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Equations Of Long Waves With A Free Surface III. The Multidimensional Case

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Abstract

Long-wave equations for an incompressible inviscid free-surface fluid in $N+1$ dimensions are derived and shown to be Hamiltonian and liftable into the space of moments.

1 Introduction

Hierarchies of integrable evolution equations are concentrated in 0 (discrete) and 1 (continuous) space dimensions. The only known integrable 2-dimensional hierarchies, first introduced in [5,6] by Manin and myself, are of free-surface type; the prototypical such system was discovered by Benney in 1973 [1]:

$$u_t = uu_x + gh_x - u_y \int_0^y u_x dy, \tag{1.1a}$$

$$h_t = (\int_0^h udy)_x. \tag{1.1b}$$

Here $-\infty < x < \infty$; t is the time variable; $u = u(x, y, t)$ is the horizontal component of velocity of an inviscous incompressible fluid; $0 \leq y \leq h$; $h = h(x, t)$ is the height of the free surface over the bottom $\{y = 0\}$; subscripts t, x, and y denote partial derivatives; the density of the fluid is taken to be 1; the gravitational acceleration g in formula (1.1a) is also taken to be 1 most of the time; and the mathematical time t in formulae (1.1) is opposite in sign to the physical time t, to make forthcoming formulae simpler.

Benny in [1] found two remarkable facts about the system (1.1):

(A) If one introduces the *moments* of the velocity $u(x, y, t)$:

$$A_n(x,t) = \int_0^h u^n(x,y,t)dy, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.2}$$

then the *integro-differential* system (1.1) implies a purely *differential* evolution system in the space of moments A_n's:

$$A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}; \tag{1.3}$$

(B) The system (1.3) has an infinite number of conserved densities $H_n \in A_n + \mathbf{Z}[g; A_0, ..., A_{n-2}]$:

$$H_0 = A_0, \quad H_1 = A_1, \; H_2 = A_2 + gA_0^2, ... \tag{1.4}$$

Both of these facts can be generalized considerably. In this note I shall re-examine the *nature* of the Benney system (1.1) by deriving an $(N+1)$-dimensional version of it for the case when the external potential is arbitrary and not just gravitational. We shall see that the resulting $(N+1)$-dimensional free-surface system, *integro-differential* as expected, again implies a purely *differential* evolution system in the space of moments.

2 Incompressible Fluids With A Free Surface

We start off the Euler equations for an incompressible inviscid fluid in $N+1$ dimensions. Denote the space coordinates by $(x_\alpha) = (x_i; y), \; 1 \leq \alpha \leq N+1, \; 1 \leq i \leq N, \; y = x_{n+1}$, and set

$$\partial_\alpha = \partial/\partial x_\alpha, \quad (\cdot)_{,\alpha} = \partial_\alpha(\cdot), \quad \partial_i = \partial/\partial x_i, \quad (\cdot)_{,i} = \partial_i(\cdot). \tag{2.1}$$

The Euler equations are:

$$u_{\alpha,t} - u_\beta u_{\alpha,\beta} = (P - U), \tag{2.2$_\alpha$}$$
$$u_{\alpha,\alpha} = 0 \; , \tag{2.3}$$
$$h_t = (\int_0^h u_i dy)_{,i} \; , \tag{2.4}$$
$$u_{N+1} \mid_{y=0} = 0 \; , \tag{2.5}$$
$$P \mid_{y=h} = P_0 = const \; . \tag{2.6}$$

Here $\boldsymbol{u} = (u_\alpha) = (u_1, ..., u_{N+1})$ is the velocity vector, $P = P(x_1, ..., x_{N+1}; t)$ is the pressure, $U = U(x_1, ..., x_{N+1})$ is the potential, $h = h(x_1, ..., x_N; t)$ is the height of the free surface over the horizontal (for inessential simplicity) bottom $\{y = 0\}$; and we sum on the repeated indices.

The system (2.2-6) is clearly *non-local*, having the non-holonomic incompressibility constrain (2.3) imposed upon it, and with no separate equation for the time evolution of the unknown pressure function P given. It is only after one makes a "long-wave approximation" to this not-evolutional system that one ends up with a genuine evolution system like (1.1).

