Journal of Nonlinear Mathematical Physics Volume 13, Number 3 (2006), 315–322 LETTER

A note on q-Bernoulli numbers and polynomials

Taekyun KIM

EECS, Kyungpook National University, Taegu 702-701, S. Korea
E-mail: tkim@knu.ac.kr; tkim64@hanmail.net

Received September 13, 2005; Accepted in Revised Form January 2, 2006

Abstract

Recently, B. A. Kupershmidt have constructed a reflection symmetries of q-Bernoulli polynomials (see [9]). In this paper we give another construction of a q-Bernoulli polynomials, which form Barnes' multiple Bernoulli polynomials at $q = 1$, cf. [1, 13, 14]. By using q-Volkenborn integration, we can also investigate the properties of the reflection symmetries of these' q-Bernoulli polynomials.

1 Introduction

Let $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of integers, the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$ and let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number or a p-adic number. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. In this paper, we use the notation:

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, cf. [2, 3, 4 ,5 ,6, 7, 8]. For any positive integer N, we set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$$

and this can be extended to a distribution on X. This distribution yields an integral for each non-negative integer m:

$$\int_{\mathbb{Z}_p}[x]^m d\mu_q(x) = \int_X [a]^m d\mu_q(a) = \frac{1}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i \frac{i+1}{[i+1]},$$

cf. [5, 20].

The multiple Barnes' Bernoulli polynomials were defined by

$$\left(\prod_{j=1}^{r}\frac{w_j}{e^{w_j t}-1}\right) t^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x|w_1, w_2, \cdots, w_r)\frac{t^n}{n!}, \tag{1.1}$$

for each $w_j > 0, 0 < t < 1$, cf. [1, 13, 14, 20].

The numbers $B_n^{(r)}(w_1, w_2, \cdots, w_r) = B_n^{(r)}(0|w_1, w_2, \cdots, w_r)$ are called Barnes' multiple Bernoulli numbers. Barnes employed a representation for his multiple zeta function in terms of contour integrals, generalizing the Hankel integral representation for the gamma function, cf. [1, 13, 14]. These Barnes' multiple zeta function interpolates Barnes' multiple Bernoulli numbers at negative integers, cf. [1, 10, 11, 12, 13, 14, 20]. In [13], K Ota have studied Kummer-type congruences for derivatives of these' Barnes' multiple Bernoulli polynomials. The identities for Barnes' multiple Bernoulli polynomials are now intensively studied by mathematicians and physicists, cf. [7, 10, 11, 12, 13, 15, 16, 17, 19, 20]. As a side remark, we note that special values of Barnes' multiple zeta function at positive integers have come to the foreground in the recent years, both in connection with theoretical physics (Feynman diagrams) and the theory of mixed Tate motives, cf. [10, 14, 15, 16, 17, 18, 19, 20]. The computation of Feynman diagrams has conforented physicists with classes of integrals that are usually hard to be evaluated, both analytically and numerically, cf. [10, 11, 12, 14, 15, 16, 17].

Throughout this paper, we assume that $\alpha_1, \cdots, \alpha_k$ are taken in the positive integers and let $w \in \mathbb{Z}_p$. Now, we can rewrite the multiple q-Bernoulli numbers, polynomials as follows, cf. [4]:

$$\beta_n^{(r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [w + \alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r),$$

and

$$\beta_n^{(r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r).$$

Originally q-Bernoulli numbers and polynomials were introduced by L. Carlitz in 1948 [2], but they do not seem to be the most natural ones; in particular, they dont't appear as the values at non positive integers of various Riemann and Hurwitz q-zeta functions (see [9: p. 412]). In [8], Koblitz have studied the q-analogue of p-adic L-function which are interpolating q-Bernoulli numbers at negative integers and some properties of Carlitz's q-Bernoulli numbers are studies by many authors, cf. [2, 3, 7, 8, 15, 16, 17, 18, 19, 20, 21]. In the recent paper (see [9]), B A Kupershmidt constructed a reflection symmetries of q-Bernoulli polynomials. These properties are seem to be interesting and worthwhile in the areas of number theory and mathematical physics. In this paper we give another construction of a q-Bernoulli polynomials, which are Barnes' multiple Bernoulli polynomials at $q = 1$. By using q-Volkenborn integration, we can also investigate the properties of the reflection symmetries of q-Bernoulli polynomials. Finally, we give the new explicit formulas which are related to these numbers.

