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Statistical Inference for Lindley Model based on Type II Censored Data

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In this paper, the moment-based, maximum likelihood and Bayes estimators for the unknown parameter of the Lindley model based on Type II censored data are discussed. The expectation maximization (EM) algorithm and direct maximization methods are used to obtained the maximum likelihood estimator (MLE). Existence and uniqueness of the moment-based and maximum likelihood estimators are discussed and a bias corrected estimator based on parametric bootstrap is developed. For Bayesian estimation, since the Bayes estimator cannot be obtained in an explicit form, two approximations based on Lindley and the importance sampling methods are used. Asymptotic confidence intervals, bootstrap confidence intervals and credible intervals are also proposed. Based on Type II censored data, the prediction of future observations is discussed. The analysis of a real data has been presented for illustrative purposes. Finally, Monte Carlo simulations are performed to compare the performances of the proposed estimation methods.

Keywords: Lindley model; Type II censoring; Maximum likelihood estimator; Bayes estimator; Importance Sampling.

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1. Introduction

The Lindley distribution has the probability density function (pdf)

$$f(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \quad ,x>0,\ \ \theta>0. \tag{1.1}$$

and the cumulative distribution function (cdf)

$$F(x;\theta)=1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}, \quad ,x>0,\ \ \theta>0. \tag{1.2}$$

The Lindley distribution was originally introduced by Lindley [15] in the context of Bayesian statistics. In recent years, this distribution has been studied and generalized by several authors, see, for example, Ghitany et al. [12], Zakerzadeh and Dolati [27], Ghitany et al. [11], Bakouch et al. [7], Ghitany et al. [10], Torabi et al. [23], Asgharzadeh et al. [4] and Asgharzadeh et al. [5]. Estimation for the Lindley distribution were discussed by Krishna and Kumar [14], Ali et al. [2], Gupta and Singh [13], Al-Mutairi et al. [3]. Recently, Asgharzadeh et al. [6] discussed the inferential methods for the Lindley distribution based on record data.

In this paper, we aim to study the point and interval estimation of the parameter in the Lindley distribution and to study the prediction of future failures based on Type II censored data. We obtain the moment-based estimator (MBE), maximum likelihood estimator (MLE) and the Bayes estimator for the parameter of Lindley distribution. The existence and uniqueness of the MBE and MLE are discussed. The prediction of future observations is also developed based on Type II censored data. Although the estimation of parameter of the Lindley distribution has been discussed extensively in the literature, a comprehensive comparison of different methods for estimation has not been done. Moreover, another contribution of our work is the development of the MBE and proofing the existence and uniqueness of the MBE and MLE. In addition, the Bayesian approaches proposed in this paper are also different from the existing methods. Specifically, we attempt using the importance sampling method to compute the predictive density and the corresponding credible interval. Nevertheless, we have also discussed the prediction problem for future failures which has not been considered before.

This paper is organized as follows. In Section 2, we discuss the point and interval estimation methods based on frequentist approach. We propose a moment-based estimation method and develop the EM algorithm for the computation of the MLE. The existence and uniqueness of the MBE and MLE are discussed. Difference confidence interval construction methods for the model parameter are then discussed. Bootstrap method based on the MLE for point and interval estimation is also discussed in Section 2. Bayesian estimator and the corresponding credible interval are developed in Section 3. In Section 4, we derive the prediction of future observations based on Type II censored data. In Section 5, a real data analysis is presented to illustrate the methodologies developed here. In Section 6, a Monte Carlo simulation study is used to study the performances of the proposed methodologies and recommendations are provided. Finally, in Section 7, we discuss how the methodologies developed in this paper can be extended to other censoring schemes.

2. Point and Interval Estimation based on Frequentist Approach

Suppose it is planned that the life-testing experiment will be terminated as soon as the m-th (where m is pre-fixed) failure is observed. Then, only the first m failures out of n units under test will be observed. The data obtained from such an experiment will be referred to as a *Type II censored sample*. Suppose $X_{1:n} < X_{2:n} < \cdots < X_{m:n}$ is an ordered Type II censored sample from a population with pdf $f(x;\theta)$ and cdf $F(x;\theta)$ in Eqs. (1.1) and (1.2), respectively. For notation simplicity, we denote the observed values of $X_{1:n} < X_{2:n} < \cdots < X_{m:n}$ by $x_1 < x_2 < \cdots < x_m$ instead of $x_{1:n} < x_{2:n} < \cdots < x_{m:n}$.

2.1. *Moment-based estimation*

Moment-based estimation of the shape parameter θ of the Lindley distribution can be obtained through the spacings of the transformed Type II censored sample $X_{1:n} < X_{2:n} < \cdots < X_{m:n}$ as follows. Let us define

$$Y_{i:n} = -\ln[1-F(X_{i:n},\theta)] = \theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right), \quad i = 1,2,\ldots,m,$$

then $Y_{1:n} < Y_{2:n} < \cdots < Y_{m:n}$ are Type II censored sample from a standard exponential distribution. Moreover, the spacings

$$\begin{cases} Z_1 = nY_{1:n} \\ Z_2 = (n-1)(Y_{2:n} - Y_{1:n}) \\ \vdots \\ Z_m = (n-m+1)(Y_{m:n} - Y_{m-1:n}) \end{cases}$$

are independently and identically distributed random variables from a standard exponential distribution. Therefore, we have

$$\begin{aligned} Q(\theta) = 2\sum_{i=1}^{m} Z_i &= 2\sum_{i=1}^{m-1} Y_{i:n} + 2(n-m+1)Y_{m:n}] \\ &= 2\sum_{i=1}^{m} c_i Y_{i:n} \\ &= 2\sum_{i=1}^{m} c_i \left[\theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right)\right] \end{aligned}$$

has a chi-square distribution with $2m$ degrees of freedom, where $c_1 = \cdots = c_{m-1} = 1$ and $c_m = n-m+1$. Now, since $\overline{Z} = (\sum_{i=1}^{m} Z_i)/m \to 1$ in probability as $m \to \infty$, by setting the left hand side equals to 1, the moment-based estimate (MBE) of θ, denoted as $\hat{\theta}_{MB}$, can be computed as the solution of the nonlinear equation

$$\sum_{i=1}^{m} c_i \left[\theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right)\right] = m.$$

Note that

$$Q(\theta) = 2\sum_{i=1}^{m} c_i \left[\theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right)\right]$$

is a continuous function of θ on $(0,\infty)$ and

$$\frac{d}{d\theta}\left[\theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right)\right] = \frac{\theta(X_{i:n}^2 + \theta X_{i:n} + \theta X_{i:n}^2 + 2X_{i:n} + 1)}{(1+\theta)(1+\theta+\theta X_{i:n})} \geq 0.$$

This implies that $\frac{d}{d\theta}Q(\theta) \geq 0$ and consequently, $Q(\theta)$ is an monotonic increasing function in θ. Therefore, the existence and uniqueness of the MBE of θ, which is a solution to $Q(\theta) = 2m$, is guaranteed.

