Journal of Nonlinear Mathematical Physics Volume 12, Supplement 2 (2005), 316–332 SIDE VI

On a special two-dimensional lattice by Blaszak and Szum: pfaffianization and molecule solutions

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This article is a part of the special issue titled "Symmetries and Integrability of Difference Equations (SIDE VI)"

Abstract

In this paper, we first present the Casorati and grammian determinant solutions to a special two-dimensional lattice by Blaszak and Szum. Then, by using the pfaffianization procedure of Hirota and Ohta, a new integrable coupled system is generated from the special lattice. Moreover, gram-type pfaffian solutions to the pfaffianized system are proposed. Finally, the bi-directional wronskian solutions to the corresponding molecule equation are given.

1 Introduction

In the early 1990s, Hirota and Ohta[1, 2] developed a procedure for generalizing equations from the KP hierarchy to produce coupled systems of equations, which we now call pfaffianization. These pfaffianized equations appear as coupled systems of the original equations and have soliton solutions expressed by pfaffians. Such a procedure has been successfully applied to the DS equations[3], the discrete KP equation[4], the self-dual Yang-Mills equation[6], the two-dimensional Toda lattice[7], the semi-discrete Toda equation[8], the differential-difference KP equation[9], etc.. Besides, the pfaffianized KP hierarchies have been investigated in [10]. The key points involved in this procedure are to first express N-soliton solutions of an original equation in the form of Wronskian, Casorati or Grammian type determinant, then to construct a pfaffian with elements satisfying the pfaffianized form of the dispersion relation given in the determinant solutions and finally to seek coupled bilinear equations whose solutions are these pfaffians.

In [2] , the so-called molecule solutions for the two-dimensional Toda molecule equation have been expressed as bi-directional Wronskians. A determinant solution whose determinant size appears as discrete independent variable of the equation is sometimes called "molecule type" solution. Similarly, in our paper we can consider the molecule solutions for the special two-dimensional molecule equation by Blaszak and Szum. We have found the bi-directional Wronskian solutions for the special two-dimensional molecule equation .

The first part of this paper is to apply the pfaffianization procedure to a special lattice by Blaszak and Szum. We will first present its Casorati and Grammian solutions, respectively. Then by pfaffianization, we derive the pfaffianized form of the bilinear lattice and propose the Gram-type pfaffian solutions to the coupled system. The second part is to look for the molecule solutions for the corresponding molecule equation, which are expressed as bi-directional Wronskians.

2 Properties of Pfaffian

For the sake of self-sufficiency, let us review some necessary properties of Pfaffian. Pfaffians are antisymmetric functions with respect to its independent variables:

$$\mathrm{pf}(a,b) = -\mathrm{pf}(b,a), \quad \text{for any} \;\; a \;\; \text{and} \;\; b.$$

A 2n-th degree pfaffian is defined by the expansion rule

$$\mathrm{pf}(1,2,\cdots,2n) = \sum_{j=2}^{2n}(-1)^j \mathrm{pf}(1,j)\,\mathrm{pf}(2,3,\cdots,\hat{j},\cdots,2n),$$

where $\hat{j}$ denotes the absence of the letter j. For example, if $n = 2$, we have

$$\mathrm{pf}(1,2,3,4) = \mathrm{pf}(1,2)\,\mathrm{pf}(3,4) - \mathrm{pf}(1,3)\,\mathrm{pf}(2,4) + \mathrm{pf}(1,4)\,\mathrm{pf}(2,3).$$

There exist various kinds of pfaffian identities, with the so-called "Plücker relation for pfaffians" [5] particularly useful in this letter:

$$\begin{aligned}
&\mathrm{pf}(\alpha_1,\alpha_2,\alpha_3,1,2,\cdots,2n-1)\,\mathrm{pf}(\alpha_4,1,2,\cdots,2n-1)\\
&\quad -\mathrm{pf}(\alpha_1,1,2,\cdots,2n-1)\,\mathrm{pf}(\alpha_2,\alpha_3,\alpha_4,1,2,\cdots,2n-1)\\
&\quad +\mathrm{pf}(\alpha_2,1,2,\cdots,2n-1)\,\mathrm{pf}(\alpha_1,\alpha_3,\alpha_4,1,2,\cdots,2n-1)\\
&\quad -\mathrm{pf}(\alpha_3,1,2,\cdots,2n)\,\mathrm{pf}(\alpha_1,\alpha_2,\alpha_4,1,2,\cdots,2n-1) = 0.
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\mathrm{pf}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,1,2,\cdots,2n)\mathrm{pf}(1,2,\cdots,2n)\\
&\quad -\mathrm{pf}(\alpha_1,\alpha_2,1,2,\cdots,2n)\mathrm{pf}(\alpha_3,\alpha_4,1,2,\cdots,2n)\\
&\quad +\mathrm{pf}(\alpha_1,\alpha_3,1,2,\cdots,2n)\mathrm{pf}(\alpha_2,\alpha_4,1,2,\cdots,2n)\\
&\quad -\mathrm{pf}(\alpha_1,\alpha_4,1,2,\cdots,2n)\mathrm{pf}(\alpha_2,a_3,1,2,\cdots,2n) = 0
\end{aligned} \tag{2.2}$$

3 Determinant solutions to a special lattice

In [11], the following lattice is constructed by Blaszak and Szum as an application of the so-called "central extension procedure and operand formalism"

$$\frac{\partial u(n)}{\partial t} = u(n)\mathcal{H}^{-1}p(n-1), \tag{3.1}$$

$$\frac{\partial v(n)}{\partial t} = u(n+1) - u(n) + (E+1)^{-1}\frac{\partial p(n)}{\partial y}, \tag{3.2}$$

$$\frac{\partial p(n)}{\partial t} = v(n+1) - v(n) - p(n)\mathcal{H}^{-1}p(n). \tag{3.3}$$

