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Slashed Moment Exponential Distribution.

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In this article, we introduce a new class of slash distribution, the slashed moment exponential distribution. The proposed distribution can be seen as an extension of the moment exponential distribution proposed in Dara and Ahmad (2012). This extension is more flexible than the moment exponential distribution in terms of kurtosis of distribution. It arises as the ratio of two independent random variables, a moment exponential distribution in the numerator and a power of uniform distribution in the denominator. We study some probability properties, discuss moment and maximum likelihood estimations and derive the reliability function and hazard function. Finally, we present two real data applications indicating that the new distribution can improve the moment exponential and exponentiated moment exponential distributions in fitting real data.

Keywords: Exponentiated moment exponential distribution; kurtosis; moment exponential distribution; slashed moment exponential distribution.

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1. Introduction

A random variable X follows a moment exponential (ME) distribution, denoted as $X \sim ME(\beta)$, if its probability density function (PDF) is given by

$$f_X(x;\beta) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}}, \ x > 0, \ \beta > 0, \tag{1.1}$$

The rth distributional moment associated with the ME distribution is given by

$$\mu_r = E(X^r) = \beta^r \Gamma(r+2), \ r = 1,2,3,... \tag{1.2}$$

where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ is the gamma function. For more details of the ME distribution see Dara and Ahmad (2012).

Hasnain et al. (2015) introduced an extension of ME distribution called exponentiated moment exponential distribution. A random variable X follows an exponentiated moment exponential distribution, denoted as $X \sim EME(\alpha, \beta)$, if its probability density function (PDF) is given by

$$f_X(x;\alpha,\beta) = \frac{\alpha}{\beta^2} \left(1 - \left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}\right)^{\alpha-1} x e^{-\frac{x}{\beta}}, \quad x > 0, \quad \beta, \alpha > 0, \tag{1.3}$$

If $\alpha = 1$, the exponentiated moment exponential distribution reduces to the classical moment exponential distribution.

Gómez et al. (2007) and Gómez and Venegas (2008) introduced the class of slash-elliptical distributions. This class of distributions can be regarded as an extension of the class of elliptical distributions studied in Fang et al. (1990).

A random variable T follows a slash-elliptical distribution with location parameter μ and scale parameter σ, denoted as $T \sim SEl(t;\mu,\sigma,q)$, if it can be represented as

$$T = \sigma \frac{X}{U^{1/q}} + \mu, \tag{1.4}$$

where $X \sim El(0,1,q)$ and $U \sim U(0,1)$ are independent and $q > 0$.

This idea also can be applied to positive random variables as can be seen in (Olivares-Pacheco et al., 2010; Olmos et al., 2012; Iriarte et al., 2014; etc.).

In this work, we introduce an extension of the moment exponential distribution. This extension is more flexible than the moment exponential distribution in terms of kurtosis of distribution. We defined a slashed moment exponential random variable by a quotient between two independent random variables, one (numerator) from the moment exponential family and the other (denominator) a power of the uniform distribution. The new model can be used as an alternative model to the moment exponential and exponentiated moment exponential distributions.

The article is organized as follows. In Section 2 we present the stochastic representation, the density function and some mathematical properties of the new model. In addition, the reliability function and hazard rate function are derived. In Section 3 we discuss the moment and maximum likelihood estimations. In addition, we conduct a simulation study to illustrate the behavior of maximum likelihood estimates. Section 4 presents two applications to real data sets. Final conclusions are reported in Section 5.

2. Slashed moment exponential distribution

In this section we present the stochastic representation, the density function and some mathematical properties of the new distribution.

2.1. *Stochastic representation*

Definition 2.1. A random variable T has a slashed moment exponential distribution, denoted as $T \sim SME(\beta, q)$, if it can be represented as

$$T = \frac{X}{U^{1/q}}, \ q > 0, \tag{2.1}$$

where $X \sim ME(\beta)$ and $U \sim U(0,1)$ are independent.