2 An extension of q-Bernoulli numbers and polynomials

For $h \in \mathbb{Z}$, we define the extension of Changhee q-Bernoulli polynomials, numbers as follows:

$$\begin{aligned}&\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r)\\&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}} q^{\sum_{i=1}^{r}(h-i)x_i}[w+\alpha_1x_1+\cdots+\alpha_rx_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r),\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\beta_n^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)\\&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}} [\alpha_1x_1+\cdots+\alpha_rx_r]^n q^{\sum_{i=1}^{r}(h-i)x_i} d\mu_q(x_1)\cdots d\mu_q(x_r).\end{aligned}\tag{2.2}$$

These can be written as

$$\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r)=\sum_{j=0}^{n}\binom{n}{j}q^{wj}\beta_j^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)[w]^{n-j}.$$

By (2.2), we have

$$\begin{aligned}\beta_n^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^n q^{\sum_{i=1}^{r}(h-i)x_i}d\mu_q(x_1)\cdots d\mu_q(x_r)\\&=\frac{1}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{(j\alpha_1+h)(j\alpha_2+h-1)\cdots(j\alpha_r+h-r+1)}{[j\alpha_1+h][j\alpha_2+h-1]\cdots[j\alpha_r+h-r+1]}.\end{aligned}\tag{2.3}$$

Therefore we obtain the following :

Theorem 1. *For any $n \geq 0$, we have*

$$\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r)=\frac{1}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-q^w)^j\prod_{l=1}^{r}\left(\frac{j\alpha_l+h-l+1}{[j\alpha_l+h-l+1]}\right).$$

Remark 1. Note that

$$\beta_0^{(2,1)}(q|1)=\frac{2}{[2]},\quad \beta_1^{(2,1)}(q|1)=\frac{2q+1}{[2][3]},\quad \beta_2^{(2,1)}(q|1)=\frac{2q^2}{[3][4]},$$
$$\beta_3^{(2,1)}(q|1)=-\frac{q^2(q-1)(2[3]+q)}{[3][4][5]},\cdots,$$
$$\beta_0^{(h,1)}(q|1)=\frac{h}{[h]},$$
$$\beta_1^{(h,1)}(q|1)=-\frac{(1+q+\cdots+q^{h-1})+q(1+q+\cdots+q^{h-2})+\cdots+q^{h-1}}{[h][h+1]}\cdots$$
$$\beta_0^{(2,2)}(q|1,1)=\frac{2!}{[2][1]},\quad \beta_1^{(2,2)}(q|1,1)=-\frac{2(q+2)}{[2][3]},$$
$$\beta_2^{(2,2)}(q|1,1)=-\frac{2((q-1)^2+5q)}{[3][4]},\cdots,$$
$$\beta_0^{(r,r)}(q|\underbrace{1,\cdots,1}_{r\text{ times}})=-\frac{r!}{[r][r-1]\cdots[2][1]}.$$

Remark 2. By the definition of $\beta_n^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)$, we note that

$$\beta_n^{(h,r)}(0,q|\alpha_1,\alpha_2,\cdots,\alpha_r)=\beta_n^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r).$$

By (2.1), (2.2), it is easy to see that

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^n q^{\sum_{i=1}^r(h-i)x_i}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=(q-1)\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^{n+1} q^{\sum_{i=1}^r(h-\alpha_i-i)x_i}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$+\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^n q^{\sum_{i=1}^r(h-\alpha_i-i)x_i}d\mu_q(x_1)\cdots d\mu_q(x_r).$$

Thus we have

$$\beta_m^{(h,r)}(q|\underbrace{1,1,\cdots,1}_{r\text{ times}})=(q-1)\beta_m^{(h-1,r)}(q|\underbrace{1,1,\cdots,1}_{r\text{ times}})+\beta_m^{(h-1,r)}(q|\underbrace{1,1,\cdots,1}_{r\text{ times}}).$$

