Mathematical Model for the most Efficient Elevation of Leaves Absorbing Sunlight

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Abstract. The changes of leaves elevation are mainly influenced by sunlight. This paper makes the ASD (Area exposed to the Sunlight per Day) exposure model of leaves, which is analysis respectively from the tree only a leaf and two leaves, further proving more pieces of the leaves. We get the conclusion that the half of the sun's rays effective angle is the best elevation for any leaves. It provides the general model of leaves elevation changes with theoretical guidance and technical support.

Introduction

If we observe the trees around us carefully, we can find the elevations of leaves are different. What is the reason of making them grow in that way? The main function of the leaves is to absorb the sunlight for photosynthesis [1], so the change of the leaf elevation also is mainly for effective absorption of sunlight.

At present, a few scholars do research on the best elevation, because there are many factors in trees environment which cause the leaf elevation to vary, such as tree gene, geographic position, temperature, influence by surrounding animals which make it difficult to work out a general mathematics model of the best elevation.

This paper simplified model, starting with the leaves of the 2d independent tree (the 2d independent tree refers to a tree of which the leaves can not be influenced by other trees).in the leaves only by the influence of elevation light as assume, make the leaves of ASD exposure model, and get rules of the elevation changes.

Calculation of ASD

Plants can increase the volume and dry weight through photosynthesis [2-3]. The influence of light intensity on photosynthesis is very big, so we analyze the relationship between the sunlight and angle through fully absorbing the sunlight of the leaves.

When in the research of relationship between overlapping leaves and to the absorption of sunlight, we adopt standard of the overall ASD. In the calculation of ASD, in order to make the sun energy received in every moment have uniform standard to measure, We will assume that the ray of sunlight, that shines from one certain angle, is a vector and we will resolve the vector into two directions, one is perpendicular to the surface of the leaf and another is parallel to the surface of the leaf, and we just consider the energy of the vertical one. The function and schematic diagram we defined are as follow:

$$\text{Standard light area} = \text{Actual lighte area} \times \sin \gamma \quad (1)$$

Figure.1 The schematic diagram of sunlight energy conversion

Absorbing sunlight of a leaf

Because the leaves only have one side of the photosynthesis, the leaf is simplified to a line segment (Fig. 2). Discussing ASD with the change of the leaf angle, we calculate the ASD of the leaf through the integral. And finally, we determine the best angle for a leaf to absorb sunlight according to the biggest ASD.

Figure.2 The schematic diagram of a leaf to absorb sunlight

Here, we assume that there is an angle $\theta = \frac{\pi}{4}$ between sunlight and horizontal plane at 9 o'clock in the morning. According to fig. 3, along with the changes of angle α , we can obtain some analyses: when $0 \leq \alpha < \frac{\pi}{4}$ and the angle θ between sunlight and horizontal plane range from $\alpha + \frac{\pi}{2}$ to $\frac{3\pi}{4}$, the sunlight blazed down on the back of the leaf and can't be used for the leaf, so in this case we will not consider this part; when $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} + \alpha$, angle of the sunlight and the leaf is $(\frac{\pi}{2} - \theta + \alpha)$, and then we calculate the ASD through integral. Other situations are considered in the same way. From those, we can obtain the computational expression of ASD in the following paper [4].

$$S(\alpha) = \begin{cases} \frac{12}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2} + \alpha} S_0 \sin(\frac{\pi}{2} - \theta + \alpha) d\theta & 0 \leq \alpha < \frac{\pi}{4} \\ \frac{12}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} S_0 \sin(\frac{\pi}{2} + \theta - \alpha) d\theta & \frac{\pi}{4} \leq \alpha < \frac{3\pi}{4} \\ \frac{12}{\pi} \int_{\alpha - \frac{\pi}{2}}^{\frac{3\pi}{4}} S_0 \sin(\frac{\pi}{2} + \theta - \alpha) d\theta & \frac{3\pi}{4} \leq \alpha < \frac{5\pi}{4} \\ 0 & \frac{5\pi}{4} \leq \alpha < \frac{7\pi}{4} \\ \frac{12}{\pi} \int_{\frac{\pi}{4}}^{2\pi - \alpha} S_0 \sin(\alpha - \theta - \frac{3\pi}{2}) d\theta & \frac{7\pi}{4} \leq \alpha \leq 2\pi \end{cases} \quad (2)$$

Where, S_0 is the area of a leaf, $S(\alpha)$ represents the value of ASD.