2.2. *Maximum likelihood estimation*

The likelihood function can be written as

$$L(\theta|\mathbf{x}) = \frac{n!}{(n-m)!}\left\{\prod_{i=1}^{m} f(x_i;\theta)\right\}[1-F(x_m;\theta)]^{n-m},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. For the Lindley distribution with pdf in (1.1) and cdf in (1.2), the likelihood function is given by

$$\begin{aligned} L(\theta|\mathbf{x}) &= C\prod_{i=1}^{m}\left[\frac{\theta^2}{1+\theta}(1+x_i)e^{-\theta x_i}\right]\left[\frac{1+\theta+\theta x_m}{1+\theta}e^{-\theta x_m}\right]^{n-m} \\ &= \frac{C\theta^{2m}}{(1+\theta)^n}\prod_{i=1}^{m}(1+x_i)[1+\theta+\theta x_m]^{n-m}e^{-\theta[\sum_{i=1}^{m} x_i+(n-m)x_m]}, \end{aligned}$$

where C is a normalizing constant independent of the parameter θ. Hence, the log-likelihood function can be expressed as

$$\begin{aligned} \ln L(\theta|\mathbf{x}) &= \text{constant} + 2m\ln\theta - n\ln(1+\theta) + (n-m)\ln(1+\theta+\theta x_m) \\ &- \theta\left[\sum_{i=1}^{m} x_i + (n-m)x_m\right]. \end{aligned}$$

From the log-likelihood function, we obtain the normal equation as

$$\frac{d\ln L(\theta)}{d\theta} = \frac{2m}{\theta} - \frac{n}{1+\theta} + \frac{(n-m)(1+x_m)}{1+\theta+\theta x_m} - \left[\sum_{i=1}^{m} x_i + (n-m)x_m\right] = 0.$$

The MLE of θ, denoted as $\hat{\theta}_{ML}$, can be obtained by solving the normal equation. We have

$$-\frac{d^2\ln L(\theta)}{d\theta^2} = \frac{2m}{\theta^2} - \frac{n}{(1+\theta)^2} + \frac{(n-m)(1+x_m)^2}{(1+\theta+\theta x_m)^2} \tag{2.1}$$

and the observed Fisher information can be obtained by substituting $\hat{\theta}$ in place of θ in Eq. (2.1). The asymptotic variance of the MLE, denoted by σ^2, can be computed by taking the inverse of the observed Fisher information.

The following theorem shows the existence and uniqueness of the MLE of θ based on Type II censored sample.

Theorem 1. *Suppose we have observed the Type II censored sample $\mathbf{x} = (x_1, \cdots, x_m)$, where X follows the Lindley distribution. Then, the MLE of θ exists and is unique if $m > \frac{n}{2}$.*

Proof. Define $\varphi(\theta) = \frac{d \ln L(\theta)}{d\theta}$. We note that

$$\lim_{\theta \to 0} \varphi(\theta) = +\infty, \quad \lim_{\theta \to +\infty} \varphi(\theta) = -\sum_{i=1}^{m} x_i - (n-m)x_m < 0,$$

and

$$\varphi'(\theta) = \frac{d^2 \ln L(\theta)}{d\theta^2} = \frac{(n-2m)\theta^2 - 4m\theta - 2m}{\theta^2(1+\theta)^2} - \frac{(n-m)(1+x_m)^2}{(1+\theta+\theta x_m)^2} < 0.$$

Therefore, $\varphi(\theta)$ is a continuous function on $(0,\infty)$ which decreases monotonically from $+\infty$ to negative values. Therefore, the MLE of θ which is a solution to $\varphi(\theta) = 0$, exists and is unique, if $m > \frac{n}{2}$. In other words, the MLE exists and it is unique, if the censoring proportion is less than 50%.

Besides obtaining the MLE by using direct optimization using numerical methods such as the Newton-Raphson method, the expectation-maximization (EM) algorithm is considered here. Since the MLE based on complete sample is in closed-from and the moments of the left-truncated Lindley distribution can be computed easily, the EM algorithm has the advantage over direct maximization here.

2.2.1. *EM algorithm*

Consider the random variable Y follows the left-truncated Lindley distribution truncated at T with pdf

$$\begin{aligned} g(y) &= \frac{f(y)}{1-F(T)} \\ &= \frac{\theta^2}{\theta+1+\theta T}(1+y)e^{-\theta(y-T)}, y > T, \theta > 0, \end{aligned}$$

the expected value of Y is

$$E(Y) = \frac{1}{\theta(\theta+1+\theta T)}\left[2+(2T+1)\theta+T(1+T)\theta^2\right].$$

Let Y_s, $s = 1,2,\ldots,(n-m)$, be the censored lifetimes under the Type II censoring scheme, then Y_s follows the left-truncated Lindley distribution with $T = x_m$. Suppose $\theta_{(h)}$ is the estimate of θ in the h-th iteration, then in the E-step of the $(h+1)$-th iteration of the EM-algorithm, one requires to compute

$$E(Y_s) = \frac{1}{\theta_{(h)}(\theta_{(h)}+1+\theta_{(h)}x_m)}\left[2+(2x_m+1)\theta_{(h)}+x_m(1+x_m)\theta_{(h)}^2\right],$$

$s = 1,2,\ldots,(n-m)$. Since the log-likelihood function of θ based on a complete sample $x_1,x_2,\ldots,x_n$ is

$$\ln L(\theta|\mathbf{x}) = \text{constant} + 2n\ln\theta - n\ln\ln(1+\theta) - \theta\sum_{i=1}^{n} x_i$$

which gives the MLE of θ as

$$\hat{\theta} = \frac{-(\bar{X}-1)+\sqrt{(\bar{X}-1)^2+8\bar{X}}}{2\bar{X}},$$

where $\bar{X} = \sum_{i=1}^{n} x_i/n$, then, in the $(h+1)$-th iteration of the M-step in the EM-algorithm, the value of $\theta_{(h+1)}$ is computed by the following formula:

$$\hat{\theta}_{(h+1)} = \frac{-(\bar{X}_{(h)} - 1) + \sqrt{(\bar{X}_{(h)} - 1)^2 + 8\bar{X}_{(h)}}}{2\bar{X}_{(h)}},$$

where $\bar{X}_{(h)} = \frac{1}{n}\left[\sum_{i=1}^{m} x_i + (n-m)E(Y_s)\right]$. The MLE of θ can be obtained by repeating the E-step and M-step until convergence occurs. A reasonable starting value for $\theta_{(0)}$ is the estimate of the parameter based on the "pseudo-complete" sample by replacing the censored observations Y_s by $x_m, s = 1,2,\ldots,(n-m)$.

Louis [17] developed a procedure for extracting the observed information matrix when the EM-algorithm is used to find the MLE in incomplete data problem. The idea of the procedure can be expressed by the *missing information principle* (see, for example, Louis [17] and Tanner [22]):

$$\begin{aligned}\text{Observed information } &= \text{Complete information} - \text{Missing information}\\ &= I_C(\theta) - I_M(\theta).\end{aligned}$$

The complete information based on a complete sample of size n is

$$I_C(\theta) = \frac{2n}{\theta} - \frac{n}{(\theta+1)^2}$$

and the missing information based on a Type II censored sample with effective sample size m is

$$I_M(\theta) = \frac{2(n-m)}{\theta} - \frac{(n-m)(1+x_m)^2}{(\theta+1+\theta x_m)^2}.$$

This result in the same expression of the observed information presented in Eq. (2.1) and the asymptotic variance of $\hat{\theta}$ can be estimated as $Var(\hat{\theta}) = 1/I(\hat{\theta})$, where $I(\theta) = I_C(\theta) - I_M(\theta)$.

