having compact support, then this property will be inherited by m at all times $t \in [0,T)$.

Proposition 1. *Assume that $u_0 \in \mathbb{H}^4(\mathbb{R})$ is such that $m_0 = u_0 - u_{0,xx}$ has compact support. If $T = T(u_0) > 0$ is the maximal existence time of the unique solution $u(x,t)$ to (1.1) with initial data $u_0(x)$, then for any $t \in [0,T)$ the C^1 function $x \mapsto m(x,t)$ has compact support.*

Proof. Let us associate to the function m the family $\{\varphi(\cdot,t)\}_{t\in[0,T)}$ of increasing C^2 diffeomorphisms of the line defined by

$$\varphi_t(x,t) = u(\varphi(x,t),t), \qquad t \in [0,T), \tag{2.2}$$

with

$$\varphi(x,0) = x, \qquad x \in \mathbb{R}. \tag{2.3}$$

The claimed smoothness of the functions φ follows from classical results on the dependence on parameters of solutions of differential equations [1].

Using (1.1) and (2.2)–(2.3), and by differentiating with respect to t, one can easily check the following identity:

$$m(\varphi(x,t),t) \cdot \varphi_x^2(x,t) = m(x,0), \qquad x \in \mathbb{R}, t \in [0,T). \tag{2.4}$$

Additionally, from (2.2)–(2.3) we infer

$$\varphi_x(x,t) = \exp\left(\int_0^t u_x(\varphi(x,s),s)\,ds\right), \qquad x \in \mathbb{R}, t \in [0,T). \tag{2.5}$$

It follows that if m_0 is supported in the compact interval $[a,b]$, then since $\varphi_x(x,t) > 0$ on $\mathbb{R} \times [0,T)$ from (2.5), we can conclude from (2.4) that $m(x,t)$ has its support in the interval $[\varphi(a,t), \varphi(b,t)]$.

$\blacksquare$

Remark 1. Relation (2.4) is not accidental: it represents the conservation of momentum in the physical variables (see [10, 11]).

We next show that although $m = u - u_{xx}$ has compact support for all $t \in [0,T)$ if m_0 does, where T is the maximal existence time of the solution, this property *does not* carry over to the function u.

Theorem 1. *Assume that the function $u_0 \in \mathbb{H}^4(\mathbb{R})$ has compact support. Let $T > 0$ be the maximal existence time of the unique solution $u(x,t)$ with initial data $u_0(x)$. If at every $t \in [0,T)$ the C^2 function $x \mapsto u(x,t)$ has compact support, then u is identically zero.*

For $(x,t) \in \mathbb{R} \times [0,T)$, let $m = u - u_{xx}$. Clearly $m(x,0)$ has compact support since u_0 does. By Proposition 1 the C^1 function $x \mapsto m(x,t)$ has compact support for all $t \in [0,T)$. Given u, it is easy to find m. The reverse is also true by means of taking Fourier transforms [21], namely

$$2u(x) = e^{-x} \int_{-\infty}^{x} e^{y} m(y)\, dy + e^{x} \int_{x}^{\infty} e^{-y} m(y)\, dy. \tag{2.6}$$

In order to prove Theorem 1 we will use the following result.

Proposition 2. *Let $u \in C^2(\mathbb{R}) \cap \mathbb{H}^2(\mathbb{R})$ be such that $m = u - u_{xx}$ has compact support. Then u has compact support if and only if*

$$\int_{\mathbb{R}} e^{x} m(x)\, dx = \int_{\mathbb{R}} e^{-x} m(x)\, dx = 0. \tag{2.7}$$

Proof. Assume u has compact support and let $0 < N_m \in \mathbb{R}$ be such that $m(x) \equiv 0$ for all $|x| > N_m$. We now examine $u(x)$ when $|x| > N_m$— firstly when $x > 0$, then for $x < 0$.

Case 1: $x > N_m$. We write (2.6) as the sum of three integrals, denoted $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$:

$$2u(x) = e^{-x} \int_{-\infty}^{-x} e^{y} m(y)\, dy + \ e^{-x} \int_{-x}^{x} e^{y} m(y)\, dy + \ e^{x} \int_{x}^{\infty} e^{-y} m(y)\, dy.$$

Note the integrals $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ are well-defined as m has compact support. It is obvious that $\mathcal{I}_1 \equiv \mathcal{I}_3 \equiv 0$ since $m(y) = 0,\ |y| > N_m$, and so we observe that $u(x) = 0$ with $x > N_m$ if and only if $e^{-x} \int_{-x}^{x} e^{y} m(y)\, dy = 0$, that is, if and only if $\int_{-x}^{x} e^{y} m(y)\, dy = 0$.