Absorbing sunlight of two leaves

Assumption that there are two leaves on the tree, the situation can be illustrated as the following Fig. 3:

Figure.3 The schematic diagram of two leaves to absorb sunlight

When there are two leaves on the tree, we need to consider the mutual effects between the two leaves, for one leaf may overlap the other one.

When $\alpha - \frac{\pi}{2} \geq \theta$, the left leaf can't absorb sunshine with its back to the sun. And the right leaf is sheltered partly. So we need to consider the relation between the two angles to compute ASD.

Here, we just consider one situation when θ meets $\frac{\pi}{4} \leq \theta < \frac{\pi}{2}$. When θ meets $\frac{\pi}{2} \leq \theta \leq \pi$, the situation is similar.

● For leaf A, without the shelter of the leaf B, so the result is the same with the situation that there is only one leaf, then we get

$$S_A = \begin{cases} \frac{1}{15} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} S_0 \sin(\frac{\pi}{2} - \theta + \alpha) d\theta & 0 \leq \alpha < \frac{3\pi}{4} \\ \frac{1}{15} \int_{\alpha - \frac{\pi}{2}}^{\frac{\pi}{2}} S_0 \sin(\frac{\pi}{2} - \theta + \alpha) d\theta & \frac{3\pi}{4} \leq \alpha \leq \pi \end{cases} \quad (3)$$

Where, S_A is the ASD value of the leaf A. The angles α, θ are shown in the picture.

● For the leaf B, the situation can be classified as the following:

➤ When $0 \leq \alpha < \frac{3\pi}{4}$, the leaf A is without the shelter of the leaf B. we get that

$$S_B = \begin{cases} \frac{1}{15} \int_{\frac{\pi}{2} - \beta}^{\frac{\pi}{2}} S_0 \sin(\theta + \beta - \frac{\pi}{2}) d\theta & 0 \leq \beta < \frac{\pi}{4} \\ \frac{1}{15} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} S_0 \sin(\theta + \beta - \frac{\pi}{2}) d\theta & \frac{\pi}{4} \leq \beta \leq \pi \end{cases} \quad (4)$$

Where S_B is the ASD value of the leaf B. The angles β, θ are shown in the picture.

➤ When $\frac{3\pi}{4} \leq \alpha \leq \pi$, the leaf A can partly shelter the leaf B, there are two situations, which can be classified as follows:

✧ When $\alpha - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the leaf A is without the shelter of the leaf B. Then we get

$$S_B = \begin{cases} 0 & 0 \leq \beta < \frac{\pi}{2} - \theta \\ \frac{1}{15} \int_{\alpha - \frac{\pi}{2}}^{\frac{\pi}{2}} S_0 \sin(\frac{3\pi}{2} - \theta - \beta) d\theta & \frac{\pi}{2} - \theta \leq \beta \leq \pi \end{cases} \quad (5)$$

✧ When $\frac{\pi}{4} \leq \theta < \alpha - \frac{\pi}{2}$, if $\beta = \alpha - 2\theta$, the leaf A just wholly covers the leaf B, so we

can make it critical point, we concludes that

$$S_B = \begin{cases} 0 & 0 \leq \beta < \alpha - 2\theta \\ \int_{\frac{\pi}{4}}^{\alpha - \frac{\pi}{2}} S_B^* \sin(\beta + \theta - \frac{\pi}{2}) d\theta & \alpha - 2\theta \leq \beta \leq \pi \end{cases} \quad (6)$$

Where S_B^* is the area that is exposed to the sunlight on the leaf B, which is illustrated as the following Fig. 4:

Figure.4 The analysis chart of shadow area

According to the analysis chart of shadow area (Fig. 4 (a)), we can get the S_B^* :

$$S_B^* = \begin{cases} \frac{1}{8} D_0 L_0 - 2 \int_{\frac{L_0}{2}}^L \frac{D_0}{L_0} (L - L_0) dL & 0 \leq \frac{\sin(\alpha - \theta - \frac{\pi}{2})}{\sin(\theta + \beta - \frac{\pi}{2})} < \frac{1}{2} \\ 2 \int_0^{L^*} \frac{D_0}{L_0} L dL & \frac{1}{2} \leq \frac{\sin(\alpha - \theta - \frac{\pi}{2})}{\sin(\theta + \beta - \frac{\pi}{2})} \leq 1 \end{cases} \quad (7)$$

According to the Sine Theorem, we get that (Fig. 4 (b))

$$\frac{L_0 - L^*}{\sin(\alpha - \theta - \frac{\pi}{2})} = \frac{L_0}{\sin(\theta + \beta - \frac{\pi}{2})} \quad (8)$$

We proceed, we can get that

$$L^* = L_0 - \frac{\sin(\alpha - \theta - \frac{\pi}{2})}{\sin(\theta + \beta - \frac{\pi}{2})} L_0 \quad (9)$$

Where L^* is the length of leaf B's sunlight-exposing part, and L_0 is the length of the leaf.