In [12], by setting $w(n) = (E+1)^{-1}p(n)$, the lattice (3.1)-(3.3) is rewritten as

$$\frac{\partial u(n)}{\partial t} = u(n)(w(n) - w(n-1)), \tag{3.4}$$

$$\frac{\partial v(n)}{\partial t} = u(n+1) - u(n) + \frac{\partial w(n)}{\partial y}, \tag{3.5}$$

$$\frac{\partial w(n+1)}{\partial t} + \frac{\partial w(n)}{\partial t} = v(n+1) - v(n) - w(n+1)^2 + w(n)^2. \tag{3.6}$$

By the dependent variable transformation

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \quad v(n) = \frac{D_t^2 f(n)\cdot f(n+1)}{f(n)f(n+1)}, \quad w(n) = \left(\frac{f(n+1)}{f(n)}\right)$$

and introducing an auxiliary variable z, the bilinear form for eqs. (3.4)-(3.6) are also presented in [12]

$$(D_z e^{(1/2)D_n} - D_t^2 e^{(1/2)D_n})\tau(n)\cdot\tau(n) = 0, \tag{3.7}$$

$$(D_t D_z - D_t D_y)\tau(n)\cdot\tau(n) = 4\sinh^2(\frac{1}{2}D_n)\tau(n)\cdot\tau(n), \tag{3.8}$$

where the bilinear operators D_y^m, D_t^k and $\exp(\delta D_n)$ are defined by [13]

$$D_y^m D_t^k a\cdot b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(y,t)b(y',t')\Bigg|_{y'=y,t'=t},$$

$$\exp(\delta D_n)a\cdot b \equiv a(n+\delta)b(n-\delta).$$

3.1 Casorati solutions to the lattice (3.7)-(3.8)

In the following, we show that soliton solutions of the lattice (3.7)-(3.8) can be written in the compact form using the Casorati determinant

$$\tau_n = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}, \tag{3.9}$$

where $\phi_i(n)$ satisfies the following relations

$$\frac{\partial \phi_i(n)}{\partial y} = \phi_i(n+2) + \phi_i(n-1), \frac{\partial \phi_i(n)}{\partial z} = \phi_i(n+2), \frac{\partial \phi_i(n)}{\partial t} = \phi_i(n+1). \tag{3.10}$$

A particular solution of (3.10) is obtained by choosing "exponential type" functions as

$$\phi_i(n) = c_i p_i^n e^{(p_i^2+p_i^{-1})y + p_i^2 z + p_i t} + d_i q_i^n e^{(q_i^2+q_i^{-1})y + q_i^2 z + q_i t},$$

where $p_i, q_i, c_i, d_i (i = 1, \cdots, N)$ are arbitrary constants.

Following Nimmo and Freeman's notation [14, 15], we denote τ_n in (3.9) as

$$\tau(n) = |\ 0,\ \ 1,\ \ 2,\ \ \cdots,\ \ N-1\ | = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}.$$

Using the dispersion relation (3.10), we can easily obtain the following formulae

$$\begin{aligned}
&\tau(n+1) = |\ 1,\ 2,\ \cdots,\ N\ |, \qquad \tau(n-1) = |\ -1,\ 0,\ \cdots,\ N-2\ |,\\
&\frac{\partial\tau(n)}{\partial t} = |\ 0,\ 1,\ \cdots,\ N-2,\ N\ |, \quad \frac{\partial\tau(n+1)}{\partial t} = |\ 1,\ 2,\ \cdots,\ N-1,\ N+1\ |,\\
&\frac{\partial^2\tau(n)}{\partial t^2} = |\ 0,\ 1,\ \cdots,\ N-3,\ N-1,\ N\ | + |\ 0,\ 1,\ \cdots,\ N-2,\ N+1\ |,\\
&\frac{\partial^2\tau(n+1)}{\partial t^2} = |\ 1,\ 2,\ \cdots,\ N-2,\ N,\ N+1\ | + |\ 1,\ 2,\ \cdots,\ N-1,\ N+2\ |,\\
&\frac{\partial\tau(n)}{\partial z} = -|\ 0,\ 1,\ \cdots,\ N-3,\ N-1,\ N\ | + |\ 0,\ 1,\ \cdots,\ N-2,\ N+1\ |,\\
&\frac{\partial\tau(n+1)}{\partial z} = -|\ 1,\ 2,\ \cdots,\ N-2,\ N,\ N+1\ | + |\ 1,\ 2,\ \cdots,\ N-1,\ N+2\ |,\\
&\frac{\partial^2\tau(n)}{\partial z\partial t} = -|\ 0,\ 1,\ \cdots,\ N-4,\ N-2,\ N-1,\ N\ | + |\ 0,\ 1,\ \cdots,\ N-2,\ N+2\ |,\\
&\frac{\partial\tau(n)}{\partial y} = |\ -1,\ 1,\ \cdots,\ N-1\ | - |\ 0,\ \cdots,\ N-3,\ N-1,\ N\ |\\
&\qquad + |\ 0,\ \cdots,\ N-2,\ N+1\ |,\\
&\frac{\partial^2\tau(n)}{\partial y\partial t} = |\ 0,\ 1,\ \cdots,\ N-1\ | + |\ -1,\ 1,\ \cdots,\ N-2,\ N,\ |\\
&\qquad - |\ 0,\ 1,\ \cdots,\ N-4,\ N-2,\ N-1,\ N\ | + |\ 0,\ 1,\ \cdots,\ N-2,\ N+2\ |.
\end{aligned}$$

Substitution of the above expressions into eqs. (3.7) and (3.8) will lead to the following Plücker relations respectively

$$\begin{aligned}
&|\ 0,\ 1,\ \cdots,\ N-1\ | \times |\ 1,\ \cdots,\ N-2,\ N,\ N+1\ |\\
&\qquad + |\ 0,\ \cdots,\ N-2,\ N+1\ | \times |\ 1,\ 2,\ \cdots,\ N\ |\\
&\qquad - |\ 0,\ \cdots,\ N-2,\ N\ | \times |\ 1,\ \cdots,\ N-1,\ N+1\ | = 0, \qquad (3.11)\\
&|\ -1,\ 0,\ \cdots,\ N-2\ | \times |\ 1,\ 2,\ \cdots,\ N\ |\\
&\qquad - |\ -1,\ 1,\ \cdots,\ N-1\ | \times |\ 0,\ 1,\ \cdots,\ N-2,\ N\ |\\
&\qquad + |\ 0,\ 1,\ \cdots,\ N-1\ | \times |\ -1,\ 1,\ \cdots,\ N-2,\ N\ | = 0. \qquad (3.12)
\end{aligned}$$

Therefore we have confirmed that $\tau(n)$ satisfies eqs. (3.7) and (3.8).