2.2. *Density function*

Proposition 2.1. *Let $T \sim SME(\beta, q)$. Then, the density function of T is given by*

$$f_T(t;\beta,q) = q\beta^q t^{-(q+1)} \gamma\left(q+2, \frac{t}{\beta}\right), \ t > 0, \tag{2.2}$$

where $\beta, q > 0$ and $\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u} du$ is the incomplete gamma function.

Proof. By using the stochastic representation in (2.1) and the Jacobian method, we have the pdf of T, which can be expressed as

$$f_T(t;\beta,q) = \int_0^1 \frac{q}{\beta^2} t w^{q+1} e^{-\frac{tw}{\beta}} dw.$$

Now, by letting $u = \frac{tw}{\beta}$, the density function $f_T(t;\beta,q)$ reduces to

$$\begin{aligned} f_T(t;\beta,q) &= q\beta^q t^{-(q+1)} \int_0^{t/\beta} u^{q+1} e^{-u} du \\ &= q\beta^q t^{-(q+1)} \gamma\left(q+2, \frac{t}{\beta}\right) \ \square. \end{aligned}$$

Figure 1 depicts some of the shapes that the slashed moment exponential distribution can take for different values of the parameter q.

2.3. *Some properties*

In this subsection some basic properties of the slashed moment exponential distribution are considered.
Let $T \sim SME(\beta, q)$, then

(1) $\lim_{q\to\infty} f_T(t;\beta,q) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}}$,

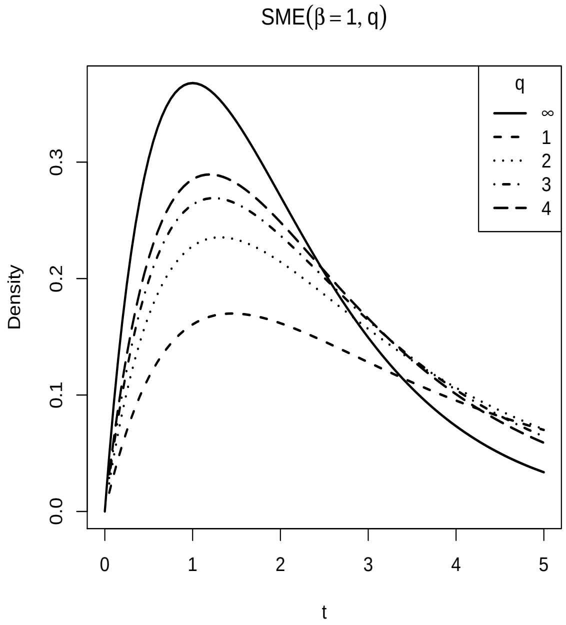

Fig. 1. Plot of the slashed moment exponential distribution, $SME(\beta = 1, q)$.

(2) $F_T(k;\beta,q) = 1 - \left(1+\frac{k}{\beta}\right)e^{-\frac{k}{\beta}} - \frac{k}{q}f_T(k;\beta,q).$

Remark 2.1. Property 1 reveals that as $q \to \infty$ the slashed moment exponential converges to the ordinary moment exponential distribution.

2.4. *Reliability analysis*

The reliability function $R_T(t)$, which is the probability of an item not failing prior to some time t, is defined by $R_T(t) = 1 - F_T(t)$. The reliability function of a slashed moment exponential distribution is given by

$$R_T(t) = \left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}} + \frac{t}{q}f_T(t;\beta,q), \tag{2.3}$$

where $f_T(t;\beta,q)$ is given in (2.2). An interesting characteristic of a random variable is its hazard rate function defined by $h_T(t) = \frac{f_T(t)}{1-F_T(t)}$ which is an important quantity, characterizing the life-time of a certain phenomenon. It can be loosely interpreted as the conditional probability of failure at time t, given it has survived to time t. The hazard rate function for a slashed moment exponential random variable is given by

$$h_T(t) = \frac{f_T(t;\beta,q)}{\left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}} + \frac{t}{q}f_T(t;\beta,q)}, \tag{2.4}$$

where $f_T(t;\beta,q)$ is given in (2.2). Figure 2 displays some plots of the reliability function and the hazard rate function of a slashed moment exponential distribution.