3 Long-wave Approximation

Pick arbitrary non-zero constants $\lambda_1, ..., \lambda_{N+1}$, and generalize the system (2.3-6) by keeping equations (2.3-6) unchanged and replacing equation (2.2_α) by the equation

$$\lambda_\alpha(u_{\alpha,t} - u_\beta u_{\alpha,\beta}) = (P - U)_{,\alpha}\,, \qquad \text{no sum on } \alpha. \tag{3.1_α}$$

Our original system (2.2-6) results when

$$\lambda_1 = ... = \lambda_N = \lambda_{N+1} = 1. \tag{3.2}$$

Now set

$$E = E(\boldsymbol{\lambda}) = \int_0^h dy(\frac{\lambda_\alpha}{2} u_\alpha^2 - U). \tag{3.3}$$

This is an analog of the energy density for our extended system {(3.1), (2.2-6)}, because
Proposition 3.4.

$$E_{,t} = \left\{ \int_0^h dy(-U - P_0 + P + \frac{\lambda_\beta}{2} u_\beta^2) u_i \right\}_{,i}. \tag{3.5}$$

Proof. We have:

$$E_{,t} = \frac{\lambda_\alpha}{2} u_\alpha^2|_h h_t - U|_h h_t + \tag{3.6a}$$
$$+ \int_0^h dy \lambda_\beta u_\beta \left\{ \lambda_\beta^{-1}(P - U)_{,\beta} + u_\alpha u_{\beta,\alpha} \right\}. \tag{3.6b}$$

Let us transform separately each of the two summands in the expression (3.6b).

$$\begin{aligned}
1) \quad & \int_0^h dy u_\beta (P - U)_{,\beta} \ \text{[by (2.3)]} = \int_0^h dy \left\{ u_\beta(P - U) \right\}_{,\beta} = \\
&= \left\{ u_{N+1}(P - U) \right\}\Big|_0^h + \left\{ \int_0^h dy u_i(P - U) \right\}_{,i} - \left\{ (P - U) u_i \right\}\Big|_h h_{,i} = \\
&= \left(U\Big|_h - P_0 \right)\left(-u_{N+1}\Big|_h + u_i|_h h_{,i} \right) + \left\{ \int_0^h dy u_i(P - U) \right\}_{,i}.
\end{aligned} \tag{3.7}$$

But formulae (2.3,5) imply that

$$u_{N+1} = -\int_0^y dy u_{i,i}, \tag{3.8}$$

so that

$$u_{N+1}|_h = -\int_0^h dy u_{i,i} \; = -\Big(\int_0^h dy u_i\Big)_{,i} + u_i|_h h_{,i} \quad \text{[by (2.4)]} \; =$$
$$= -h_t + u_i|_h h_{,i}\,. \tag{3.9}$$

Hence, the expression (3.7) equals to

$$\Big(U|_h - P_0\Big) h_{,t} + \Big\{\int_0^h dy u_i \Big(P - U\Big)\Big\}_{,i}\,. \tag{3.10}$$

$$2) \quad \int_0^h dy \lambda_\beta u_\beta u_\alpha u_{\beta,\alpha} \text{ [by (2.3)]} = \int_0^h dy \Big(\frac{\lambda_\beta}{2} u_\beta^2 u_\alpha\Big)_{,\alpha} =$$
$$= \frac{\lambda_\beta}{2} u_\beta^2 u_{N+1}|_0^h + \Big(\int_0^h dy \frac{\lambda_\beta}{2} u_\beta^2 u_i\Big)_{,i} - \frac{\lambda_\beta}{2} u_\beta^2|_h u_i|_h h_{,i} \quad \text{[by (2.5), (3.9)]} =$$
$$= \frac{\lambda_\beta}{2} u_\beta^2|_h (-h_t) + \Big(\int_0^h dy \frac{\lambda_\beta}{2} u_\beta^2 u_i)_{,i}\,. \tag{3.11}$$

Collecting together expressions (3.6a,10,11), we arrive at formula (3.3). ■

We now set

$$\lambda_1 = \ldots = \lambda_N = 1, \quad \lambda_{N+1} = \epsilon, \tag{3.12}$$

consider ϵ as an asymptotic parameter, and keep only zero-order in ϵ terms in the resulting asymptotic expansion.

The equation (3.1_{N+1}) then becomes:

$$(P - U)_{,N+1} = 0, \tag{3.13}$$

which can be rewritten with the help of (3.6) as

$$P - U = P_0 + V, \tag{3.14}$$

$$V = V(x_1, \ldots, x_n, h) = -U|_h. \tag{3.15}$$

We thus arrive at the purely evolution system:

$$u_{i,t} = u_j u_{i,j} - u_{i,y} \int_0^y u_{j,j} dy + V_{,i} \quad 1 \le i \le N, \tag{3.16a}$$

$$h_t = (\int_0^h u_j dy)_{,j}\,, \tag{3.16b}$$

$$u_i = u_i(\boldsymbol{x}, y, t), \; h = h(\boldsymbol{x}, t), \quad \boldsymbol{x} = (x_1, \ldots, x_N), \tag{3.16c}$$

where we used formula (3.8) for u_{N+1}.