To sum up, we get

$$S(\alpha, \beta) = S_A + S_B \quad (10)$$

The best angle for a leaf to absorb sunlight

A LEAF

According to the relationship of $S(\alpha)$ and α from the computational expression of ASD in formulas (2), we get the best angle for a leaf to absorb sunlight with the help of Matlab.

Figure.5 the relationship graph between ASD and the angle of a leaf

In Fig. 5, the value of ASD increases firstly and then decreases with the increase of angle α , and achieves maximum in angle of $\frac{\pi}{2}$. Therefore, for a leaf, the best angle to absorb sunlight is $\frac{\pi}{2}$.

Two leaves

According to the formulas (3) to (10), we can get the relation between ASH and the angles α, β by the numerical simulation method, with the help of Matlab [5]. Following is result.

Figure.6 The relation graph between ASH and the angles α, β

In Fig. 6, ASH stands for the overall area exposed to the sunlight per half day.

According to the graphs above, we can conclude that $S(\alpha, \beta)$ has only one extreme point, we can get the coordinate of the extreme point from the section graphics. If we just consider the relation between ASH and α (Fig. 7(a)). When $\alpha = \frac{5\pi}{8}$, we can get the maximum of the ASH. If

we just consider the relation between ASH and the angle β (Fig. 7 (b)). when $\beta = \frac{3\pi}{8}$, we can get the maximum of the ASH. For the maximum point is single, we get the maximum of ASH at the point $\left(\frac{5\pi}{8}, \frac{3\pi}{8}\right)$.

(a) $\alpha - ASH$

(b) $\beta - ASH$

Figure.7 section graphics of ASH

Model analyses

From the analyses and graphs above, we can get the following conclusions:

(1) On the model of two leaves, we just consider the situation of $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$, so the efficient angles of symmetrical leaves are different. It shows that the best elevation of leaf A is $\frac{5\pi}{8}$ when $\theta_1 \in [\frac{\pi}{4}, \frac{\pi}{2}]$; while best elevation of leaf B is $\frac{3\pi}{8}$ when $\theta_2 \in [\frac{\pi}{2}, \frac{3\pi}{4}]$. Its essence is to narrow the illumination angle, solving the best single leaf of elevation. The calculating of leaf A and leaf B's efficient angles are independent. Best for both light elevation angle of average angle.

(2) We use general formula to express the above conclusions. If the scope of the angle of the sunlight is the interval $[\theta_1, \theta_2]$, then we get that the best angle is $\frac{\theta_1 + \theta_2}{2}$. So the surface of the leaf is perpendicular to the axis of symmetry of the rotating area of the sunlight (Fig. 8)

Figure.8 The diagrams of the best angle of two leaves to absorb sunlight

(3) The leaf connection angle is π when there are two leaves in the tree. Why is not $(\frac{\pi}{2}, \frac{\pi}{2})$ for the elevation of two leaves? Such exposure rate should be improved in that condition. The reason is that its essence is to narrow the illumination limit angle, which leads to the slight change of the best leaf elevation.

Due to leaves A and B can get sunlight all of time at the best elevation, so illumination angle is $\theta_1 = \theta_2 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. Thus, the elevation of the leaves should be adjusted to $(\frac{\pi}{2}, \frac{\pi}{2})$.

When illumination angle changes, the best elevations leaves A and B still make themselves get sunlight all of time. The scope of the angle of the sunlight is the interval $[\theta_1, \theta_2]$, then we get that the best angle is $[\frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 + \theta_2}{2}]$.

(4) Expand to the situation of more pieces of the leaves. From the above conclusions, the elevation is only related to illumination effective angle. In the leafy cases, the best elevation angle between the leaves may be covered. Change illumination effective angle $[\theta_1, \theta_2]$ into $[\theta_1^*, \theta_2^*]$, so the best elevation of leaf(i) is $\frac{\theta_1^*(i) + \theta_2^*(i)}{2}$.

Conclusion

This article aims to provide the general model of leaves elevation changes. Suppose that the leaves of the change in elevation only by the influence of light, 2d independent trees exist a general rule: the elevation is the half sum of the upper limit and the lower limit of illumination effective

angle. That illumination effective angle of leaf(i) is $[\theta_1, \theta_2]$, so the best elevation is $\frac{\theta_1(i) + \theta_2(i)}{2}$.

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