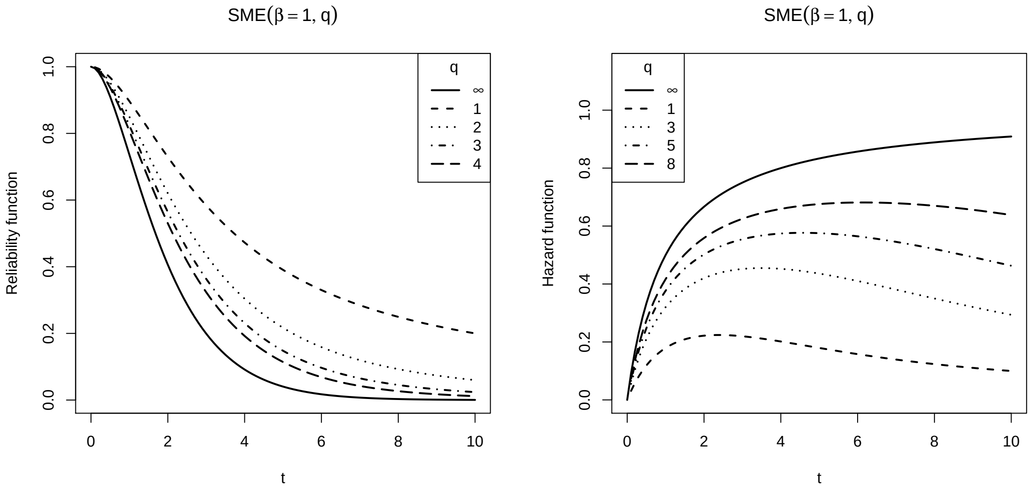

Fig. 2. Plot of the reliability function and hazard function for a slashed moment exponential distribution, $SME(\beta = 1, q)$.

2.5. *Moments and related measures*

In this subsection we derive, distributional moments of the slashed moment exponential distribution, an important requirement in any statistical analysis. Some of the important features and characteristics of a distribution can be studied through moments, which can be used to derive asymmetry and kurtosis coefficients.

Proposition 2.2. *Let $T \sim SME(\beta, q)$. Then, for $r = 1, 2, \ldots$ and $q > r$ it follows that r-th moment is given by*

$$\mu_r = E(X^r) = \frac{q\beta^r}{q-r}\Gamma(r+2), \tag{2.5}$$

where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du$ is the gamma function.

Proof. Using the stochastic representation for the distribution given in (2.1), we have that

$$\mu_r = E(T^r) = E\left(\left(\frac{X}{U^{\frac{1}{q}}}\right)^r\right) = E(X^r)\,E\left(U^{-\frac{r}{q}}\right),$$

where it follows that $E\left(U^{-\frac{r}{q}}\right) = \frac{q}{q-r}$, $q > r$ and $E(X^r) = \beta^r\Gamma(r+2)$ are the moments for the distribution $ME(\beta)$ $\square$.

Corollary 2.1. *Let $T \sim SME(\beta, q)$, then it follows that*

$$E(T) = \frac{2q\beta}{q-1},\ q > 1, \quad \text{and} \quad Var(T) = q\beta^2\left[\frac{6}{q-2} - \frac{4q}{(q-1)^2}\right],\ q > 2.$$

Corollary 2.2. *If $T \sim SME(\beta, q)$, then the coefficients of asymmetry ($\sqrt{\beta_1}$) and kurtosis (β_2) are, respectively,*

$$\sqrt{\beta_1} = \frac{(q-2)^{1/2}A}{(2q)^{1/2}(q-3)C^{3/2}},\ q > 3,$$

$$\text{and}\quad \beta_2 = \frac{(q-2)B}{q(q-3)(q-4)C^2},\ q > 4,$$

where

$$\begin{aligned} A &= 12(q-1)^3(q-2) - 18q(q-1)^2(q-3) + 8q^2(q-2)(q-3), \\ B &= 30(q-1)^4(q-2)(q-3) - 48q(q-1)^3(q-2)(q-4) + 36q^2(q-1)^2(q-3)(q-4) \\ &\quad - 12q^3(q-2)(q-3)(q-4), \\ C &= 3(q-1)^2 - 2q(q-2). \end{aligned}$$