For the case

$$N=1,\ U=-gy^2/2, \quad V=gh^2/2, \tag{3.17}$$

we recover the original Benny system (1.1).

We now proceed to show that the system (3.16) is a Hamiltonian system which induces a purely differential evolution in the space of moments.

4 The Evolution Of Moments

For a multiindex

$$\sigma=(\sigma(1),...,\sigma(N))=(\sigma_1,...,\sigma_N)\ \in \mathbf{Z}^N_{\geq 0}, \tag{4.1}$$

set

$$A_\sigma=A_\sigma(x,t)=\int_0^h u^\sigma dy, \tag{4.2}$$

where

$$u^\sigma=u_1^{\sigma(1)}...u_N^{\sigma(N)}. \tag{4.3}$$

In particular,

$$A_0=h. \tag{4.4}$$

Proposition 4.5.

$$A_{\sigma,t}=A_{\sigma+1_j,j}+\sigma_j A_{\sigma-1_j}V_{,j}\ ,\ \sigma\in\mathbf{Z}^N_{\geq 0}, \tag{4.6}$$

where 1_j is the multiindex

$$1_j=(0,...,1,...,0) \tag{4.7}$$

with only nonzero entry being 1 at the j^{th} place.

Proof. We have:

$$A_{\sigma,t}=(\int_0^h dyu^\sigma)_t=u^\sigma|_h h_t+\int_0^h dy\sigma_i u^{\sigma-1_i}u_{i,t}=$$

$$=u^\sigma|_h(u_j|_h h_{,j}+\int_0^h dyu_{j,j})+ \tag{4.8a}$$

$$+\int_0^h dy\sigma_i u^{\sigma-1_i}(u_j u_{i,j}-u_{i,y}\int_0^y u_{j,j}dy+V_{,i}\,). \tag{4.8b}$$

We now transform some of the five summands making up the expression (4.8).

The Second summand in (4.8b) becomes:

$$
\begin{aligned}
&-\int_0^h dy(u^\sigma)_y \int_0^y dyu_{j,j} = -\int_0^h dy\{(u^\sigma \int_0^y dyu_{j,j})_y - u^\sigma u_{j,j}\} = \\
&= -u^\sigma|_h \int_0^h dyu_{j,j} + \int_0^h dyu^\sigma u_{j,j}.
\end{aligned}
\tag{4.9}
$$

The second summand in (4.8a) and the first summand in (4.9) cancel out. The first summand in (4.8b) and the second summand in (4.9) combine into

$$
\begin{aligned}
&-\int_0^h dy\Big((u^\sigma)_{,j}\, u_j + u^\sigma u_{j,j}\Big) = \int_0^h dy(u^\sigma u_j)_{,j} = \\
&= \Big(\int_0^h dyu^\sigma u_j\Big)_{,j} - (u^\sigma u_j)|_h h_{,j}\,.
\end{aligned}
\tag{4.10}
$$

The first summand in (4.8a) and the second summand in (4.10) cancel out. What remains, the first summand in (4.10) and the third summand in (4.8b), make up the RHS of (4.6). ∎

5 Hamiltonian Properties Of The Evolution Of Moments

In the space of moments A_σ's, consider the following matrix:

$$
B_{\sigma|\mu} = \sigma_i A_{\sigma+\mu-1_i}\partial_i + \partial_i \mu_i A_{\sigma+\mu-1_i}. \tag{5.1}
$$

We shall verify in a moment that this is a Hamiltonian matrix.

Let us now check that our long-wave system (4.6) is Hamiltonian. Take as the Hamiltonian the remainder of the total energy $E(\boldsymbol{\lambda})$ (3.3) after the asymptotic expansion (3.12) has been made:

$$
H = \frac{1}{2}\sum_i A_{2_i} - \int_0^h dyU. \tag{5.2}
$$

Then

$$
\frac{\delta H}{\delta A_\mu} = \frac{\partial H}{\partial A_\mu} = \frac{1}{2}\sum_j \delta^\mu_{2_j} + V\delta^\mu_0. \tag{5.3}
$$

Thus, the corresponding motion equations,

$$
A_{\sigma,t} = B_{\sigma|\mu}\Big(\frac{\delta H}{\delta A_\mu}\Big), \tag{5.4}
$$

become, by formula (5.1):

$$A_{\sigma,t} = \sigma_i A_{\sigma+\mu-1_i} \partial_i (V \delta_0^\mu) + \partial_i (\mu_i A_{\sigma+\mu-1_i} \frac{1}{2} \sum_j \delta_{2_j}^\mu) =$$
$$= \sigma_i A_{\sigma-1_i} V_{,i} + A_{\sigma+1_i,i}. \tag{5.5}$$

These are exactly our motion equations (4.6).