Therefore, since $u(x)$ had compact support, we can infer that

$$\lim_{x \to \infty} \int_{-x}^{x} e^{y} m(y)\, dy = \int_{\mathbb{R}} e^{y} m(y)\, dy = 0.$$

Case 2: $-x > N_m$. We now decompose (2.6) in terms of the integrals I_1, I_2, I_3:

$$2u(x) = e^{-x} \int_{-\infty}^{x} e^{y} m(y)\, dy + \ e^{x} \int_{x}^{-x} e^{-y} m(y)\, dy + \ e^{x} \int_{-x}^{\infty} e^{-y} m(y)\, dy.$$

As before, the integrals $I_1, I_2,$ and I_3 are well-defined because m has compact support, and we also have $I_1 \equiv I_3 \equiv 0$. We again infer that $\int_{\mathbb{R}} e^{-y} m(y)\, dy = 0$.

Therefore, if $u(x)$ has compact support then (2.7) holds.

We now prove the converse. It is given that m has compact support, which means that there is a constant $N > 0$ such that $m(x) = 0$ for all $|x| > N$. Assume that (2.7) holds. Therefore

$$\int_{\mathbb{R}} e^{x} m(x)\, dx = \int_{-N}^{N} e^{x} m(x)\, dx = 0 \tag{A}$$

$$\int_{\mathbb{R}} e^{-x} m(x) dx = \int_{-N}^{N} e^{-x} m(x)\, dx = 0. \tag{B}$$

Pick $x > N$. We can now write equation (2.6) as

$$2u(x) = e^{-x} \int_{-N}^{N} e^{y} m(y)\, dy + e^{x} \int_{x}^{\infty} e^{-y} m(y)\, dy = 0,$$

if we take into account (A) and the fact that $m(y) = 0$ for all $y \geq x > N$.

Similarly, for $x < -N$, equation (2.6) becomes

$$2u(x) = e^{-x} \int_{-\infty}^{x} e^{y} m(y)\, dy + e^{x} \int_{-N}^{N} e^{-y} m(y)\, dy = 0.$$

So u has compact support. ■

Proof of Theorem 1. We now prove the theorem using the result that u has compact support if and only if (2.7) holds. We assume that u has compact support, and show that this implies $u \equiv 0$.

Let us now write (1.1) in the form

$$m_t + 2u_x m + u m_x = 0$$

and differentiate the left hand side of (2.7) with respect to t to get the result:

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} e^{x} m(x,t)\, dx &= \int_{\mathbb{R}} e^{x} m_t\, dx = -2 \int_{\mathbb{R}} e^{x} m u_x\, dx - \int_{\mathbb{R}} e^{x} m_x u\, dx \\
&= -2 \int_{\mathbb{R}} e^{x} m u_x\, dx + \int_{\mathbb{R}} e^{x} m u_x\, dx + \int_{\mathbb{R}} e^{x} m u\, dx \\
&= - \int_{\mathbb{R}} e^{x} m u_x\, dx + \int_{\mathbb{R}} e^{x} m u\, dx \\
&= - \int_{\mathbb{R}} u u_x\, dx + \int_{\mathbb{R}} u u_{xx}\, dx + \int_{\mathbb{R}} e^{x} u^2\, dx - \int_{\mathbb{R}} e^{x} u u_{xx}\, dx \\
&= - \int_{\mathbb{R}} e^{x} u u_x\, dx - \frac{1}{2} \int_{\mathbb{R}} e^{x} u_x^2\, dx + \int_{\mathbb{R}} e^{x} u^2\, dx + \int_{\mathbb{R}} e^{x} u_x (u + u_x)\, dx \\
&= \int_{\mathbb{R}} e^{x} u^2\, dx + \frac{1}{2} \int_{\mathbb{R}} e^{x} u_x^2\, dx
\end{aligned}$$

where all boundary terms after integration by parts vanish as both $m(\cdot, t)$ and, by assumption, $u(\cdot, t)$ have compact support for all $t \in [0, T)$. Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}} e^{x} m(x,t) = \int_{\mathbb{R}} e^{x} \left(u^2 + \frac{1}{2} u_x^2 \right) dx, \quad t \in [0, T). \tag{2.8}$$

The expression under the integral on the right hand side of this relation must be identically zero by (2.7). This implies that both of the terms must be identically zero, and in particular $u \equiv 0$. This completes the proof. ■

Remark 2. If $u_0 \not\equiv 0$ is a function in $\mathbb{H}^4(\mathbb{R})$ with compact support, then the classical solution $u(\cdot, t)$ of (1.1) looses instantly the property of having compact support. To see this we go through the same argument as above, this time restricting our attention to an arbitrarily small time interval $[0, \varepsilon)$.

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