It is easy to see that

$$\underbrace{\int_X \cdots \int_X}_{r \text{ times}} [w+\alpha_1 x_1+\cdots+\alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1)\cdots d\mu_q(x_r)$$

$$= [d]^{n-r} \sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r}$$

$$\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\frac{w+\alpha_1 i_1+\cdots+\alpha_r i_r}{d}+\alpha_1 x_1+\cdots+\alpha_r x_r; q^d]^n$$

$$\times q^{x_1(h-1)d+\cdot+x_r(h-r)d} d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_r). \tag{2.4}$$

From (2.2), (2.4), we have the following:

Theorem 2. *For any positive integer n, we have*

$$\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = [d]^{n-r} \sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r}$$

$$\times \beta_n^{(h,r)}(\frac{w+\alpha_1 i_1+\cdots+\alpha_r i_r}{d}, q^d|\alpha_1,\cdots,\alpha_r).$$

Moreover,

$$\beta_n^{(h,r)}(wd,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = [d]^{n-r} \sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r}$$

$$\times \beta_n^{(h,r)}(w+\frac{\alpha_1 i_1+\cdots+\alpha_r i_r}{d}, q^d|\alpha_1,\cdots,\alpha_r).$$

Remark 3. Note that

$$\lim_{q\to 1} \beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = B_n^{(r)}(w|\alpha_1,\alpha_2,\cdots,\alpha_r), \text{ (see Eq. (1.1))}\ .$$

Hence, $\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r)$ can be considered by the q-analogue of Barnes' multiple Bernoulli polynomials. Now, we will give the inverse formula of Eq. (2.3).

Indeed we see

$$\sum_{i=0}^{n} \binom{n}{i} (q-1)^i \beta_i^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1+\cdots+\alpha_r x_r]^i q^{\sum_{j=1}^r (h-j)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r)$$

$$= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{n(\alpha_1 x_1+\cdots+\alpha_r x_r)} q^{\sum_{j=1}^r (h-j)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r)$$

$$= \frac{(n\alpha_1+h)(n\alpha_2+h-1)\cdots(n\alpha_r+h-r+1)}{[n\alpha_1+h][n\alpha_2+h-1]\cdots[n\alpha_r+h-r+1]}.$$

Therefore we obtain the following:

Theorem 3. *For $h \in \mathbb{Z}$, we have*

$$\sum_{i=0}^{m} \binom{m}{i} (q-1)^i \beta_i^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \prod_{j=1}^{r} \left(\frac{m\alpha_j + h - j + 1}{[m\alpha_j + h - j + 1]} \right).$$

Let χ be a Dirichlet character with conductor $d \in \mathbb{Z}_{\geq \not{\,}}$. Then we define the generalized Changhee q-Bernoulli numbers as follows: For $m \geq 0$,

$$\begin{aligned} &\beta_{m,\chi}^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) \\ &= \underbrace{\int_X \cdots \int_X}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^m q^{\sum_{j=1}^{r}(h-j)x_j} \left(\prod_{j=1}^{r} \chi(x_j) \right) d\mu_q(x_1) \cdots d\mu_q(x_r). \quad (2.5) \end{aligned}$$

By simple calculation, we see that

$$\begin{aligned} &\underbrace{\int_X \cdots \int_X}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^m q^{\sum_{j=1}^{r}(h-j)x_j} \left(\prod_{j=1}^{r} \chi(x_j) \right) d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= [d]^{m-r} \sum_{i_1, \cdots, i_r = 0}^{d-1} q^{hi_1 + \cdots + (h-r+1)i_r} \left(\prod_{j=1}^{r} \chi(x_j) \right) \\ &\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\frac{\alpha_1 i_1 + \cdots + \alpha_r i_r}{d} + \alpha_1 x_1 + \cdots + \alpha_r x_r; q^d]^m \\ &\times q^{\sum_{j=1}^{r} x_j (h-j) d} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_r). \quad (2.6) \end{aligned}$$

By (2.1), (2.6), we have the following :

Theorem 4. *For $h \in \mathbb{Z}$, we have*

$$\begin{aligned} &\beta_{m,\chi}^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) = [d]^{m-r} \sum_{i_1, \cdots, i_r = 0}^{d-1} q^{hi_1 + \cdots + (h-r+1)i_r} \\ &\times \left(\prod_{j=1}^{r} \chi(x_j) \right) \beta_m^{(h,r)}(\frac{\alpha_1 i_1 + \cdots + \alpha_r i_r}{d}, q^d | \alpha_1, \alpha_2, \cdots, \alpha_r). \end{aligned}$$

Remark. By using our formulae in the case of $q = 1$, we can obtain many new formulas which are related to the multiple Barnes' Bernoulli numbers.