3.2 Grammian solution to the lattice (3.7)-(3.8)

Besides the Casorati solution, the lattice (3.7)-(3.8) possesses a solution $\tau(n)$ expressed in the following Gramm determinant

$$\tau(n) = \det\left|c_{ij} + \int^t f_i(n)g_j(-n)dt\right|_{1\le i,j\le N}, \qquad (3.13)$$

where each $f_i(n)$, $g_j(-n)$ satisfy the linear differential equations

$$\frac{\partial f_i(n)}{\partial y} = f_i(n+2) + f_i(n-1), \frac{\partial f_i(n)}{\partial z} = f_i(n+2), \tag{3.14}$$

$$\frac{\partial f_i(n)}{\partial t} = f_i(n+1), \frac{\partial g_i(-n)}{\partial y} = -g_i(-n+2) - g_i(-n-1), \tag{3.15}$$

$$\frac{\partial g_i(-n)}{\partial z} = -g_i(-n+2), \frac{\partial g_i(-n)}{\partial t} = -g_i(-n+1). \tag{3.16}$$

It is known that any determinant can be expressed by a pfaffian. We rewrite $\tau(n)$ as

$$\tau(n) = (1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*)_n, \tag{3.17}$$

$$(i, j^*)_n = c_{ij} + \int^t f_i(n) g_j(-n) dt, \quad (i, j) = (i^*, j^*)_n = 0, \quad c_{ij} = \text{const.}. \tag{3.18}$$

In order to prove $\tau(n)$ satisfies eqs. (3.7) and (3.8), we introduce pfaffians defined by

$$(d_m^*, i)_n = f_i(n+m), \quad (d_m, j^*)_n = g_j(-n+m), \tag{3.19}$$

$$(d_m^*, j^*)_n = (d_m, i)_n = (d_l, d_m^*)_n = (d_l, d_m)_n = (d_l^*, d_m^*)_n = 0. \tag{3.20}$$

By virtue of the above pfaffians and eqs. (3.14)-(3.16), we have the following difference or differential formulae for $\tau(n)$

$$\tau(n+1) = (\cdot)_n + (d_{-1}, d_0^*, \cdot)_n, \quad \tau(n-1) = (\cdot)_n - (d_0, d_{-1}^*, \cdot)_n,$$

$$\frac{\partial \tau(n)}{\partial t} = (d_0, d_0^*, \cdot)_n, \quad \frac{\partial \tau(n+1)}{\partial t} = (d_{-1}, d_2^*, \cdot)_n,$$

$$\frac{\partial^2 \tau(n)}{\partial t^2} = (d_0, d_1^*, \cdot)_n - (d_1, d_0^*)_n, \quad \frac{\partial^2 \tau(n+1)}{\partial t^2} = (d_{-1}, d_2^*, \cdot)_n - (d_0, d_1^*, \cdot)_n + (d_0, d_1^*, d_{-1}, d_0^*, \cdot)_n,$$

$$\frac{\partial \tau(n)}{\partial z} = (d_0, d_1^*, \cdot)_n + (d_1, d_0^*, \cdot)_n, \quad \frac{\partial \tau(n+1)}{\partial z} = (d_{-1}, d_2^*, \cdot)_n + (d_0, d_1^*, \cdot)_n + (d_0, d_1^*, d_{-1}, d_0^*, \cdot)_n,$$

$$\frac{\partial^2 \tau(n)}{\partial z \partial t} = (d_0, d_2^*, \cdot)_n - (d_2, d_0^*, \cdot)_n, \quad \frac{\partial \tau(n)}{\partial y} = (d_0, d_1^*, \cdot)_n + (d_1, d_0^*, \cdot)_n - (d_{-1}, d_{-1}^*, \cdot)_n,$$

$$\frac{\partial^2 \tau(n)}{\partial y \partial t} = (d_0, d_2^*, \cdot)_n + (d_0, d_{-1}^*, \cdot)_n - (d_2, d_0^*, \cdot)_n - (d_{-1}, d_0^*, \cdot)_n - (d_{-1}, d_{-1}^*, d_0, d_0^*, \cdot)_n.$$

Here we have denoted $\tau(n) = (1, \cdots, N, N^*, \cdots, 1^*)_n = (\cdot)_n$.

Substituting the above pfaffians into eqs. (1) and (2) respectively, after some calculations, we get the following two Jacobi identities

$$(d_0, d_1^*, d_{-1}, d_0^*, \cdot)_n (\cdot)_n - (d_0, d_1^*, \cdot)_n (d_{-1}, d_0^*, \cdot)_n + (d_0, d_0^*, \cdot)_n (d_{-1}, d_1^*, \cdot)_n \equiv 0, \tag{3.21}$$

$$(d_{-1}, d_{-1}^*, d_0, d_0^*, \cdot)_n (\cdot)_n - (d_{-1}, d_{-1}^*, \cdot)_n (d_0, d_0^*, \cdot)_n + (d_{-1}, d_0^*, \cdot)_n (d_0, d_{-1}^*, \cdot)_n \equiv 0. \tag{3.22}$$

Thus we have proved $\tau(n)$ given by eq. (3.13) is the grammian solution to eqs. (3.7) and (3.8).