2.3. *Bootstrap Estimation*

Based on the MLE described in Section 2.2, we can develop bias-adjusted estimator based on the parametric bootstrap method (Efron and Tibshirani, [9]). The detailed description of the bootstrap method and the computational formula of the bootstrap bias-adjusted estimator will be discussed in Section 2.4.3.

2.4. *Interval estimation*

2.4.1. *Exact confidence interval*

We know that the pivot

$$Q(\theta) = Q(\theta;\underline{X}) = 2\sum_{i=1}^{m} c_i \left[\theta X_{i:n} - \ln\left(\frac{1+\theta+\theta X_{i:n}}{1+\theta}\right)\right]$$

has a chi-square distribution with $2m$ degrees of freedom, where $c_1 = \cdots = c_{m-1} = 1$ and $c_m = n-m+1$. So, a $100(1-\alpha)\%$ confidence interval for θ can be constructed from the relation

$$\Pr\left(\chi^2_{\alpha/2,2m} < Q(\theta) < \chi^2_{1-\alpha/2,2m}\right) = 1-\alpha,$$

where $\chi^2_{\alpha/2,2m}$ and $\chi^2_{1-\alpha/2,2m}$ denote the lower and upper $\alpha/2$ percentage points of a chi-square distribution with $2m$ degrees of freedom. Since $Q(\theta)$ is strictly increasing in θ, an exact $100(1-\alpha)\%$ confidence interval for θ based on the pivotal quantity $Q(\theta)$ can be computed as $\left(Q^{-1}(\chi^2_{\alpha/2,2m}), Q^{-1}(\chi^2_{1-\alpha/2,2m})\right)$, where $Q^{-1}(t)$ is the solution of θ for the equation $Q(\theta) = t$.

2.4.2. *Asymptotic confidence intervals based on MLE*

Let us now consider the asymptotic confidence interval using the asymptotic normality of the MLE. Following the general asymptotic theory of MLE, the sampling distribution of

$$\frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}}$$

can be approximated by a standard normal distribution. A two-sided $100(1-\alpha)\%$ normal-approximation confidence interval for θ can then be constructed as

$$[\hat{\theta}_l, \hat{\theta}_u] = \hat{\theta} \pm z_{1-\frac{\alpha}{2}} \sqrt{Var(\hat{\theta})},$$

where z_q is the q-th percentile of the standard normal distribution. Since θ is a positive parameter and the lower end of the confidence interval in (2.2) could be less than zero, a modified confidence interval of θ can be obtained as

$$[\hat{\theta}_{l+}, \hat{\theta}_u] = \left[\max\left(0, \hat{\theta} - z_{1-\alpha/2}\sqrt{Var(\hat{\theta})}\right), \hat{\theta} + z_{1-\alpha/2}\sqrt{Var(\hat{\theta})}\right].$$

Alternatively, to ensure the resulting confidence limits for θ to be positive, it is also possible to use a logarithm transformation to obtain an approximate confidence interval for θ (see, for example, Meeker and Escobar, [18]) by approximating the distribution of $\frac{\ln(\hat{\theta})-\ln(\theta)}{\sqrt{Var(\ln(\hat{\theta}))}}$ by a standard normal distribution, where $Var(\ln(\hat{\theta}))$ can be approximated by delta method as

$$\widehat{Var}(\ln(\hat{\theta})) = \frac{\widehat{Var}(\hat{\theta})}{\hat{\theta}^2}.$$

The resulting $100(1-\alpha)\%$ approximate confidence interval for θ based on the logarithm transformed MLE is then given by

$$[\hat{\theta}_l^*, \hat{\theta}_u^*] = \left[\frac{\hat{\theta}}{\exp\left(\frac{z_{1-\frac{\alpha}{2}}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}\right)}, \hat{\theta}\cdot\exp\left(\frac{z_{1-\frac{\alpha}{2}}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}\right)\right].$$

2.4.3. *Parametric bootstrap method*

In this section, we construct confidence intervals based on the percentile parametric bootstrap method (Efron and Tibshirani, [9]) as well as develop bias-adjusted estimator based on parametric bootstrap. To obtain the percentile bootstrap confidence intervals, we use the following algorithm:

1. Based on the original sample $\mathbf{x} = (x_1, x_2, \cdots, x_m)$, obtain $\hat{\theta}$, the MLE of θ.

2. Simulate the first m order statistics from a sample of size n from Lindley distribution with parameter $\hat{\theta}$: $\mathbf{x}^{(1)} = (x_{1:n}^{(1)}, x_{2:n}^{(1)}, \ldots, x_{m:n}^{(1)})$.
3. Compute the MLE of θ based on $\mathbf{x}^{(1)}$, say $\hat{\theta}^{(1)}$.
4. Repeat Steps 2 – 3 B times and obtain $\hat{\theta}^{(1)}, \hat{\theta}^{(2)}, \ldots, \hat{\theta}^{(B)}$.
5. Arrange $\hat{\theta}^{(1)}, \hat{\theta}^{(2)}, \ldots, \hat{\theta}^{(B)}$ in ascending order and obtain $\hat{\theta}^{[1]}, \hat{\theta}^{[2]}, \ldots, \hat{\theta}^{[B]}$.

A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval of θ, say $[\theta_L^*, \theta_U^*]$, is then given by

$$\theta_L^* = \hat{\theta}^{([B\alpha/2])}, \qquad \theta_U^* = \hat{\theta}^{([B(1-\alpha/2)])},$$

Based on the bootstrap samples, we can also approximate the bias of the MLE as $\widehat{Bias} = \hat{\theta}_{(\cdot)} - \hat{\theta}$, where $\hat{\theta}_{(\cdot)} = \frac{1}{B}\sum_{i=1}^{B} \hat{\theta}^{[i]}$. The bootstrap estimator $\hat{\theta}_{(\cdot)}$ can be viewed as a bias-adjusted estimator, denoted as $\hat{\theta}_B$, because $\hat{\theta}_B = \hat{\theta} - \widehat{Bias} = \hat{\theta}_{(\cdot)}$.

3. Bayesian estimation and credible interval

In the Bayesian approach, θ is considered a random variable having a specific prior distribution. We consider the gamma prior with shape parameter a and rate parameter b, denoted as $G(a,b)$, for θ which has pdf

$$\pi(\theta; a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \quad \theta > 0, \tag{3.1}$$

where $a > 0$ and $b > 0$. Most often, the parameters a and b are obtained from the past history. The gamma distribution is chosen to be the prior distribution here because the posterior distribution of θ, given the data, can be written as a product of a gamma pdf and a term involving θ and x_m,

$$\pi(\theta|\mathbf{x}) \propto \theta^{2m+a-1} \exp\left\{-\theta\left[\sum_{i=1}^{m} x_i + (n-m)x_m + b\right]\right\} \times \frac{(1+\theta+\theta x_m)^{n-m}}{(1+\theta)^n}. \tag{3.2}$$

In fact, one can consider other probability distributions with positive support as the prior distribution of θ.

