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Note on operadic non-associative deformations

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Abstract

Deformation equation of a non-associative deformation in operad is proposed. Its integrability condition (the Bianchi identity) is considered. Algebraic meaning of the latter is explained.

Key words: Operad, deformation, Sabinin principle, Bianchi identity.
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1 Introduction and outline of the paper

Non-associativity is sometimes said to be an *algebraic equivalent* of the differential geometric concept of curvature [3]. To see the equivalence, one must represent an associator in curvature terms. In particular, this can be observed for the *geodesic loops* of a manifold with an affine connection [1, 2].

In this paper, the equivalence is clarified from an operad theoretical point of view. By using the Gerstenhaber brackets and a coboundary operator in an operad algebra, the (formal) associator can be represented as a curvature form in differential geometry. This equation is called a *deformation equation.* Its integrability condition is the Bianchi identity.

2 Operad

Let K be a unital associative commutative ring, $\operatorname{char} K \neq 2, 3$, and let C^n ($n \in \mathbb{N}$) be unital K-modules. For *homogeneous* $f \in C^n$, we refer to n as the *degree* of f and write (when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f \doteq (-1)^n$, $C^f \doteq C^n$ and $\circ_f \doteq \circ_n$. Also, it is convenient to use the *reduced* degree $|f| \doteq n-1$. Throughout the paper we assume that $\otimes \doteq \otimes_K$.

Definition 1. A linear *operad* with coefficients in K is a sequence $C \doteq \{C^n\}_{n\in\mathbb{N}}$ of unital K-modules (an $\mathbb{N}$-graded K-module), such that the following conditions hold.

(1) For $0 \leq i \leq m-1$ there exist *partial compositions*

$$\circ_i \in \operatorname{Hom}(C^m \otimes C^n, C^{m+n-1}), \qquad |\circ_i| = 0$$

(2) For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the *composition relations* hold,

$$(h \circ_i f) \circ_j g = \begin{cases} (-1)^{|f||g|}(h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i-1, \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i+|f|, \\ (-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i+f \leq j \leq |h|+|f|. \end{cases}$$

(3) There exists a unit $\mathrm{I} \in C^1$ such that

$$\mathrm{I} \circ_0 f = f = f \circ_i \mathrm{I}, \qquad 0 \leq i \leq |f|$$

In the 2nd item, the *first* and *third* parts of the defining relations turn out to be equivalent.

Example 2 (endomorphism operad [4]**).** Let L be a unital K-module and $\mathcal{E}_L^n \doteq \mathcal{E}nd_L^n \doteq \operatorname{Hom}(L^{\otimes n}, L)$. Define the partial compositions for $f \otimes g \in \mathcal{E}_L^f \otimes \mathcal{E}_L^g$ as

$$f \circ_i g \doteq (-1)^{i|g|} f \circ (\mathrm{id}_L^{\otimes i} \otimes g \otimes \mathrm{id}_L^{\otimes(|f|-i)}), \qquad 0 \leq i \leq |f|$$

Then $\mathcal{E}_L \doteq \{\mathcal{E}_L^n\}_{n \in \mathbb{N}}$ is an operad (with the unit $\mathrm{id}_L \in \mathcal{E}_L^1$) called the *endomorphism operad* of L.

Thus algebraic operations turn out to be elements of an endomorphism operad. It is convenient to call homogeneous elements of an *abstract* operad the *operations* as well.

3 Gerstenhaber brackets and associator

Definition 3 (total composition). The *total composition* $\bullet C^f \otimes C^g \to C^{f+|g|}$ is defined by

$$f \bullet g \doteq \sum_{i=0}^{|f|} f \circ_i g \quad \in C^{f+|g|}, \qquad |\bullet| = 0$$

The pair $\operatorname{Com} C \doteq \{C, \bullet\}$ is called a *composition algebra* of C.

Lemma 4 (Gerstenhaber identity). *The composition algebra multiplication* $\bullet$ *is non-associative and satisfies the* Gerstenhaber identity

$$\begin{aligned}(h, f, g) &\doteq (h \bullet f) \bullet g - h \bullet (f \bullet g) \\ &= (-1)^{|f||g|}(h, g, f)\end{aligned}$$

Definition 5 (Gerstenhaber brackets). The *Gerstenhaber brackets* $[\cdot,\cdot]$ are defined in $\operatorname{Com} C$ by

$$[f, g] \doteq f \bullet g - (-1)^{|f||g|} g \bullet f = -(-1)^{|f||g|}[g, f], \qquad |[\cdot,\cdot]| = 0$$

The *commutator algebra* of $\operatorname{Com} C$ is denoted as $\operatorname{Com}^- C \doteq \{C, [\cdot,\cdot]\}$.