Remark 2.2. Notice that as $q \to \infty$ the asymmetry and kurtosis coefficients take the values $\sqrt{2}$ and 6, respectively, which are the corresponding coefficients for the moment exponential distribution. Figure 3 depicts plots for the asymmetry and kurtosis coefficients, respectively, of the slashed moment exponential distribution.

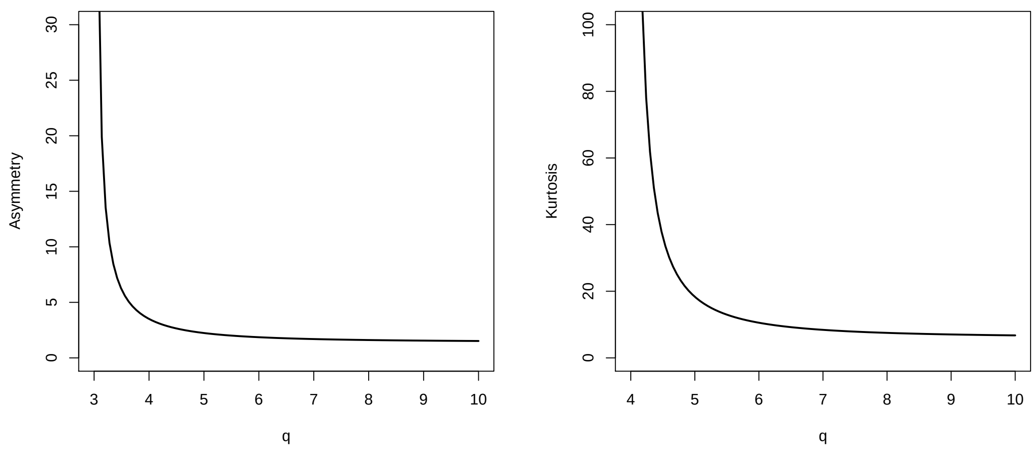

Fig. 3. Plot of the asymmetry and kurtosis coefficients for a slashed moment exponential distribution, $SME(\beta, q)$.

3. Inference

In this section we discuss moment and maximum likelihood estimation for parameters β and q for the slashed moment exponential distribution. Additionally, we conduct a small scale simulation study illustrating the ML estimations behavior for parameters β and q in small and moderate sample sizes.

3.1. *Moment estimators*

Proposition 3.1. *Let $T_1,\ldots,T_n$ a random sample for the random variable $T \sim SME(\beta,q)$. Then, moment estimators for $\theta = (\beta,q)$, are given by*

$$\widehat{\beta}_M = \frac{\overline{T}(\widehat{q}-1)}{2\widehat{q}} \;\; and \;\; \widehat{q}_M = 1+\left(\frac{2\overline{T}^2}{2\overline{T^2}-3\overline{T}^2}\right)^{1/2} \;\; ,if \; 2\overline{T^2} > 3\overline{T}^2,$$

where $\overline{T}$ is the sample mean, and $\overline{T^2}$ is the sample mean for square of the sample units.

Proof. Using (2.5), it follows that

$$E(T) = \frac{2q\beta}{q-1} \;\; and \;\; E(T^2) = \frac{6q\beta^2}{q-2} \;,if \;\; q>2, \tag{3.1}$$

and replacing $E(T)$ by $\overline{T}$ and $E(T^2)$ by $\overline{T^2}$ in (3.1), we obtain a system of equations for which the solution leads to the moment estimators $(\widehat{\beta}_M,\widehat{q}_M)$ for (β,q) □.