To obtain the Bayes estimator of θ, we consider the squared error loss (SEL) function

$$L_1(\theta, \widehat{\theta}) = (\widehat{\theta} - \theta)^2.$$

We also consider the linear exponential (LINEX) loss function

$$L_2(\theta, \widehat{\theta}) = e^{c(\widehat{\theta}-\theta)} - c(\widehat{\theta} - \theta) - 1,$$

which is one of the most popular asymmetric loss functions. This loss function was introduced by Varian [24]. For the LINEX loss function, the sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa.) For c close to zero, the LINEX loss is approximately squared error loss (SEL) and therefore almost symmetric.

The Bayes estimator of θ under the SEL, $\widehat{\theta}_{BS}$, is

$$\widehat{\theta}_{BS} = E_\theta(\theta|\mathbf{x}) = \frac{\int_0^\infty \theta\pi(\theta|\mathbf{x})d\theta}{\int_0^\infty \pi(\theta|\mathbf{x})d\theta}. \tag{3.3}$$

Also, the Bayes estimator of θ under the LINEX loss , $\widehat{\theta}_{BL}$, is

$$\widehat{\theta}_{BL} = -\frac{1}{c}\log\left(E_\theta(e^{-c\theta}|\mathbf{x})\right) = -\frac{1}{c}\log\left(\frac{\int_0^\infty e^{-c\theta}\pi(\theta|\mathbf{x})d\theta}{\int_0^\infty \pi(\theta|\mathbf{x})d\theta}\right). \tag{3.4}$$

Due to the complex form of $\pi(\theta|\mathbf{x})$, the Bayes estimators of θ cannot be obtained in closed forms. Here, we consider two methods namely (i) Lindley's approximation method; and (ii) importance sampling method to obtain the approximate Bayes estimator.

3.1. *Lindley's approximation*

The Lindley's approximation was originally proposed by Lindley [15] to approximate the ratio of two integrals such as (3.3) and (3.4). This method has been used in the literature to approximate the Bayes estimator, see, for example, Lindley [16] and Press [19].
In the one parameter case, the form of Lindley's approximation for any function of θ, say $U(\theta)$, reduces to the following form:

$$E(U(\theta)|\mathbf{x}) = U(\theta) + \frac{1}{2}\left[U_2 + 2U_1\rho_1\right][\sigma(\theta)]^2 + \frac{1}{2}l_3U_1[\sigma(\theta)]^4, \tag{3.5}$$

where $l_i = \frac{\partial^i}{\partial\theta^i}\ln L(\theta|\mathbf{x})$, for $i = 1,2,3$, $U_j = \frac{\partial^j}{\partial\theta^j}U(\theta)$, for $j = 1,2$, $\rho_1 = \frac{\partial}{\partial\theta}\rho(\theta)$, $\rho(\theta) = \ln\pi(\theta)$ and $[\sigma(\theta)]^2$ is the inverse of Fisher information in (3.3).
To apply the Lindley's approximation, we first obtain

$$l_3(\theta) = \frac{4m}{\theta^3} - \frac{2n}{(1+\theta)^3} + \frac{2(n-m)x_m(1+x_m)^2}{(1+\theta+\theta x_m)^3}.$$

From the prior density in (3.1), we observe that $\rho_1 = \frac{a-1}{\theta} - b$. Under SEL, $U(\theta) = \theta$ and for LINEX loss function $U(\theta) = e^{-c\theta}$. Substitution these in (3.5), we obtain the Bayes estimate of θ using Lindley's approximation method under SEL as

$$\widehat{\theta}_{BLS} = \hat{\theta} + \left(\frac{a-1}{\hat{\theta}} - b\right)\sigma^2(\hat{\theta}) + \frac{1}{2}l_3(\hat{\theta})\sigma^4(\hat{\theta}). \tag{3.6}$$

and under LINEX loss function as

$$\widehat{\theta}_{BLL} = \hat{\theta} - \frac{1}{c}\log\left\{1 + \left(\frac{c^2}{2} - c(\frac{a-1}{\hat{\theta}} - b)\right)\sigma^2(\hat{\theta}) - \frac{c}{2}l_3(\hat{\theta})\sigma^4(\hat{\theta})\right\}, \tag{3.7}$$

where $\hat{\theta}$ is MLE of θ.

3.2. *Importance sampling method*

Besides the Lindley's approximation method, the importance sampling procedure can be used to obtain the Bayes estimator of θ. Importance sampling is considered here instead of direct sampling because importance sampling is more efficient in the sense that the variance of the resulting estimator obtained from importance sampling is smaller. Note that the posterior density of θ can be written as

$$\pi(\theta|\mathbf{x}) \propto \pi\left(\theta; 2m+a, \sum_{i=1}^{m} x_i + (n-m)x_m + b\right) g(\theta|\mathbf{x}), \tag{3.8}$$

where $\pi(\theta;\cdot,\cdot)$ is presented in equation (3.1) and

$$g(\theta|\mathbf{x}) = \frac{(1+\theta+\theta x_m)^{n-m}}{(1+\theta)^n}. \tag{3.9}$$

We now apply the importance sampling scheme to generate samples from the posterior distribution $\pi(\theta|\mathbf{x})$ using the following algorithm:

Step 1. Generate $\theta_1, \cdots, \theta_M$ from $G\left(2m+a, \sum_{i=1}^{m} x_i + (n-m)x_m + b\right)$.

Step 2. Obtain an approximate Bayes estimate under SEL as

$$\hat{\theta}_{BMS} = \frac{\sum_{i=1}^{M} \theta_i g(\theta_i|\mathbf{x})}{\sum_{i=1}^{M} g(\theta_i|\mathbf{x})}. \tag{3.10}$$

and under LINEX loss function as

$$\hat{\theta}_{BML} = -\frac{1}{c}\log\left(\frac{\sum_{i=1}^{M} e^{-c\theta_i} g(\theta_i|\mathbf{x})}{\sum_{i=1}^{M} g(\theta_i|\mathbf{x})}\right). \tag{3.11}$$

Then, the credible interval of θ can be obtained by using the results in Chen and Shao [8]. Let $\pi(\theta|\mathbf{x})$ and $\Pi(\theta|\mathbf{x})$ be the posterior density and posterior distribution functions of θ, respectively and let $\theta^{(\beta)}$ be the β-th quantile of θ, i.e,

$$\theta^{(\beta)} = \inf\{\theta : \Pi(\theta|\mathbf{x}) \geq \beta\},\ 0 < \beta < 1. \tag{3.12}$$

For a given θ^*, we have $\Pi(\theta^*|\mathbf{x}) = E\{I_{\theta \leq \theta^*}(\theta)|\mathbf{x}\}$, where I_A is the indicator function such that $I_A(\theta) = 1$ if A is true and $I_A(\theta) = 0$ otherwise. Therefore, a simulation consistent estimator of $\Pi(\theta^*|\mathbf{x})$ can be obtained as

$$\hat{\Pi}(\theta^*|\mathbf{x}) = \frac{\frac{1}{M}\sum_{i=1}^{M} I_{\theta_i \leq \theta^*}(\theta) g(\theta_i|\mathbf{x})}{\frac{1}{M}\sum_{i=1}^{M} g(\theta_i|\mathbf{x})}. \tag{3.13}$$