Now, the matrix B (5.1) is *linear* in the field variables A_σ's. Therefore [2; 4, Ch. 5], the matrix B is Hamiltonian iff the algebra canonically attached to it by the rule

$$\mathbf{X}^t B(\mathbf{Y}) \sim A_\sigma [\mathbf{X}, \mathbf{Y}]_\sigma \tag{5.6}$$

is a Lie algebra; here $\sim$ denotes equality modulo $\sum_i Im(\partial_i)$ ("divergencies"). Hence,

$$\mathbf{X}^t B(\mathbf{Y}) = X_\sigma (\sigma_i A_{\sigma+\mu-1_i} \partial_i + \partial_i \mu_i A_{\sigma+\mu-1_i})(Y_\mu) \sim$$
$$\sim X_\sigma \sigma_i A_{\sigma+\mu-1_i} Y_{\mu,i} - X_{\sigma,i} \mu_i A_{\sigma+\mu-1_i} Y_\mu, \tag{5.7}$$

so that

$$p^\nu [\mathbf{X}, \mathbf{Y}]_\nu = p^{\sigma+\mu-1_i} (X_\sigma \sigma_i Y_{\mu,i} - Y_\mu \mu_i X_{\sigma,i}) =$$
$$= \frac{\partial}{\partial p_i}(X_\sigma p^\sigma) \cdot \frac{\partial}{\partial x_i}(Y_\mu p^\mu) - \frac{\partial}{\partial p_i}(Y_\mu p^\mu) \cdot \frac{\partial}{\partial x_i}(X_\sigma p^\sigma). \tag{5.8}$$

We see that we indeed get a Lie algebra of functions on $T^*\mathbf{R}^N$ polynomial in the p's (the coordinates in the fibers $T^*\mathbf{R}^N \to \mathbf{R}^N$.) This is a particular case of the general construction in [3] attaching a Hamiltonian matrix to a local Lie algebra. For $N = 1$, this interpretation of the Hamiltonian matrix B (5.1) is due to Lebedev [7].

For any Hamiltonian $H = H(\{A\})$, denote

$$H_\sigma = \frac{\delta H}{\delta A_\sigma}, \tag{5.9}$$

the corresponding variational derivative. Then [3, formula (22)] the evolution in the moments space

$$A_{\sigma,t} = B_{\sigma|\mu}(H_\mu) = (\sigma_i A_{\sigma+\mu-1_i} \partial_i + \partial_i \mu_i A_{\sigma+\mu-1_i})(H_\mu) \tag{5.10}$$

is implied by the evolution in the $(h, \boldsymbol{u})$-space:

$$h_t = (\mu_i A_{\mu-1_i} H_\mu)_{,i}\,, \tag{5.11a}$$
$$u_{s,t} = \mu_i u^{\mu-1_i} u_{s,i} H_\mu + u^\mu H_{\mu,s} - u_{s,y} \int_0^y dy (\mu_i u^{\mu-1_i} H_\mu)_{,i}\,. \tag{5.11b}$$

For $N = 1$, this general system first appeared in [5,6]. For general N, the system (5.11) has almost as many remarkable properties as its $N = 1$ version. We next examine one such property.

6 Local Flows In The Physical Space

When the velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, y, t)$ is y-independent, the *integro-differential* system (5.11) assumes a purely *differential* form

$$h_t = (\mu_i A^*_{\mu-1_i} H^*_\mu)_{,i}\,, \tag{6.1a}$$
$$u_{s,t} = \mu_i u^{\mu-1_i} u_{s,i} H^*_\mu + u^\mu H^*_{\mu,s}, \tag{6.1b}$$

where

$$A^*_\mu = hu^\mu, \tag{6.2a}$$
$$H^* = H(A)^* = H(A^*),\ H^*_\mu = (H_\mu)^*, \tag{6.2b}$$

and * denotes the reduction homomorphism that sends A_μ into

$$A^*_\mu = \int_0^h u^\mu dy = hu^\mu. \tag{6.3}$$

Proposition 6.4. (i) The system (6.1) is Hamiltonian, with the Hamiltonian H^*, and with the Hamiltonian matrix b:

$$b = \begin{array}{c|cc} & h & u_r \\ h & 0 & \partial_r \\ u_s & \partial_s & \dfrac{u_{s,r} - u_{r,s}}{h} \end{array} \;; \tag{6.5}$$

(ii) The homomorphism * (6.2a) is Hamiltonian between the Hamiltonian structures B (5.1) and b (6.5).