3.2. *Maximum likelihood estimators*

For a random sample $T_1,\ldots,T_n$ from the distribution $SME(\beta,q)$, the log-likelihood function can be written as

$$\ell(\beta,q) = n\log(q)+nq\log(\beta)-(q+1)\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\gamma\left(q+2,\frac{t_i}{\beta}\right), \tag{3.2}$$

so that the maximum likelihood equations are given by

$$\sum_{i=1}^{n}\frac{\gamma_1\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} = -\frac{nq}{\beta}, \tag{3.3}$$

$$\sum_{i=1}^{n}\log(t_i)-\sum_{i=1}^{n}\frac{\gamma_2\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} = \frac{n}{q}+n\log(\beta), \tag{3.4}$$

where $\gamma(\alpha,x) = \int_0^x u^{\alpha-1}e^{-u}\,du$ is the incomplete gamma function and

$$\gamma_1\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial\beta}\gamma\left(q+2,\frac{t_i}{\beta}\right) = -\frac{t_i^{q+2}}{\beta^{q+3}}e^{-\frac{t_i}{\beta}},$$

$$\gamma_2\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial q}\gamma\left(q+2,\frac{t_i}{\beta}\right) = \int_0^{t_i/\beta} u^{q+1}\log(u)e^{-u}\,du.$$

It is well known that as the sample size increases, the distribution of the MLE tends (under regularity conditions) to normal distribution with mean (β,q) and covariance matrix equal to the inverse of the Fisher (expected) information matrix. Due to the complexity of the likelihood function it is not possible to obtain its analytical expression. It is possible, however, to work with the observed information matrix, which is a consistent estimator for the expected information matrix.

3.3. *Observed information matrix*

The observed information matrix follows from the Hessian matrix, replacing unknown parameters by their MLEs. Some algebraic manipulation yields the following Hessian matrix:
Let $T \sim SME(\beta, q)$, so that the observed information matrix is given by

$$I_n(\beta,q) = \begin{pmatrix} \frac{\partial^2 l(\beta,q)}{\partial\beta^2} & \frac{\partial^2 l(\beta,q)}{\partial\beta\partial q} \\ \frac{\partial^2 l(\beta,q)}{\partial q\partial\beta} & \frac{\partial^2 l(\beta,q)}{\partial q^2} \end{pmatrix},$$

such that

$$\frac{\partial^2 l(\beta,q)}{\partial\beta^2} = -\frac{nq}{\beta^2} + \sum_{i=1}^{n} \frac{\gamma_{11}\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} - \sum_{i=1}^{n} \frac{\gamma_1^2\left(q+2,\frac{t_i}{\beta}\right)}{\gamma^2\left(q+2,\frac{t_i}{\beta}\right)},$$

$$\frac{\partial^2 l(\beta,q)}{\partial q\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{\gamma_{12}\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} - \sum_{i=1}^{n} \frac{\gamma_1\left(q+2,\frac{t_i}{\beta}\right)\gamma_2\left(q+2,\frac{t_i}{\beta}\right)}{\gamma^2\left(q+2,\frac{t_i}{\beta}\right)},$$

$$\frac{\partial^2 l(\beta,q)}{\partial\beta\partial q} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{\gamma_{21}\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} - \sum_{i=1}^{n} \frac{\gamma_1\left(q+2,\frac{t_i}{\beta}\right)\gamma_2\left(q+2,\frac{t_i}{\beta}\right)}{\gamma^2\left(q+2,\frac{t_i}{\beta}\right)},$$

$$\frac{\partial^2 l(\beta,q)}{\partial q^2} = -\frac{n}{q} + \sum_{i=1}^{n} \frac{\gamma_{22}\left(q+2,\frac{t_i}{\beta}\right)}{\gamma\left(q+2,\frac{t_i}{\beta}\right)} - \sum_{i=1}^{n} \frac{\gamma_2^2\left(q+2,\frac{t_i}{\beta}\right)}{\gamma^2\left(q+2,\frac{t_i}{\beta}\right)},$$