Let $\{\theta_{(i)}\}$, for $i = 1, \ldots, M$, be the ordered values of θ_i, and

$$w_i = \frac{g(\theta_{(i)}|\mathbf{x})}{\sum_{i=1}^{M} g(\theta_{(i)}|\mathbf{x})}$$

be the associated weight, then we have

$$\hat{\Pi}(\theta^*|\mathbf{x}) = \begin{cases} 0, & \text{for } \theta^* < \theta_{(1)}, \\ \sum_{j=1}^{i} w_j, & \text{for } \theta_{(i)} \leq \theta^* < \theta_{(i+1)}, \\ 1, & \text{for } \theta^* \geq \theta_{(M)}. \end{cases} \tag{3.14}$$

Hence, $\theta^{(\beta)}$ can be approximated by

$$\hat{\theta}^{(\beta)} = \begin{cases} \theta_{(1)}, & \text{for } \beta = 0, \\ \theta_{(i)}, & \text{for } \sum_{j=1}^{i-1} w_j < \beta \leq \sum_{j=1}^{i} w_j. \end{cases} \tag{3.15}$$

To obtain a $100(1-\beta)\%$ highest posterior density (HPD) credible interval for θ, consider intervals of the form

$$R_j = \left[\hat{\theta}^{\left(\frac{j}{M}\right)}, \hat{\theta}^{\left(\frac{j+[(1-\beta)M]}{M}\right)}\right], \tag{3.16}$$

for $j = 1,2,\ldots,M-[(1-\beta)M]$, where $[a]$ denotes the largest integer less than or equal to a. Finally, among all R_j, $j = 1,2,\ldots,M-[(1-\beta)M]$, choose the interval which has the smallest length.

4. Prediction of Future Failures

In this section, we discuss the prediction of the censored lifetimes $X_{i:n}$, $(i = m+1, m+2, \ldots, n)$, based on the observed Type II censored sample $\mathbf{x} = (x_1, x_2, \cdots, x_m)$. Due to the well-known Markovian property of Type II right censored order statistics, the density of $Y_s = X_{(m+s):n}$ given $X_m = x_m$ is the same as the density of the s-th order statistic of a sample of size $n-m$ from the population with the right truncated density $f(y)/(1-F(x_m)), y \geq x_m$. Therefore, the conditional density of Y_s given $X_m = x_m$ is given by

$$f_s(y|x_m) = s\binom{n-m}{s} f(y)\,[F(y)-F(x_m)]^{s-1}\,[1-F(y)]^{n-m-s} \times [1-F(x_m)]^{-(n-m)}. \tag{4.1}$$

For the Lindley distribution with pdf and cdf in (1.1) and (1.2), respectively, equation (4.1) can be written as

$$f(y|x_m,\theta) = s\binom{n-m}{s}\theta^2(1+y)(1+\theta+\theta y)^{n-m-s} e^{-\theta(n-m-s+1)y} e^{\theta(n-m)x_m} \times (1+\theta+\theta x_m)^{-(n-m)} \left[(1+\theta+\theta x_m)e^{-\theta x_m} - (1+\theta+\theta y)e^{-\theta y}\right]^{s-1}. \tag{4.2}$$

Using the binomial expansion

$$\left[(1+\theta+\theta x_m)e^{-\theta x_m} - (1+\theta+\theta y)e^{-\theta y}\right]^{s-1} = \sum_{k=1}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}(1+\theta+\theta x_m)^k e^{-\theta k x_m}(1+\theta+\theta y)^{s-k-1}e^{-\theta(s-k-1)y},$$

the conditional density of $Y_s = X_{s:n-m}$ given $X_m = x_m$, can be expressed as

$$f(y|x_m,\theta) = s\binom{n-m}{s}\theta^2(1+y)\sum_{k=1}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}(1+\theta+\theta y)^{n-m-k-1} \times \exp\{-\theta(n-m-k)(y-x_m)\}\,(1+\theta+\theta x_m)^{k-n+m}. \tag{4.3}$$

The Bayes predictive density function of Y given x_m is given by

$$f_s^*(y|x_m) = \int_0^\infty f(y|x_m,\theta)\pi(\theta|x_m)d\theta. \tag{4.4}$$

Based on the priors, the joint posterior density function of θ given the data is

$$\pi(\theta|x_m) \propto \pi\left(\theta; 2m+a, \sum_{i=1}^{m} x_i + (n-m)x_m + b\right) g(\theta|x_m) \tag{4.5}$$

Substituting (4.5) in (4.4), the predictive density function $f_s^*(y|x_m)$ can be obtained as

$$f_s^*(y|x_m) = \int_0^\infty f(y|x_m,\theta) G\left(2m+a, \sum_{i=1}^{m} x_i + (n-m)x_m + b\right) g(\theta|x_m) d\theta. \tag{4.6}$$

The Bayesian point predictors can be obtained from the predictive density function $f_s^*(y|x_m)$ and given the loss function. The Bayesian point predictor of $Y_s = X_{s:n-m}$ under the SEL is

$$\widehat{Y}_{SEP} = \int_{x_m}^\infty y f_s^*(y|x_m) dy. \tag{4.7}$$

Since (4.7) can not be computed explicitly, we adopt here the following algorithm to obtain the Bayesian point predictor:

Step 1. Generate $\theta_1, \cdots, \theta_M$ from $G(2m+a, \sum_{i=1}^m x_i + (n-m)x_m + b)$.

Step 2. Obtain the simulation consistent estimators of $f_s^*(y|x_m)$ by the importance sampling technique as

$$\widehat{f_s^*}(y|x_m) = \frac{\sum_{i=1}^M f(y|x_m,\theta_i) g(\theta_i|x_m)}{\sum_{i=1}^M g(\theta_i|x_m)}. \tag{4.8}$$

Step 3. By using (3.9), (4.7) and (4.8), the Bayes predictor of the future failure under SEL, $\widehat{Y}_{s,SP}$, can be obtained as

$$\widehat{Y}_{s,SP} = \frac{\sum_{i=1}^M I(x_m,\theta_i) g(\theta_i|x_m)}{\sum_{i=1}^M g(\theta_i|x_m)}, \tag{4.9}$$

and under LINEX loss function as

$$\widehat{Y}_{s,LP} = -\frac{1}{c} \log\left(\frac{\sum_{i=1}^M J(x_m,\theta_i) g(\theta_i|x_m)}{\sum_{i=1}^M g(\theta_i|x_m)}\right), \tag{4.10}$$

where $I(x_m,\theta)$ and $J(x_m,\theta)$ are given by

$$I(x_m,\theta) = s\binom{n-m}{s}\theta^2 \sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}\frac{e^{\theta(n-m-k)x_m}}{(1+\theta+\theta x_m)^{n-m-k}} \times \int_{x_m}^\infty y(1+y)(1+\theta+\theta y)^{n-m-k-1} e^{-\theta(n-m-k)y} dy.$$

and

$$J(x_m,\theta) = s\binom{n-m}{s}\theta^2 \sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}\frac{e^{\theta(n-m-k)x_m}}{(1+\theta+\theta x_m)^{n-m-k}} \times \int_{x_m}^\infty e^{-cy}(1+y)(1+\theta+\theta y)^{n-m-k-1} e^{-\theta(n-m-k)y} dy.$$

respectively.