Proof. (i) We have:

$$\frac{\delta H^*}{\delta h} = \left(\frac{\delta H}{\delta A_\mu}\right)^* \frac{\partial A^*_\mu}{\partial h} = H^*_\mu u^\mu, \tag{6.6}$$

$$\frac{\delta H^*}{\delta u_k} = \left(\frac{\delta H}{\delta A_\mu}\right)^* \frac{\partial A^*_\mu}{\partial u_k} = H^*_\mu \frac{\partial (hu^\mu)}{\partial u_k} =$$
$$= H^*_\mu \mu_k A^*_{\mu-1_k} = H^*_\mu h \mu_k u^{\mu-1_k}. \tag{6.7}$$

The not-yet verified as Hamiltonian matrix b (6.5) produces the motion equations

$$h_t = \left(\frac{\delta H^*}{\delta u_r}\right)_{,r} \tag{6.8a}$$
$$u_{s,t} = \left(\frac{\delta H^*}{\delta h}\right)_{,s} + \frac{1}{h}(u_{s,r} - u_{r,s})\frac{\delta H^*}{\delta u_r}. \tag{6.8b}$$

By formula (6.7a), equations (6.1a) and (6.8a) are identical.

To show that equations (6.1b) and (5.8b) coincide, we need to verify that

$$\mu_i u^{\mu-1_i} u_{s,i} H^*_\mu + u^\mu H^*_{\mu,s} \overset{?}{=} \left(\frac{\delta H^*}{\delta h}\right)_{,s} + \frac{1}{h}(u_{s,r} - u_{r,s})\frac{\delta H^*}{\delta u_r}$$
$$[\text{by } (6.6,7)] = (u^\mu H^*_\mu)_{,s} + (u_{s,r} - u_{r,s}) H^*_\mu \mu_r u^{\mu-1_r}, \tag{6.9}$$

which is equivalent to (no sum on μ):

$$\mu_i u^{\mu-1_i} u_{s,i} \overset{?}{=} (u^\mu)_{,s} + (u_{s,r} - u_{r,s})\mu_r u^{\mu-1_r}, \tag{6.10}$$

which is obvious.

Now, the reason the matrix b (6.5) is Hamiltonian lies in the origin of that matrix. Consider the subalgebra of Hamiltonians in the A-space which depend upon the A_μ's with $|\mu| \leq 1$, where

$$|\mu| = |(\mu_1, \cdots, \mu_N)| = \Sigma_i \mu_i. \tag{6.11}$$

Formula (5.1) shows that this is indeed a Hamiltonian subalgebra, governed by the Hamiltonian matrix

$$\tilde{b} = \begin{array}{c c} & \begin{array}{cc} A_0 & A_{1_j} \end{array} \\ \begin{array}{c} A_0 \\ A_{1_i} \end{array} & \begin{pmatrix} 0 & \partial_j A_0 \\ A_0 \partial_i & A_{1_j}\partial_i + \partial_j A_{1_i} \end{pmatrix} \end{array}. \tag{6.12}$$

Passing to the coordinates

$$h = A_0, \quad u_j = A_0^{-1} A_{1_j} \quad , \quad 1 \leq j \leq N, \tag{6.13}$$

we recover the Hamiltonian matrix b (6.5);

(ii) We have to verify that

$$JbJ^t = B^*, \tag{6.14}$$

where J is the Frechét derivative of the homomorphism *:

$$J = \begin{array}{c c} & \begin{array}{ccc} h & & u_i \end{array} \\ A_\sigma & \begin{pmatrix} u^\sigma & | & \sigma_i h u^{\sigma-1} \end{pmatrix} \end{array}. \tag{6.15}$$

Multiplying JbJ^t through, formula (6.14) reduces to the identity

$$\sigma_i h u^{\sigma-1_i}\partial_i u^\mu + u^\sigma \partial_j \mu_j h u^{\mu-1j} + \sigma_i u^{\sigma-1_i}(u_{i,j} - u_{j,i})\mu_j h u^{\mu-1_j} \overset{?}{=}$$
$$\overset{?}{=} \sigma_i h u^{\sigma+\mu-1_i}\partial_i + \partial_j \mu_j h u^{\sigma+\mu-1_j}, \tag{6.16}$$

which is obvious. ■

For $N = 1$, we recover results from [5,6].

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