where γ_1 and γ_2 are given in previous subsection, and

$$\gamma_{11}\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial\beta}\gamma_1\left(q+2,\frac{t_i}{\beta}\right) = \frac{t_i^{q+2}}{\beta^{q+4}}(q+3-t_i)e^{-\frac{t_i}{\beta}},$$

$$\gamma_{12}\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial q}\gamma_1\left(q+2,\frac{t_i}{\beta}\right) = -\frac{t_i^{q+2}}{\beta^{q+3}}\log\left(\frac{t_i}{\beta}\right)e^{-\frac{t_i}{\beta}},$$

$$\gamma_{21}\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial\beta}\gamma_2\left(q+2,\frac{t_i}{\beta}\right) = -\frac{t_i^{q+2}}{\beta^{q+3}}\log\left(\frac{t_i}{\beta}\right)e^{-\frac{t_i}{\beta}},$$

$$\gamma_{22}\left(q+2,\frac{t_i}{\beta}\right) = \frac{\partial}{\partial q}\gamma_2\left(q+2,\frac{t_i}{\beta}\right) = \int_0^{t_i/\beta} u^{q+1}\log^2(u)e^{-u}\,du.$$

3.4. *Simulation study*

In this subsection, simulation is performed to illustrate the behavior of the ML estimators for parameters β and q. We generate 1000 random samples of sizes $n = 50$, $n = 100$ and $n = 200$ from the distribution $SME(\beta,q)$ for fixed values of the parameters. Random numbers $T \sim SME(\beta,q)$ can be generated as

(1) Generate $Y \sim Uniform(0,1)$
(2) Set $X = -b\left(W_{-1}(\frac{Y-1}{e})+1\right)$
(3) Generate $U \sim Uniform(0,1)$
(4) Set $T = XU^{-1/q}$.

where W_{-1} is the negative branch of the LambertW function, see Corless et al. (1996). Measures and empirical standard deviations(SD) are presented in Table 1. Here, the parameters are well estimated and the estimates are asymptotically unbiased.

Table 1. Maximum likelihood estimators for samples generated for several values of the parameters β and q.

Maximum Likelihood Estimators							
Parameters		$n=50$		$n=100$		$n=200$	
β	q	$\widehat{\beta}$ (SD)	$\widehat{q}$ (SD)	$\widehat{\beta}$ (SD)	$\widehat{q}$ (SD)	$\widehat{\beta}$ (SD)	$\widehat{q}$ (SD)
0.5	0.5	0.559 (0.130)	0.548 (0.107)	0.531 (0.090)	0.523 (0.063)	0.521 (0.061)	0.514 (0.043)
	1.0	0.527 (0.121)	1.091 (0.311)	0.519 (0.076)	1.050 (0.178)	0.507 (0.054)	1.023 (0.119)
	1.5	0.531 (0.111)	1.720 (0.596)	0.518 (0.072)	1.634 (0.394)	0.503 (0.052)	1.536 (0.233)
1.0	1.0	1.048 (0.240)	1.084 (0.306)	1.024 (0.158)	1.049 (0.180)	1.010 (0.108)	1.016 (0.115)
	1.5	1.064 (0.227)	1.744 (0.696)	1.028 (0.147)	1.595 (0.376)	1.011 (0.102)	1.547 (0.234)
	2.0	1.065 (0.226)	2.342 (0.971)	1.019 (0.150)	2.182 (0.666)	1.018 (0.106)	2.127 (0.433)
2.0	1.5	2.111 (0.446)	1.745 (0.690)	2.041 (0.304)	1.598 (0.455)	2.024 (0.208)	1.555 (0.238)
	2.0	2.108 (0.433)	2.287 (0.924)	2.057 (0.308)	2.241 (0.789)	2.037 (0.212)	2.134 (0.469)
	2.5	2.099 (0.408)	2.741 (0.950)	2.058 (0.310)	2.770 (0.899)	2.030 (0.214)	2.701 (0.711)

4. Illustrations

In this section we analyze two real data sets using slashed moment exponential distribution. In addition, we analyze the data sets using the moment exponential and exponentiated moment exponential distributions. In each illustration, we deliver evidence that the slashed moment exponential distribution can present a better fit to the data than the other distributions.