Bayesian prediction intervals are obtained from the Bayes predictive density $f_s^*(y|x_m)$. The $100(1-\alpha)\%$ Bayesian prediction interval for Y_s is given by $(L(x_m), U(x_m))$, where $L(x_m)$ and $U(x_m)$ can be obtained by solving the following two nonlinear equations simultaneously

$$\Pr(Y > L(x_m)|x_m) = \int_{L(x_m)}^{\infty} f_s^*(y|x_m)dy = 1 - \frac{\alpha}{2}, \tag{4.11}$$

$$\Pr(Y > U(x_m)|x_m) = \int_{U(x_m)}^{\infty} f_s^*(y|x_m)dy = \frac{\alpha}{2}. \tag{4.12}$$

By using $\widehat{f_s^*}(y|x_m)$ defined in (4.8) to approximate $f_s^*(y|x_m)$, we can approximate $L(x_m)$ and $U(x_m)$ as

$$1 - \frac{\alpha}{2} = \frac{\frac{1}{M}\sum_{i=1}^{M} K(L(x_m), \theta_i) g(\theta_i|x_m)}{\frac{1}{M}\sum_{i=1}^{M} g(\theta_i|x_m)}, \tag{4.13}$$

and

$$\frac{\alpha}{2} = \frac{\frac{1}{M}\sum_{i=1}^{M} K(U(x_m), \theta_i) g(\theta_i|x_m)}{\frac{1}{M}\sum_{i=1}^{M} g(\theta_i|x_m)}, \tag{4.14}$$

where

$$K(z,\theta) = s\binom{n-m}{s}\theta^2 \sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}\frac{e^{\theta(n-m-k)z}}{(1+\theta+\theta z)^{n-m-k}} \times \int_z^{\infty}(1+y)(1+\theta+\theta y)^{n-m-k-1}e^{-\theta(n-m-k)y}dy. \tag{4.15}$$

5. Real Data Analysis

To illustrate the methodologies developed in this paper, we present the analysis of a real data set here. The following data set presented in Wang [25] contains the failure time of 18 electronic devices under a life test. This data set has been analyzed recently by a number of authors in different studies (see, for example, Xie et al. [26], Rao [20], Rezaei and Tahmasbi [21] and Abd-Elrahman [1]).

5	11	21	31	46	75	98	122	145
165	195	224	245	293	321	330	350	420

To check the validity of using Lindley distribution to fit this data set, Kolmogorov-Smirnov (K-S) test is applied. The the K-S statistic of the distance between the fitted and the empirical distribution function (based on the parameter $\theta = 0.01156$) is 0.1751 and the corresponding p-value is 0.5961. Therefore, it is reasonable to use the Lindley distribution to fit the data.

Suppose that the life test ended when the 15-th observation is observed, i.e., we observe a Type II censored sample with $n = 18$ and $m = 15$. Based on the estimation methods presented in Sections 2 – 4, the point and interval estimates of the Lindley parameter θ are summarized in Table 1. We also compute the Bayesian point and interval predictors for the future lifetimes. The results are presented in Table 2. Note that for computing Bayes estimators and 95% HPD credible intervals, since we do not have any prior information, we used the non-informative prior with parameters $a = b = 0$.

Table 1. Point and interval estimates of the Lindley parameter in the illustrative example based on different methods.

Method		Point Estimate
BME		0.01042
MLE		0.01086
Bootstrap		0.01120
Bayes (Lindley)	Square Error	0.01086
	LINEX($c=-1$)	0.01085
	LINEX($c=0.1$)	0.01085
	LINEX($c=1$)	0.01085
Bayes (Importance Sampling)	Square Error	0.01076
	LINEX($c=-1$)	0.01076
	LINEX($c=0.1$)	0.01076
	LINEX($c=1$)	0.01076
Method		**Interval Estimate**
Exact method		(0.00702, 0.01435)
Normal-approximation of MLE		(0.00634, 0.01312)
Normal-approximation of log-MLE		(0.00687, 0.01378)
Percentile Bootstrap		(0.00788, 0.01588)
HPD credible interval		(0.00756, 0.01440)

Table 2. Bayesian point and interval predictors of the censored lifetimes in the illustrative example

	Observed value	Point prediction				Interval prediction
		SE	LINEX			
			$c=-1$	$c=0.1$	$c=1$	
$s=1$	330	341.578	337.064	340.261	346.548	(301.991 , 370.368)
$s=2$	350	364.889	355.618	365.247	374.167	(312.414 , 506.451)
$s=3$	420	437.427	424.641	439.051	451.062	(388.772 , 540.697)

6. Monte Carlo Simulation Study

To evaluate the performance of different estimation procedures developed in this paper, a Monte Carlo simulation study with different settings is used. Different values of the parameter ($\theta = 0.5$, and $\theta = 1$), different prior distributions ($G(3,1)$ and $G(0,0)$) for the Bayesian methods, and different sample sizes are considered. Note that $G(3,1)$ is an informative prior while $G(0,0)$ is an noninformative prior. Tables 3 and 4 present the estimated biases and mean squared errors (MSEs) of the different point estimators of θ based on 5,000 replications. Specifically, the estimated biases and MSEs are computed as

$$\widehat{Bias} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$$

and

$$\widehat{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2,$$

where $\hat{\theta}_i$ is the estimate of θ obtained in the i-th simulation, where $i = 1,\ldots,N$ and $N = 5,000$.

For interval estimation, we computed the 95% confidence intervals (CIs) for θ based on the exact method, and the asymptotic distributions of the MLE and logarithm of MLE. Furthermore, we computed the bootstrap percentile confidence interval and the HPD credible intervals. The performances of different interval estimation methods are compared in terms of their simulated average widths and simulated coverage probabilities based on 5,000 replications. The results are reported in Tables 5.

Table 3. Estimated biases and MSEs of the BME, MLE, bootstrap and Bayes estimators of $\theta = 0.5$.