4 The coupled system of the lattice (3.7)-(3.8)

In this section, we will first apply the pfaffianization procedure to the lattice (3.7)-(3.8), then we will present the Gram-type pfaffian solution to its pfaffianized system. For this purpose, we need the following pfaffian identities

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \alpha, \beta, \gamma, \delta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}) \\
&\qquad -\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \alpha, \beta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \gamma, \delta) \\
&\qquad +\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \alpha, \gamma)\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \beta, \delta) \\
&\qquad -\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \alpha, \delta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-2}, \beta, \gamma) = 0,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \alpha, \beta, \gamma)\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \delta) \\
&\qquad -\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \alpha, \beta, \delta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \gamma) \\
&\qquad +\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \alpha, \gamma, \delta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \beta) \\
&\qquad -\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, \beta, \gamma, \delta)\mathrm{pf}(a_1, a_2, \cdots, a_{N-1}, , \alpha) = 0,
\end{aligned} \tag{4.2}$$

4.1 Pfaffianization of the lattice (3.7)-(3.8)

As mentioned in section 1, in order to pfaffianize the lattice (3.7)-(3.8), we require a pfaffian with elements satisfying the pfaffianized form of the dispersion relation (3.10). Hence the entries in our pfaffian are chosen to satisfy

$$(i, j)_{n+1} = (i + 1, j + 1)_n, \tag{4.3}$$

$$\frac{\partial}{\partial y}(i, j)_n = (i + 2, j)_n + (i, j + 2)_n + (i - 1, j)_n + (i, j - 1)_n, \tag{4.4}$$

$$\frac{\partial}{\partial z}(i, j)_n = (i + 2, j)_n + (i, j + 2)_n, \quad \frac{\partial}{\partial t}(i, j)_n = (i + 1, j)_n + (i, j + 1)_n. \tag{4.5}$$

In much the same way as [4], if we take

$$\tau(n) = (1, 2, \cdots, N)_n, \quad \text{with } N \text{ being even,} \tag{4.6}$$

then we can calculate

$$\tau(n+1) = (2,3,\cdots,N+1)_n, \quad \tau(n-1) = (0,1,\cdots,N-1)_n, \tag{4.7}$$

$$\frac{\partial\tau(n)}{\partial t} = (1,2,\cdots,N-1,N+1)_n, \quad \frac{\partial\tau(n+1)}{\partial t} = (2,3,\cdots,N,N+2)_n, \tag{4.8}$$

$$\frac{\partial^2\tau(n)}{\partial t^2} = (1,2,\cdots,N-2,N,N+1)_n + (1,2,\cdots,N-1,N+2)_n, \tag{4.9}$$

$$\frac{\partial^2\tau(n+1)}{\partial t^2} = (2,3,\cdots,N-1,N+1,N+2)_n + (2,3,\cdots,N,N+3)_n, \tag{4.10}$$

$$\frac{\partial\tau(n)}{\partial z} = -(1,2,\cdots,N-2,N,N+1)_n + (1,2,\cdots,N-1,N+2)_n, \tag{4.11}$$

$$\frac{\partial\tau(n+1)}{\partial z} = -(2,3,\cdots,N-1,N+1,N+2)_n + (2,3,\cdots,N,N+3)_n, \tag{4.12}$$

$$\frac{\partial^2\tau(n)}{\partial z\partial t} = -(1,2,\cdots,N-3,N-1,N,N+1)_n + (1,2,\cdots,N-1,N+3)_n, \tag{4.13}$$

$$\frac{\partial\tau(n)}{\partial y} = (0,2,\cdots,N)_n - (1,\cdots,N-2,N,N+1)_n + (1,\cdots,N-1,N+2)_n, \tag{4.14}$$

$$\frac{\partial^2\tau(n)}{\partial y\partial t} = (1,2,\cdots,N)_n + (0,2,\cdots,N-1,N+1)_n \tag{4.15}$$

$$-(1,2,\cdots,N-3,N-1,N,N+1)_n + (1,2,\cdots,N-1,N+3)_n. \tag{4.16}$$

For simplicity, here we have denoted $\mathrm{pf}(1,2,\cdots,N)_n$ to be $(1,2,\cdots,N)_n$ without any confusion. These pfaffians no longer satisfy the bilinear form of the lattice (3.7)-(3.8). Following the Hirota-Ohta's procedure, we now introduce two new variables $g(n)$ and $\hat{g}(n)$ defined by

$$g(n) = (0,1,\cdots,N+1)_n, \quad \hat{g}(n) = (2,3,\cdots,N-1)_n. \tag{4.17}$$

Then we can show that $\tau(n)$, $g(n)$ and $\hat{g}(n)$ satisfy the following four bilinear equations:

$$(D_z - D_t^2)e^{\frac{1}{2}D_n}\tau(n)\cdot\tau(n) = -2e^{\frac{1}{2}D_n}g(n)\cdot\hat{g}(n), \tag{4.18}$$

$$(D_tD_z - D_tD_y - 4\sinh^2(\frac{1}{2}D_n))\tau(n)\cdot\tau(n) = -2g(n)\hat{g}(n), \tag{4.19}$$

$$(D_y - D_z)e^{\frac{1}{2}D_n}g(n)\cdot\tau(n) = D_te^{-\frac{1}{2}D_n}g(n)\cdot\tau(n), \tag{4.20}$$

$$(D_y - D_z)e^{\frac{1}{2}D_n}\tau(n)\cdot\hat{g}(n) = D_te^{-\frac{1}{2}D_n}\tau(n)\cdot\hat{g}(n). \tag{4.21}$$

In fact, for $g(n)$ and $\hat{g}(n)$, we have

$$\frac{\partial \tau(n+1)}{\partial y} = (1,3,\cdots,N+1)_n - (2,\cdots,N-1,N+1,N+2)_n + (2,\cdots,N,N+3)_n, \tag{4.22}$$

$$\frac{\partial g(n+1)}{\partial y} = (0,2\cdots,N+2)_n - (1,\cdots,N,N+2,N+3)_n + (1,\cdots,N+1,N+4)_n \tag{4.23}$$

$$\frac{\partial g(n+1)}{\partial z} = (1,\cdots,N+1,N+4)_n - (1,\cdots,N,N+2,N+3)_n, \tag{4.24}$$

$$\frac{\partial g(n)}{\partial t} = (0,\cdots,N,N+2)_n, \quad \hat{g}(n+1) = (3,\cdots,N)_n, \tag{4.25}$$

$$\frac{\partial \hat{g}(n+1)}{\partial t} = (3,\cdots,N-1,N+1)_n, \tag{4.26}$$

$$\frac{\partial \hat{g}(n)}{\partial z} = (2,\cdots,N-2,N+1)_n - (2,\cdots,N-3,N-1,N)_n, \tag{4.27}$$

$$\frac{\partial \hat{g}(n)}{\partial y} = (1,3,\cdots,N-1)_n - (2,\cdots,N-3,N-1,N)_n + (2,\cdots,N-2,N+1)_n. \tag{4.28}$$