Theorem 6. $\mathrm{Com}^- C$ *is a* graded Lie algebra.

Proof. The anti-symmetry of the Gerstenhaber brackets is evident. To prove the (graded) *Jacobi identity*

$$(-1)^{|f||h|}[[f,g],h]+(-1)^{|g||f|}[[g,h],f]+(-1)^{|h||g|}[[h,f],g]=0$$

use the G erstenhaber identity. ■

Let $\{L,\mu\}$ be a non-associative algebra with a multiplication $\mu : L\otimes L\to L$. The multiplication μ can be seen as an element of the component $\mathcal{E}_L^2$ of an endomorphism operad $\mathcal{E}_L$. One can easily check that the *associator* of μ reads

$$A \doteq \mu\circ(\mu\otimes \mathrm{id}_L-\mathrm{id}_L\otimes\mu)=\mu\bullet\mu=\frac{1}{2}[\mu,\mu]\doteq\mu^2,\qquad \mu\in\mathcal{E}_L^2$$

So the total composition and Gerstenhaber brackets can be used for representing the associator in operadic terms. This was first noticed by Gerstenhaber [4].

Proposition 7. *If K is a field of characteristic* 0*, then every binary operation $\mu\in C^2$ generates a power-associative subalgebra in* $\mathrm{Com}\, C$.

Proof. Use the Albert criterion [5] that a power-associative algebra over a field K of characteristic 0 can be given by the identities

$$\mu^2\bullet\mu=\mu\bullet\mu^2,\qquad (\mu^2\bullet\mu)\bullet\mu=\mu^2\bullet\mu^2$$

Both identities easily follow from the corresponding Gerstenhaber identities

$$(\mu,\mu,\mu)=0,\qquad (\mu^2,\mu,\mu)=0$$

■

4 Coboundary operator

Let $h\in C$ be an operation from an operad C. By using the Gerstenhaber brackets, define an *adjoint representation* $h\mapsto\partial_h$ of $\mathrm{Com}^- C$ by

$$\partial_h f\doteq \mathrm{ad}_h^{right} f\doteq[f,h],\qquad |\partial_h|=|h|$$

It follows from the Jacobi identity in $\mathrm{Com}^- C$ that ∂_h is a (right) derivation of $\mathrm{Com}^- C$,

$$\partial_h[f,g]=[f,\partial_h g]+(-1)^{|g||h|}[\partial_h f,g]$$

and the following commutation relation holds:

$$[\partial_f,\partial_g]\doteq\partial_f\partial_g-(-1)^{|f||g|}\partial_g\partial_f=\partial_{[g,f]}$$

Let $h\doteq\mu\in C^2$ be a *binary* operation. Then, since $|\mu|=1$ is *odd*, one has

$$\partial_\mu^2=\frac{1}{2}[\partial_\mu,\partial_\mu]=\frac{1}{2}\partial_{[\mu,\mu]}=\partial_{\frac{1}{2}[\mu,\mu]}=\partial_{\mu\bullet\mu}=\partial_{\mu^2}=\partial_A$$

So associativity $\mu^2=0$ implies $\partial_\mu^2=0$. In this case, ∂_μ is called a *coboundary* operator. In particular, for $C=\mathcal{E}_L$ one obtains the *Hochschild coboundary operator* [6]

$$-\partial_\mu f=\mu\circ(\mathrm{id}_L\otimes f)-\sum_{i=0}^{|f|}(-1)^i f\circ(\mathrm{id}_L^{\otimes i}\otimes\mu\otimes\mathrm{id}_L^{\otimes(|f|-i)})+(-1)^{|f|}\mu\circ(f\otimes\mathrm{id}_L)$$

5 Deformation equation

Definition 8 (deformation). For an operad C, let $\mu, \mu_0 \in C^2$ be two binary operations. The difference $\omega \doteq \mu - \mu_0$ is called a *deformation.*

Let $\partial \doteq \partial_{\mu_0}$ and denote the (formal) associators of μ and μ_0 as follows:

$$A \doteq \mu \bullet \mu = \frac{1}{2}[\mu,\mu], \qquad A_0 \doteq \mu_0 \bullet \mu_0 = \frac{1}{2}[\mu_0,\mu_0]$$

Definition 9 (associative deformation). The deformation is called *associative* if $A = 0 = A_0$.