						Prior 1: $G(0,0)$								Prior 2: $G(1,3)$							
						Lindley Method				MCMC Method				Lindley Method				MCMC Method			
n	m		$\widehat{\theta}_{BM}$	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{B}$	$\widehat{\theta}_{BLS}$	$\widehat{\theta}_{BLL}$			$\widehat{\theta}_{BMS}$	$\widehat{\theta}_{BML}$			$\widehat{\theta}_{BLS}$	$\widehat{\theta}_{BLL}$			$\widehat{\theta}_{BMS}$	$\widehat{\theta}_{BML}$		
							-1	0.1	1		-1	0.1	1		-1	0.1	1		-1	0.1	1
20	15	Bias	0.045	0.042	0.001	0.057	0.073	0.056	0.042	0.061	0.078	0.060	0.046	0.011	0.016	0.010	0.000	0.061	0.078	0.060	0.046
		MSE	0.046	0.043	0.041	0.038	0.045	0.038	0.032	0.040	0.049	0.040	0.034	0.017	0.021	0.017	0.016	0.040	0.049	0.040	0.034
30	20	Bias	0.034	0.029	-0.005	0.026	0.034	0.026	0.019	0.027	0.034	0.026	0.020	0.009	0.016	0.008	0.002	0.027	0.034	0.026	0.020
		MSE	0.031	0.030	0.028	0.016	0.018	0.016	0.015	0.016	0.018	0.016	0.015	0.012	0.013	0.012	0.011	0.016	0.018	0.016	0.015
	25	Bias	0.026	0.025	-0.003	0.028	0.035	0.028	0.022	0.029	0.036	0.029	0.023	0.012	0.018	0.011	0.005	0.029	0.036	0.029	0.023
		MSE	0.025	0.025	0.024	0.016	0.017	0.016	0.015	0.016	0.018	0.016	0.015	0.012	0.013	0.012	0.011	0.016	0.018	0.016	0.015
40	25	Bias	0.026	0.023	-0.003	0.018	0.022	0.017	0.013	0.018	0.023	0.018	0.014	0.007	0.012	0.007	0.003	0.018	0.023	0.018	0.014
		MSE	0.024	0.024	0.021	0.010	0.010	0.010	0.009	0.010	0.011	0.010	0.009	0.008	0.009	0.008	0.008	0.010	0.011	0.010	0.009
	30	Bias	0.021	0.017	-0.001	0.014	0.018	0.014	0.011	0.014	0.018	0.014	0.011	0.007	0.010	0.007	0.004	0.014	0.018	0.014	0.011
		MSE	0.020	0.019	0.018	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.007
	35	Bias	0.018	0.016	-0.001	0.011	0.014	0.011	0.009	0.011	0.014	0.011	0.009	0.005	0.008	0.005	0.003	0.011	0.014	0.011	0.009
		MSE	0.017	0.017	0.016	0.005	0.006	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
50	30	Bias	0.021	0.018	-0.002	0.011	0.013	0.010	0.008	0.011	0.014	0.011	0.008	0.005	0.008	0.005	0.003	0.011	0.014	0.011	0.008
		MSE	0.016	0.019	0.016	0.005	0.005	0.005	0.005	0.005	0.006	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.005	0.005
	35	Bias	0.018	0.014	-0.003	0.009	0.011	0.009	0.007	0.009	0.012	0.009	0.007	0.004	0.007	0.004	0.002	0.009	0.012	0.009	0.007
		MSE	0.016	0.014	0.014	0.005	0.005	0.005	0.004	0.005	0.005	0.005	0.005	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005
	40	Bias	0.017	0.013	-0.003	0.008	0.010	0.008	0.006	0.009	0.011	0.008	0.007	0.004	0.006	0.004	0.002	0.009	0.011	0.008	0.007
		MSE	0.015	0.014	0.014	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004	0.004
	45	Bias	0.014	0.014	-0.003	0.010	0.012	0.010	0.008	0.010	0.012	0.010	0.008	0.005	0.008	0.005	0.003	0.010	0.012	0.010	0.008
		MSE	0.013	0.013	0.013	0.004	0.005	0.004	0.004	0.004	0.005	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.004	0.004

Table 4. Estimated biases and MSEs of the BME, MLE, bootstrap and Bayes estimators of $\theta = 1$.

						Prior 1: $G(0,0)$								Prior 2: $G(1,3)$							
						Lindley Method				MCMC Method				Lindley Method				MCMC Method			
n	m		$\widehat{\theta}_{BM}$	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{B}$	$\widehat{\theta}_{BLS}$	$\widehat{\theta}_{BLL}$			$\widehat{\theta}_{BMS}$	$\widehat{\theta}_{BML}$			$\widehat{\theta}_{BLS}$	$\widehat{\theta}_{BLL}$			$\widehat{\theta}_{BMS}$	$\widehat{\theta}_{BML}$		
							-1	0.1	1		-1	0.1	1		-1	0.1	1		-1	0.1	1
20	15	Bias	0.019	0.020	0.001	0.044	0.065	0.042	0.024	0.047	0.068	0.045	0.027	-0.042	-0.025	-0.044	-0.056	0.047	0.068	0.045	0.027
		MSE	0.010	0.010	0.009	0.046	0.052	0.045	0.041	0.047	0.054	0.047	0.043	0.028	0.028	0.028	0.028	0.047	0.054	0.047	0.043
30	20	Bias	0.015	0.012	-0.002	0.031	0.045	0.029	0.016	0.033	0.048	0.031	0.019	-0.030	-0.016	-0.031	-0.041	0.033	0.048	0.031	0.019
		MSE	0.007	0.070	0.006	0.031	0.035	0.031	0.029	0.033	0.036	0.032	0.030	0.022	0.023	0.022	0.022	0.033	0.036	0.032	0.030
	25	Bias	0.012	0.011	-0.001	0.028	0.040	0.027	0.016	0.033	0.044	0.032	0.022	-0.021	-0.010	-0.022	-0.031	0.033	0.044	0.032	0.022
		MSE	0.005	0.005	0.004	0.025	0.028	0.025	0.024	0.028	0.030	0.028	0.026	0.019	0.020	0.019	0.019	0.028	0.030	0.028	0.026
40	25	Bias	0.011	0.008	-0.001	0.023	0.035	0.022	0.012	0.026	0.037	0.025	0.015	-0.023	-0.013	-0.024	-0.033	0.026	0.037	0.025	0.015
		MSE	0.005	0.005	0.004	0.024	0.025	0.024	0.022	0.025	0.026	0.024	0.023	0.018	0.019	0.018	0.018	0.025	0.026	0.024	0.023
	30	Bias	0.009	0.009	-0.000	0.022	0.032	0.021	0.012	0.027	0.036	0.026	0.018	-0.017	-0.008	-0.018	-0.026	0.027	0.036	0.026	0.018
		MSE	0.004	0.004	0.004	0.021	0.022	0.020	0.019	0.023	0.024	0.023	0.022	0.016	0.017	0.016	0.016	0.023	0.024	0.023	0.022
	35	Bias	0.008	0.006	-0.001	0.022	0.030	0.021	0.013	0.031	0.038	0.030	0.024	-0.013	-0.005	-0.013	-0.020	0.031	0.038	0.030	0.024
		MSE	0.004	0.003	0.003	0.018	0.019	0.018	0.017	0.022	0.023	0.022	0.021	0.015	0.015	0.015	0.015	0.022	0.023	0.022	0.021
50	30	Bias	0.009	0.071	-0.001	0.020	0.030	0.019	0.011	0.023	0.032	0.022	0.014	-0.018	-0.009	-0.018	-0.026	0.023	0.032	0.022	0.014
		MSE	0.004	0.004	0.003	0.019	0.020	0.019	0.018	0.020	0.021	0.020	0.019	0.015	0.016	0.015	0.015	0.020	0.021	0.020	0.019
	35	Bias	0.008	0.070	-0.001	0.018	0.026	0.017	0.010	0.022	0.030	0.021	0.015	-0.015	-0.007	-0.016	-0.022	0.022	0.030	0.021	0.015
		MSE	0.003	0.003	0.003	0.017	0.018	0.017	0.016	0.019	0.019	0.019	0.018	0.014	0.014	0.014	0.014	0.019	0.019	0.019	0.018
	40	Bias	0.007	0.006	-0.001	0.014	0.021	0.014	0.007	0.023	0.030	0.023	0.018	-0.015	-0.008	-0.015	-0.021	0.023	0.030	0.023	0.018
		MSE	0.003	0.003	0.003	0.015	0.015	0.015	0.014	0.018	0.018	0.018	0.017	0.013	0.013	0.013	0.012	0.018	0.018	0.018	0.017
	45	Bias	0.006	0.005	-0.001	0.016	0.022	0.015	0.009	0.033	0.038	0.032	0.028	-0.010	-0.004	-0.011	-0.016	0.033	0.038	0.032	0.028
		MSE	0.003	0.003	0.003	0.013	0.014	0.013	0.013	0.018	0.018	0.018	0.017	0.011	0.012	0.011	0.011	0.018	0.018	0.018	0.017