Substitution of the above pfaffian expressions into eqs. (4.18)-(4.21) will lead to the pfaffian identities (4.1) and (4.2) respectively. For example, eq. (3.7) is reduced to the following pfaffian identitiy

$$\begin{aligned}&(1,\cdots,N,N+2,N+1)_n(2,\cdots,N-1)_n = (1,2,\cdots,N)_n(2,\cdots,N-1,N+2,N+1)_n\\&-(2,\cdots,N,N+1)_n(1,\cdots,N-1,N+2)_n + (1,\cdots,N-1,N+1)_n(2,\cdots,N,N+2)_n.\end{aligned}$$

Therefore, eqs. (4.18)-(4.21) constitute the bilinear form of the pfaffianized system of the lattice (3.7)-(3.8). Since the lattice (3.7)-(3.8) can be considered as a reduction of the pfaffianized system (3.7)-(3.8), we call the pfaffianized system the coupled system of the lattice (3.7)-(3.8).

If we wish to consider solutions to the coupled system of the lattice (3.7)-(3.8) (4.18)-(4.21), then we may choose entries in the pfaffians to be expressed in the form

$$(i,j)_n = \sum_{l=1}^{M}[\Phi_l(n+i)\Psi_l(n+j) - \Phi_l(n+j)\Psi_l(n+i)],$$

where $\Phi_l(m)$ and $\Psi_l(m)$ satisfy

$$\begin{aligned}&\frac{\partial}{\partial t}\Phi_l(m) = \Phi_l(m+1), \quad \frac{\partial}{\partial t}\Psi_l(m) = -\Psi_l(m+1),\\&\frac{\partial}{\partial z}\Phi_l(m) = \Phi_l(m+2), \quad \frac{\partial}{\partial z}\Psi_l(m) = \Psi_l(m+2),\\&\frac{\partial}{\partial y}\Phi_l(m) = \Phi_l(m+2) + \Phi_l(m-1), \quad \frac{\partial}{\partial y}\Psi_l(m) = \Psi_l(m+2) + \Psi_l(m-1).\end{aligned}$$

By the dependent variable transformation

$$u(n) = \frac{\tau(n+1)\tau(n-1)}{\tau(n)^2}, \quad v(n) = \frac{D_z\tau(n+1)\cdot\tau(n)}{\tau(n+1)\tau(n)}, \tag{4.29}$$

$$p(n-1) = \frac{g(n)}{\tau(n-1)}, \quad q(n+1) = \frac{\hat{g}(n)}{\tau(n+1)}, \quad w(n) = \left(\ln\frac{\tau(n+1)}{\tau(n)}\right)_t, \tag{4.30}$$

we may transform the coupled system of the lattice (3.7)-(3.8) (4.18)-(4.21) into

$$\frac{\partial w(n+1)}{\partial t} + \frac{\partial w(n)}{\partial t} - v(n+1) + v(n) \tag{4.31}$$

$$+w(n+1)^2 - w(n)^2 - 2p(n+1)q(n+1) + 2p(n)q(n) = 0, \tag{4.32}$$

$$\frac{\partial v(n)}{\partial t} - \frac{\partial w(n)}{\partial y} - u(n+1) + u(n) \tag{4.33}$$

$$+u(n+1)p(n)q(n+2) - u(n)p(n-1)q(n+1) = 0, \tag{4.34}$$

$$\frac{\partial p(n)}{\partial y} - \frac{\partial p(n)}{\partial z} - u(n)\frac{\partial p(n-1)}{\partial t} + u(n)p(n-1)(w(n) + w(n-1)) = 0, \tag{4.35}$$

$$\frac{\partial q(n)}{\partial y} - \frac{\partial q(n)}{\partial z} - u(n)\frac{\partial q(n+1)}{\partial t} - u(n)q(n+1)(w(n) + w(n-1)) = 0. \tag{4.36}$$

By taking $g(n) = \hat{g}(n) = 0$, it is obvious that the coupled system (4.31)-(4.36) or (4.18)-(4.21) of the lattice (3.7)-(3.8) can be easily reduced to the lattice (3.4)-(3.6) or eqs. (3.7)-(3.8) respectively.

4.2 Gram-type pfaffian solutions to the coupled system of the lattice (3.7)-(3.8)

It is known that the coupled KP equation derived by pfaffianization has solutions expressed in the form of Gram-type pfaffians. Similarly, we can expect the coupled system of the lattice (3.7)-(3.8) possesses the Gram-type pfaffian solution

$$\tau(n) = (1, \cdots, 2N)_n, \tag{4.37}$$

$$g(n) = (d_{-1}, d_0, 1, \cdots, 2N)_n, \tag{4.38}$$

$$\hat{g}(n) = (c_{-1}, c_0, 1, \cdots, 2N)_n. \tag{4.39}$$

Each pfaffian element is defined by

$$(i,j)_n = c_{ij} + \int^t (f_i(n)g_j(-n) - f_j(n)g_i(-n))dt, \quad c_{ij} = -c_{ji}, \tag{4.40}$$

$$(d_m, i)_n = f_i(n+m), \quad (c_i, j)_n = g_i(-n+m), \quad (d_l, d_m)_n = (c_l, c_m)_n = (d_m, c_l)_n = 0, \tag{4.41}$$

where $f_i(n)$ and $g_j(-n)$ satisfy the same equations as (3.14) and (3.16).