Theorem 10 (deformation equation). *One has*

$$\boxed{\underbrace{A - A_0}_{deformation} \quad = \quad \underbrace{\partial\omega + \frac{1}{2}[\omega,\omega]}_{operadic\ curvature}}$$

Proof. Calculate

$$\begin{aligned}
A &= \frac{1}{2}[\mu,\mu] \\
&= \frac{1}{2}[\mu_0 + \omega, \mu_0 + \omega] \\
&= \frac{1}{2}[\mu_0,\mu_0] + \frac{1}{2}[\mu_0,\omega] + \frac{1}{2}[\omega,\mu_0] + \frac{1}{2}[\omega,\omega] \\
&= A_0 - \frac{1}{2}(-1)^{|\mu_0||\omega|}[\omega,\mu_0] + \frac{1}{2}[\omega,\mu_0] + \frac{1}{2}[\omega,\omega] \\
&= A_0 + [\omega,\mu_0] + \frac{1}{2}[\omega,\omega] \\
&= A_0 + \partial\omega + \frac{1}{2}[\omega,\omega]
\end{aligned}$$

■

6 Sabinin's principle

The deformation equation can be seen as a differential equation for ω with given associators A_0, A. Note that if the associator is fixed, i. e. $A = A_0$, we obtain the *Maurer-Cartan equation*, well-known from the theory of *associative* deformations:

$$A = A_0 \quad \Longleftrightarrow \quad \partial\omega + \frac{1}{2}[\omega,\omega] = 0$$

Thus the deformation equation may be called the *generalized* Maurer-Cartan equation as well. The *Maurer-Cartan expression*

$$\partial\omega + \frac{1}{2}[\omega,\omega]$$

is a well-known defining form for curvature in modern differential geometry. One can see that the associator (deformation) is a formal (operadic) *curvature* while the deformation is working as a *connection.* By reformulating the *Sabinin principle*, one can say that associator is an *operadic* equivalent of the curvature.

7 Bianchi identity

By following a differential geometric analogy, one can state the

Theorem 11 (Bianchi identity). *The associator of the deformed algebra satisfies the Bianchi identity*

$$\boxed{\partial A + [A, \omega] = 0}$$

Proof. First differentiate the deformation equation,

$$\begin{aligned}
\partial(A - A_0) &= \partial^2\omega + \frac{1}{2}\partial[\omega,\omega] \\
&= \partial^2\omega + \frac{1}{2}(-1)^{|\partial||\omega|}[\partial\omega,\omega] + \frac{1}{2}[\omega,\partial\omega] \\
&= \partial^2\omega - \frac{1}{2}[\partial\omega,\omega] + \frac{1}{2}[\omega,\partial\omega] \\
&= \partial^2\omega - \frac{1}{2}[\partial\omega,\omega] - \frac{1}{2}(-1)^{|\partial\omega||\omega|}[\partial\omega,\omega] \\
&= \partial^2\omega - [\partial\omega,\omega]
\end{aligned}$$

Again using the deformation equation, we obtain

$$\begin{aligned}
\partial(A - A_0) &= \partial^2\omega - [\partial\omega,\omega] \\
&= \partial^2\omega - [A - A_0 - \frac{1}{2}[\omega,\omega],\omega] \\
&= \partial^2\omega - [A - A_0,\omega] + \frac{1}{2}[[\omega,\omega],\omega]
\end{aligned}$$

It follows from the Jacobi identity that

$$\partial A_0 = [A_0,\mu_0] = \frac{1}{2}[[\mu_0,\mu_0],\mu_0] = 0, \qquad [[\omega,\omega],\omega] = 0$$

By using these relations we obtain

$$\partial A = \partial^2\omega - [A - A_0,\omega]$$

Recall that $\partial^2 = \partial_{A_o}$ and calculate

$$\begin{aligned}
\partial A + [A,\omega] &= \partial_{A_0}\omega + [A_0,\omega] = [\omega, A_0] + [A_0,\omega] \\
&= -(-1)^{|\omega||A_0|}[A_0,\omega] + [A_0,\omega] \\
&= 0
\end{aligned}$$

■

Remark 12. To clarify algebraic meaning of the Bianchi identity, let us give another proof of the Bianchi identity:

$$\partial A + [A,\omega] = [A,\mu_0] + [A,\mu - \mu_0] = [A,\mu] = \frac{1}{2}[[\mu,\mu],\mu] = 0$$

where the latter equality is evident from the Jacobi identity. But $A \doteq \mu \bullet \mu$ and so the Bianchi identity strikingly reads

$$(\mu \bullet \mu) \bullet \mu = \mu \bullet (\mu \bullet \mu)$$

The latter identity can be easily seen from the Gerstenhaber identity.

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