4.1. *Illustration 1*

The first data set corresponds to the stress-rupture life of Kevlar 49/epoxy which is subject to constant pressure at 90% stress level until failure, so we have complete data with the exact failure times. For previous studies with this data set, see Andrews and Herzberg (1985), Barlow et al. (1984). Table 2 shows some descriptive statistics from the dataset, where b_1 and b_2 are sample skewness and kurtosis coefficients, respectively.

Table 2. Summary statistics for data set.

n	$\overline{X}$	s^2	b_1	b_2
101	1.024	1.252	2.957	16.379

From the results in Subsection 3.1, the moment estimates for the parameters of the SME distribution are $\widehat{\beta}_M = 2.510$ and $\widehat{q}_M = 2.155$. Using these estimates as starting values for the Newton-Raphson procedure, maximum likelihood estimates were computed. Table 3 depicts parameter estimates with the respective standard errors for SME, EME and ME models using the maximum likelihood (ML) approach. Standard errors (SE) were computed using the inverse of the Hessian matrix. The results of the SME fitting were compared to those provided by the ME and EME distributions. We consider the usual Akaike information criterion (AIC) introduced by Akaike (1974) and Bayesian information criterion (BIC) proposed by Schwarz (1978), which are defined as AIC=$2k - 2\log L$ and BIC=$k\log n - 2\log L$, respectively, where k is the number of parameters in the model, n is the sample size and $\log L$ is the maximized value of the likelihood function. Table 3 shows the corresponding AIC and BIC values for each fitted distribution. For these data, AIC and BIC show that the SME model provides a better fit. Figure 4 presents the histogram for the data with the fitted densities.

Table 3. Maximum likelihood estimates for SME, EME and ME models and AIC and BIC values for the data set.

Parameters	SME (SE)	EME (SE)	ME (SE)
β	2.310 (0.308)	6.488 (0.694)	4.682 (0.292)
α	-	0.570 (0.067)	-
q	1.851 (0.381)	-	-
$\log L$	-411.7963	-414.5258	-426.7965
AIC	827.592	833.051	855.593
BIC	833.296	838.755	858.445

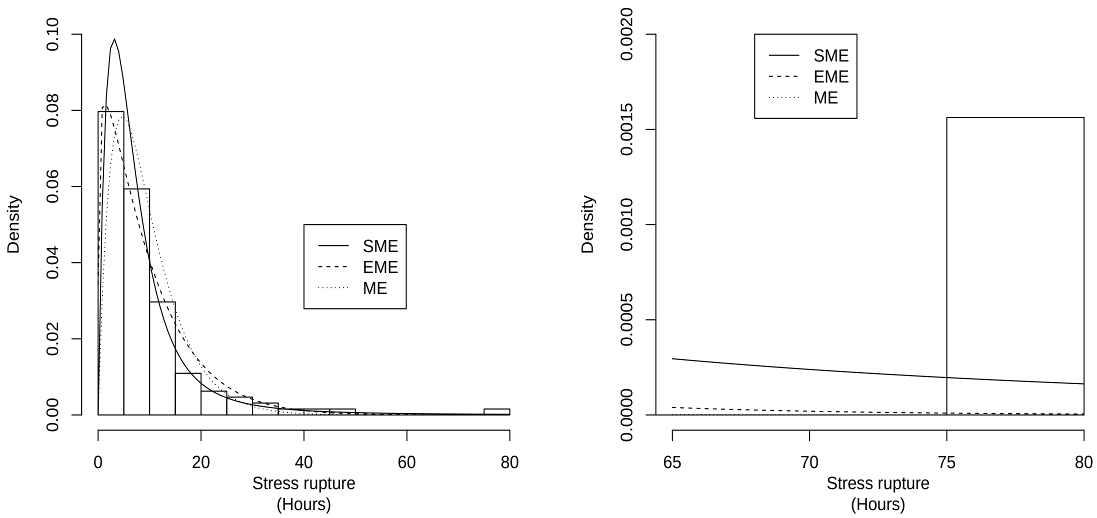

Fig. 4. Left panel: Models fitted by the maximum likelihood approach for data set. Right panel: Plots of the tails for the models.