Based on the above pfaffian elements and (3.14)-(3.16), we get the following differentials in y, z , t and differences in n for $\tau(n)$, $g(n)$ and $\hat{g}(n)$

$$\tau(n+1) = (\cdot)_n + (c_{-1}, d_0, \cdot)_n, \quad \tau(n-1) = (\cdot)_n - (c_0, d_{-1}, \cdot)_n, \tag{4.42}$$

$$\frac{\partial \tau(n)}{\partial t} = (c_0, d_0, \cdot)_n, \quad \frac{\partial \tau(n+1)}{\partial t} = (c_{-1}, d_1, \cdot)_n, \quad \frac{\partial^2 \tau(n)}{\partial t^2} = (c_0, d_1, \cdot)_n - (c_1, d_0, \cdot)_n, \tag{4.43}$$

$$\frac{\partial^2 \tau(n+1)}{\partial t^2} = (c_{-1}, d_2, \cdot)_n - (c_0, d_1, \cdot)_n - (c_0, d_1, c_{-1}, d_0, \cdot)_n, \tag{4.44}$$

$$\frac{\partial \tau(n)}{\partial z} = (c_0, d_1, \cdot)_n + (c_1, d_0 . \cdot)_n, \tag{4.45}$$

$$\frac{\partial \tau(n+1)}{\partial z} = (c_{-1}, d_2, \cdot)_n + (c_0, d_1, \cdot)_n + (c_0, d_1, c_{-1}, d_0, \cdot)_n, \tag{4.46}$$

$$\frac{\partial \tau(n)}{\partial y} = (c_0, d_1, \cdot)_n + (c_1, d_0, \cdot)_n - (c_{-1}, d_{-1}, \cdot)_n, \tag{4.47}$$

$$\frac{\partial \tau(n+1)}{\partial y} = (c_{-1}, d_2, \cdot)_n + (c_0, d_1, \cdot)_n - (c_{-2}, d_0, \cdot)_n + (c_0, d_1, c_{-1}, d_0, \cdot)_n, \tag{4.48}$$

$$\frac{\partial^2 \tau(n)}{\partial z \partial t} = (c_0, c_0, \cdot)_n - (c_2, d_0, \cdot)_n, \tag{4.49}$$

$$\frac{\partial^2 \tau(n)}{\partial y \partial t} = (c_0, d_2, \cdot)_n - (c_2, d_0, \cdot)_n - (c_{-1}, d_0, \cdot)_n + (c_0, d_{-1}, \cdot)_n - (c_0, d_1, c_{-1}, d_0, \cdot)_n, \tag{4.50}$$

$$g(n+1) = (d_0, d_1, \cdot)_n, \quad \frac{\partial g(n)}{\partial t} = (d_{-1}, d_1, \cdot)_n, \quad \frac{\partial g(n+1)}{\partial z} = (d_2, d_1, \cdot)_n + (d_0, d_3, \cdot)_n, \tag{4.51}$$

$$\frac{\partial g(n+1)}{\partial y} = (d_2, d_1, \cdot)_n + (d_{-1} . d_1, \cdot)_n + (d_0, d_3, \cdot)_n - (d_0, d_1, c_{-1}, d_{-1})_n, \tag{4.52}$$

$$\hat{g}(n+1) = (c_{-2}, c_{-1}, \cdot)_n, \quad \frac{\partial \hat{g}(n+1)}{\partial t} = (c_0, c_{-2}, \cdot)_n + (c_{-2}, c_{-1}, c_0, d_0, \cdot)_n, \tag{4.53}$$

$$\frac{\partial \hat{g}(n)}{\partial z} = (c_0, c_1, \cdot)_n + (c_2, c_{-1}, \cdot)_n + (c_{-1}, c_0, c_1, d_0, \cdot)_n, \tag{4.54}$$

$$\frac{\partial \hat{g}(n)}{\partial y} = (c_0, c_1, \cdot)_n + (c_0, c_{-2}, \cdot)_n + (c_2, c_{-1}, \cdot)_n + (c_{-1}, c_0, c_1, d_0, \cdot)_n. \tag{4.55}$$

Here we have denoted $(1, 2, \cdots, 2N)_n = (\cdot)_n$. By employing the above pfaffian expressions, eq. (4.18) is reduced to the following pfaffian identities

$$\begin{aligned} &(c_{-1}, c_0, d_0, d_1, \cdot)_n (\cdot)_n - (c_{-1}, c_0, \cdot)_n (d_0, d_1, \cdot)_n \\ &\quad + (c_{-1}, d_0, \cdot)_n (c_0, d_1, \cdot)_n - (c_{-1}, d_1, \cdot)_n (c_0, d_0, \cdot)_n \equiv 0. \end{aligned} \tag{4.56}$$

Similarly, eqs. (4.19)-(4.21) can be also reduced to pfaffian identities. So far, we have proved that the coupled system of the lattice (3.7)-(3.8) has the Gram-type pfaffian solution given by eqs. (4.37)-(4.39).

5 Bi-directional Wronskian expression of molecule solutions to the corresponding molecule equation

In [2], the molecule solutions for two-dimensional Toda molecule equation have been obtained, which are expressed as bi-directional Wronskian. We can follow the same procedure to find the molecule solutions for the special molecule equation presented by Blaszak and Szum above. By the dependent variable transformation

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \quad v(n) = \frac{D_t^2 f(n) \cdot f(n+1)}{f(n)f(n+1)}, \quad w(n) = \left(\ln \frac{f(n+1)}{f(n)}\right)_t, \tag{5.1}$$

eqs. (3.4)-(3.6) become

$$\begin{aligned} &D_t(D_t^2 f(n) \cdot f(n+1)) \cdot f(n)f(n+1) \\ &\quad = f(n+2)f^3(n) - f(n-1)f^3(n+1) + D_y(D_t f(n+1) \cdot f(n)) \cdot f(n+1)f(n). \end{aligned} \tag{5.2}$$