Acknowledgments. *The author wishes to express his sincere gratitude to the referee for his/her valuable and worthwhile comments and Prof. N. Euler for his cooperations. This paper was supported by Kyungpook National University. This paper is dedicated to the memory of Sun-Yi Park.*

References

[1] Barnes E W, On the theory of the multiple gamma functions, *Trans. Camb. Philos. Soc.*, **19** (1904), 374–425.

[2] Carlitz L, q-Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987–1000.

[3] Kim T, On a q-analogue of the p-adic log gamma functions and related integrals, *J. Number Theory* **76** (1999), 320–329.

[4] Kim T, Non-Arichimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials, *Russ. J. Math. Phys.* **10** (2003), 91–98.

[5] Kim T, q-Volkenborn integration, *Russ. J. Math. Phys.* **9** (2002), 288–299.

[6] Kim T, q-Riemann zeta functions, *Int. J. Math. Math. Sci.* **9** (2004), 599–605.

[7] Kim T, Ryoo C S, Jang L C and Rim S H, Exploring the q-Riemann zeta function and q-Bernoulli polynomials, *Discrete Dyn. Nat. Soc.* **2** (2004), 171–181.

[8] Koblitz N, On Carlitz's q-Bernoulli numbers, *J. Number Theory* **14** (1982), 332–339.

[9] Kupershmidt B A, Refelection Symmetries of q-Bernoulli polynomials, *J. Nonlinear Math. Phys.* **12** (2005), 412–422.

[10] Kurokawa N, Multiple sine functions and Selberg zeta functions, *Proc. Jpn. Acad. Ser. A-Math. Sci.* **67** (1991), 61–67.

[11] Matsumoto K, The analytic continuation and the asymptotic behavior of certain multiple zeta function, *J. Number Theory* **101** (2003), 223–243.

[12] Nishizawa M, On a q-analogue of the multiple gamma functions, *Lett. Math. Phys.* **37** (1996), 201–209.

[13] Ota K, On Kummer-type congruences for derivatives of Barnes multiple multiple Bernoulli polynomials, *J. Number Theory* **92** (2002), 1–36.

[14] Ruijsenaars S N M, On Barnes' multiple zeta and gamma functions, *Adv. Math.* **156** (2000), 107–132.

[15] Ryoo C S, Kim T, Park D-W and Rim S-H, On the real roots of the Changhee-Barnes' q-Bernoulli polynomials, *JP J. Algebr. Number Theory Appl.* **5** (2005), 293–305.

[16] Ryoo C S, Kim T and Agarwal R P, Exploring the multiple Changhee q-Bernoulli polynomials, *Int. J. Comput. Math.* **82** (2005), 483–493.

[17] Ryoo C S, Kim T and Agarwal R P, A numerical investigation on the structure of the roots of the q-analogue of Euler-Barnes' polynomials, *J. Indian Math. Soc.* **11** (2005), 143–154.

[18] Satoh J, q-analogue of Riemann's ζ-function and q-Euler numbers, *J. Number Theory* **31** (1989), 346–362.

[19] Simsek Y, Kim T and Pyung I-S, Barnes' type multiple Changhee q-zeta functions, *Adv. Stud. Contemp. Math.* **10** (2005), 121–129.

[20] Srivastava H M, Kim T and Simsek Y, q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series, *Russ. J. Math. Phys.* **12** (2005), 241–268.

[21] TSUMURA H, A note on q-analogue of Dirichlet's series, *Proc. Jpn. Acad. Ser. A-Math. Sci.* **75** (1999), 23–25.