4.2. *Illustration 2*

The second data set was previously analyzed in Chhikara and Folks (1977). It corresponds to the 46 active repair times (in hours) for an airborne communication transceiver. Table 4 shows some descriptive statistics from the dataset.

Table 4. Summary statistics for data set.

n	$\overline{X}$	s^2	b_1	b_2
46	3.607	24.445	2.888	11.803

Based on the sample above, using the results from Subsection 3.1, we calculate the moment estimators for the parameters of the SME model: $\widehat{\beta}_M = 0.769$ and $\widehat{q}_M = 1.822$. Using these estimates as starting values for the Newton-Raphson procedure, maximum likelihood estimates were computed. Table 5 shows the corresponding AIC and BIC values for each fitted distribution. For these data, AIC and BIC show that the SME model provides a better fit. Figure 5 presents the histogram for the data with the fitted densities.

Table 5. Maximum likelihood estimates for SME, EME and ME models and AIC and BIC values for the data set.

Parameters	SME (SE)	EME (SE)	ME (SE)
β	0.528 (0.130)	2.799 (0.541)	1.705 (0.177)
α	-	0.452 (0.087)	-
q	1.215 (0.334)	-	-
$\log L$	-97.81989	-103.3944	-112.8952
AIC	199.6398	210.7888	227.7904
BIC	203.2971	214.4461	229.619

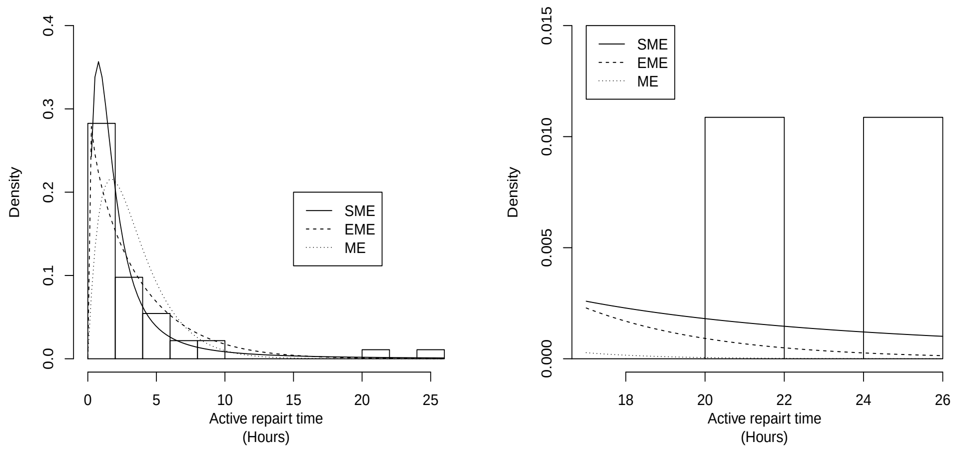

Fig. 5. Left panel: Models fitted by the maximum likelihood approach for data set. Right panel: Plots of the tails for the models.

5. Concluding remarks

This article introduces an extension of the moment exponential distribution discussed in Dara and Ahmad (2012). The extension is called the slashed moment exponential distribution. This distribution arises from the ratio between two independent random variables: the moment exponential distribution in the numerator and the power of uniform random variable in the denominator. The resulting model potentially has a larger kurtosis coefficient than the moment exponential distribution. Moment estimators for the new distribution are obtained explicitly and can be used as initial values for the computation of the maximum likelihood estimators which requires numerical procedures such as the Newton-Raphson algorithm. The derivation of the asymmetry and kurtosis coefficients illustrates the fact that the new distribution is able to fit data sets for which the moment exponential distribution is adequate but with excessive kurtosis. Applications to real data demonstrate that the new distribution can present a better fit than distributions such as the moment exponential and exponentiated moment exponential.

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