By introducing an auxiliary variable z, (5.2) also can be decoupled into the bilinear form

$$(D_z - D_t^2) f_{n+1} \cdot f_n = 0, \tag{5.3}$$

$$(D_t D_z - D_t D_y) f_n \cdot f_n = 2 f_{n+1} f_{n-1}. \tag{5.4}$$

or equivalently,

$$f_{n+1z} f_n - f_{n+1} f_{nz} - f_{n+1\,tt} f_n + 2 f_{n+1\,t} f_{nt} - f_{n+1} f_{ntt} = 0, \tag{5.5}$$

$$(f_{ntz} - f_{nty}) f_n - f_{nt}(f_{nz} - f_{ny}) = f_{n+1} f_{n-1}. \tag{5.6}$$

In what follows, we will look for the "molecule solution" to eqs. (5.3) and (5.4). We use the initial condition $f_{-1} = 0$ and $f_0 = 1$. Substituting $f_{-1} = 0$, $f_0 = 1$ into the equation (5.5) and (5.6), we get $f_1 = \phi(y, z, t)$. Here $\phi(y, z, t)$ is an arbitrary function of y, z and t satisfying the relation $\partial_z \phi = \partial_t^2 \phi$.

Let $n = 1$, from (5.6), we get

$$f_2 = \begin{vmatrix} \phi & \phi_t \\ L\phi & (L\phi)_t \end{vmatrix},$$

where $L = \partial_z - \partial_y$. Noticing $\partial_z \phi = \partial_t^2 \phi$, we have

$$\begin{aligned} &(f_{2z} - f_{2tt}) f_1 - f_2(f_{1z} + f_{1tt}) + 2 f_{2t} f_{1t} \\ &= \left(\begin{vmatrix} \phi_z & \phi_t \\ (L\phi)_z & (L\phi)_t \end{vmatrix} + \begin{vmatrix} \phi & \phi_{tz} \\ L\phi & (L\phi)_{tz} \end{vmatrix} - \begin{vmatrix} \phi_t & \phi_{tt} \\ (L\phi)_t & (L\phi)_{tt} \end{vmatrix} - \begin{vmatrix} \phi & \phi_{ttt} \\ L\phi & (L\phi)_{ttt} \end{vmatrix} \right) \phi \\ &\quad -(\phi_z + \phi_{tt}) \begin{vmatrix} \phi & \phi_t \\ L\phi & (L\phi)_t \end{vmatrix} + 2\phi_t \begin{vmatrix} \phi & \phi_{tt} \\ L\phi & (L\phi)_{tt} \end{vmatrix} = 0. \end{aligned} \tag{5.7}$$

So equation (5.5) holds for $n = 1$.

Generalizing the above result, we may deduce the following expression for f_n:

$$f_n = \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} \end{vmatrix}_{n\times n}, \tag{5.8}$$

where $\phi_i^{(j)} = \partial_t^j (L^i \phi)$, $L = \partial_z - \partial_y$.

In order to confirm this, we only have to show that f_n satisfies the eqs. (5.5) and (5.6). In fact, we have the following expressions

$$f_{n+1} = \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n)} \end{vmatrix},$$

$$f_{nz} = \begin{vmatrix} \phi_z & \phi_z^{(1)} & \cdots & \phi_z^{(n-1)} \\ \phi_1 & \phi_2^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1z} & \phi_{n-1z}^{(1)} & \cdots & \phi_{n-1z}^{(n-1)} \end{vmatrix},$$

$$f_{ny} = \begin{vmatrix} \phi_y & \phi_y^{(1)} & \cdots & \phi_y^{(n-1)} \\ \phi_1 & \phi_2^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1y} & \phi_{n-1y}^{(1)} & \cdots & \phi_{n-1y}^{(n-1)} \end{vmatrix},$$

$$f_{nt} = \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n)} \end{vmatrix},$$

$$f_{ntz} = \begin{vmatrix} \phi_z & \phi_z^{(1)} & \cdots & \phi_z^{(n-2)} & \phi_z^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1z} & \phi_{n-1z}^{(1)} & \cdots & \phi_{n-1z}^{(n-2)} & \phi_{n-1z}^{(n)} \end{vmatrix},$$

$$f_{nty} = \begin{vmatrix} \phi_y & \phi_y^{(1)} & \cdots & \phi_y^{(n-2)} & \phi_y^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1y} & \phi_{n-1y}^{(1)} & \cdots & \phi_{n-1y}^{(n-2)} & \phi_{n-1y}^{(n)} \end{vmatrix},$$

$$f_{ntz}-f_{nty}=\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-2}^{(1)} & \cdots & \phi_{n-2}^{(n-2)} & \phi_{n-2}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-2)} & \phi_n^{(n)} \end{vmatrix}, \quad f_{nz}-f_{ny}=\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-2} & \phi_{n-2}^{(1)} & \cdots & \phi_{n-2}^{(n-1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

We introduce the following $(n+1)\times(n+1), n\times n$ and $(n-1)\times(n-1)$ determinants $D, D\begin{bmatrix} i \\ j \end{bmatrix}, D\begin{bmatrix} i & j \\ k & l \end{bmatrix}$;

$$\begin{aligned} D &\equiv f_{n+1}, && (5.9)\\ D\begin{bmatrix} i \\ j \end{bmatrix} &= \text{determinant obtained by eliminating } i\text{-th row and } j\text{-th column} \\ &\quad \text{from } D, && (5.10)\\ D\begin{bmatrix} i & j \\ k & l \end{bmatrix} &= \text{determinant obtained by eliminating } i,j\text{-th rows and } k,l\text{-th columns} \\ &\quad \text{from } D. && (5.11) \end{aligned}$$

Based on the above expressions, on the one hand, we have

$$\begin{aligned} &(f_{ntz}-f_{nty})f_n - f_{nt}(f_{nz}-f_{ny}) \\ &= \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-2} & \phi_{n-2}^{(1)} & \cdots & \phi_{n-2}^{(n-2)} & \phi_{n-2}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-2)} & \phi_n^{(n)} \end{vmatrix} \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} \end{vmatrix} \\ &- \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n)} \end{vmatrix} \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-2} & \phi_{n-2}^{(1)} & \cdots & \phi_{n-2}^{(n-1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} \\ &= D\begin{bmatrix} n \\ n \end{bmatrix} D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} - D\begin{bmatrix} n+1 \\ n \end{bmatrix} D\begin{bmatrix} n \\ n+1 \end{bmatrix} \\ &= DD\begin{bmatrix} n & n+1 \\ n & n+1 \end{bmatrix} \\ &= f_{n+1}f_{n-1}, \end{aligned}$$

on the other hand, we have

$$
\begin{aligned}
f_{n+1z} &= \begin{vmatrix} \phi_z & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n)} \\ \phi_{1z} & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1z} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n)} \\ \phi_{nz} & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi_z^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_{1z}^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1z}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_{nz}^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n+1)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n+1)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n+1)} & \phi_{n-1}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n+1)} & \phi_n^{(n)} \end{vmatrix} + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n+2)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n+2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n+2)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n+2)} \end{vmatrix},
\end{aligned}
$$

$$
f_{n+1t} = \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n+1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n+1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n+1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n+1)} \end{vmatrix},
$$

$$
f_{n+1tt} = \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n)} & \phi^{(n+1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n)} & \phi_1^{(n+1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n)} & \phi_{n-1}^{(n+1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n)} & \phi_n^{(n+1)} \end{vmatrix} + \begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n+2)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n+2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n+2)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n+2)} \end{vmatrix},
$$

$$
\begin{aligned}
&(f_{n+1z}-f_{n+1tt})f_n - f_{n+1}(f_{nz}+f_{ntt}) + 2f_{n+1t}f_{nt} \\
&= -2\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n)} & \phi^{(n+1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n)} & \phi_1^{(n+1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n)} & \phi_{n-1}^{(n+1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n)} & \phi_n^{(n+1)} \end{vmatrix}
\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} \end{vmatrix} \\
&\quad -2\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n)} \end{vmatrix}
\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n+1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n+1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n+1)} \end{vmatrix} \\
&\quad +2\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-1)} & \phi^{(n+1)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-1)} & \phi_1^{(n+1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-1)} & \phi_{n-1}^{(n+1)} \\ \phi_n & \phi_n^{(1)} & \cdots & \phi_n^{(n-1)} & \phi_n^{(n+1)} \end{vmatrix}
\begin{vmatrix} \phi & \phi^{(1)} & \cdots & \phi^{(n-2)} & \phi^{(n)} \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(n-2)} & \phi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-1}^{(1)} & \cdots & \phi_{n-1}^{(n-2)} & \phi_{n-1}^{(n)} \end{vmatrix} \\
&= -2\left(\sum_{i=0}^{n}(-1)^{n+i}\phi_i^{(n+1)}D\begin{bmatrix} i+1 \\ n \end{bmatrix}\right)D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} - 2D\left(\sum_{i=0}^{n-1}(-1)^{n+i+1}\phi_i^{(n+1)}D\begin{bmatrix} i+1 & n+1 \\ n & n+1 \end{bmatrix}\right) \\
&\quad +2\left(\sum_{i=0}^{n}(-1)^{n+i}\phi_i^{(n+1)}D\begin{bmatrix} i+1 \\ n+1 \end{bmatrix}\right)D\begin{bmatrix} n+1 \\ n \end{bmatrix} \\
&= 2\sum_{i=0}^{n-1}(-1)^{n+i}\phi_i^{(n+1)}\left(-D\begin{bmatrix} i+1 \\ n \end{bmatrix}D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} + DD\begin{bmatrix} i+1 & n+1 \\ n & n+1 \end{bmatrix} + D\begin{bmatrix} i+1 \\ n+1 \end{bmatrix}D\begin{bmatrix} n+1 \\ n \end{bmatrix}\right) \\
&= 0.
\end{aligned}
$$

In the above we have used the Jacobi's formula for determinant,

$$
D\times D\begin{bmatrix} i & j \\ k & l \end{bmatrix} - D\begin{bmatrix} i \\ k \end{bmatrix}\times D\begin{bmatrix} j \\ l \end{bmatrix} + D\begin{bmatrix} i \\ l \end{bmatrix}\times D\begin{bmatrix} j \\ k \end{bmatrix} = 0. \tag{5.12}
$$

Therefore, we have the conclusion that f_n given by (5.8) is the bi-directional wronskian solution to eqs. (5.5)-(5.6).

6 Conclusion and discussions

In this paper, we have presented both Casorati and Grammian solutions to the lattice (3.7)-(3.8). Then, we have successfully applied Hirota-Ohta's pfaffinization procedure to the lattice (3.7)-(3.8) to generate a coupled system. Similar to the coupled KP equation[2], it has been shown that the coupled system also has solutions expressed in the form of

Gramm-type pfaffians. For the corresponding molecule equation (5.3)-(5.4), we have found its bi-directional wronskian solutions. Similarly, for the Leznov molecule equation

$$(D_yD_z - 2e^{D_n})f(n)\cdot f(n) = 0, \tag{6.1}$$
$$(D_yD_x - 2D_ze^{D_n})f(n)\cdot f(n) = 0, \tag{6.2}$$

following the same procedure to find molecule solution for the equation (5.3)-(5.4) above,we can find its molecule solution . The solution for eqs. (6.1)-(6.2) can be expressed as

$$\begin{aligned} f_0 &= 1 \\ f_n &= \det\left|\left(\frac{\partial}{\partial z}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{j-1}\Psi(x,y,z)\right|_{1\le i,j\le n}, \end{aligned} \tag{6.3}$$

where $\Psi(x,y,z)$ is an arbitrary function of x, y and z,which satisfies the relation $\partial_x\Psi(x,y,z) = \partial_z^2\Psi(x,y,z)$. A natural number n stands for the position of the molecule as well as a degree of Wronskian.

Acknowledgments. *This work was supported by the National Natural Science Foundation of China (grant no. 10471139), CAS President Grant and the Knowledge Innovation Program of Institute of Computational Math., AMSS, Chinese Academy of Sciences.*

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