Table 5. Estimated average widths (AW) and coverage probabilities (CP) of the interval estimates for $\theta = 1.0$.

n	m		Exact	MLE	Log-MLE	Boot	HPD	
							$G(0,0)$	$G(3,1)$
20	15	AW	0.7782	0.7801	0.7984	0.8425	0.7103	0.6541
		CP	0.9496	0.9534	0.9488	0.9250	0.9470	0.9510
30	20	AW	0.6609	0.6593	0.6706	0.6960	0.5942	0.5609
		CP	0.9492	0.9536	0.9486	0.9370	0.9450	0.9480
	25	AW	0.5970	0.5978	0.6063	0.6230	0.4879	0.4704
		CP	0.9510	0.9526	0.9456	0.9350	0.9380	0.9430
40	25	AW	0.5837	0.5832	0.5911	0.6076	0.5163	0.4946
		CP	0.9523	0.9518	0.9454	0.9460	0.9470	0.9490
	30	AW	0.5377	0.5358	0.5420	0.5559	0.4377	0.4238
		CP	0.9512	0.9542	0.9532	0.9460	0.9370	0.9410
	35	AW	0.5032	0.5026	0.5078	0.5178	0.3694	0.3609
		CP	0.9504	0.9494	0.9484	0.9370	0.9340	0.9390
50	30	AW	0.5286	0.5276	0.5335	0.5486	0.4589	0.4431
		CP	0.9483	0.9496	0.9464	0.9420	0.9450	0.9480
	35	AW	0.4933	0.4917	0.4965	0.5073	0.3961	0.3865
		CP	0.9497	0.9550	0.9522	0.9390	0.9380	0.9430
	40	AW	0.4661	0.4644	0.4685	0.4760	0.3409	0.3347
		CP	0.9500	0.9526	0.9508	0.9420	0.9340	0.9400
	45	AW	0.4434	0.4432	0.4468	0.4516	0.2928	0.2879
		CP	0.9495	0.9530	0.9538	0.9420	0.9300	0.9360

From Tables 3 and 4, it is observed the bootstrap estimator is better than the other estimators in terms of both biases and MSEs. The Bayes estimator based on non-informative prior perform better than the MLE. In addition, the MBE and the MLE performs similarly in terms of biases and MSEs. Comparing the Bayes estimators based on different prior distributions, as expected, the Bayes estimator based on the informative prior perform better than the Bayes estimator based on the non-informative prior, in terms of both biases and MSEs. Comparing the Bayes estimators obtained using Lindley's approximation and importance sampling methods, we observe that the Lindley's approximation method is better than the importance sampling method.

For interval estimation, from Table 5, it is observed that all the simulated coverage probabilities are very close to the nominal level 95%. For all interval estimators, as m increases, the estimated average width of the interval decreases. Comparing the average widths of the interval estimates, it is observed that the Bayesian credible intervals are superior to the bootstrap and asymptotic confidence intervals. The average width of the 95% confidence intervals based on the asymptotic distribution of the MLE is slightly smaller than the corresponding average lengths of the exact confidence intervals and the intervals based on the asymptotic distribution of logarithm of MLE.

Based on the simulation results, overall speaking, we would recommend the use bootstrap bias-adjusted estimator for point estimation and the use of Bayesian credible interval for interval estimation, especially when reliable prior information about the Lindley parameter is available.

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7. Extensions to Different Censoring Schemes

In this paper, we discussed the statistical inference for Lindley distribution based on Type II censored data, but the proposed methodologies can be adopted when different censoring schemes are used in the life testing experiment. Here, we describe the extensions for the hybrid and progressive Type II censored data.

7.1. *Hybrid Censoring*

The hybrid censoring scheme is a mixture of Type I and Type II censoring schemes. Let $X_{1:n} < \ldots < X_{n:n}$ denote the ordered lifetime of the experimental units. In Type I hybrid censoring scheme, the experiment stops at $T_0 = \min\{X_{m:n}, T\}$, where m and T are pre-fixed. Therefore, in Type I hybrid censoring scheme, one observes $X_{1:n} \ldots, X_{m:n}$, if $X_{m:n} < T$ (Case I), or $X_{1:n}, \ldots, X_{r:n}$ when $r < m$, and $X_{r:n} < T < X_{r+1:n}$ (Case II). Here r is a random variable and $r = 0, \ldots, m-1$.

Let $\mathbf{X} = (X_{1:n}, \ldots, X_{d:n})$ denote a Type I hybrid censored sample. Based on the observed sample $\mathbf{x} = (x_{1:d}, \ldots, x_{d:n})$, the likelihood function is

$$L(\theta|\mathbf{x}) = \frac{n!}{(n-d)!}\prod_{i=1}^{d} f(x_{i:d};\theta)[1-F(T_0;\theta)]^{n-d}, \tag{7.1}$$

where d denotes the number of failures, $d = m$ for Case I and, $d = r$ for Case II. For the Lindley distribution, we obtain the log-likelihood equation as

$$\frac{d\ln L(\theta)}{d\theta} = \frac{2d}{\theta} - \frac{n}{1+\theta} + \frac{(n-d)(1+T_0)}{1+\theta+\theta T_0} - \left[\sum_{i=1}^{d} x_{i:d} + (n-d)T_0\right] = 0,$$

which has the same form as Eq. (2.1), except m is replaced by d, and x_m is replaced by T_0. Therefore, the MLE of θ can be obtained as described in Section 2.2.

To compute the Bayes estimator and credible interval of θ, it is assumed that θ have the same prior as described in Eq. (3.1). We then obtain the posterior distribution of θ given the data as

$$\begin{aligned}\pi(\theta|\mathbf{x}) \propto \theta^{2d+a-1}\exp\left\{-\theta\left[\sum_{i=1}^{d} x_{i:d} + (n-d)T_0 + b\right]\right\} \\ \times \frac{(1+\theta+\theta T_0)^{n-d}}{(1+\theta)^n},\end{aligned}$$

which has the same form as Eq. (3.2), except m is replaced by d, and x_m is replaced by T_0. Therefore, using the methods described in Section 3, the Bayes estimator and the corresponding credible interval can be obtained.

Now, we discuss the Bayesian prediction of $Y = X_{s+d:n} (s = 1, \ldots, n-d)$ of all the $n-d$ censored units based on observed data $\underline{X} = (X_{1:n}, \ldots, X_{d:n})$. The conditional density of $Y = X_{s+d:n}$ given $\underline{X} = \underline{x}$, for $y \geq T_0$, is

$$\begin{aligned}f_s(y|\mathbf{x}) = s\binom{n-d}{s} f(y)\,[F(y)-F(T_0)]^{s-1}\,[1-F(y)]^{n-d-s} \\ \times\,[1-F(T_0)]^{-(n-d)},\end{aligned}$$

which have the same form as Eq. (4.1). Therefore, using the same method as described in Section 4, Bayesian point and interval predictors can be obtained.

For the case of Type II hybrid censoring scheme in which the experiment stops at $T_0' = \max\{X_{m:n}, T\}$, the similar results can be obtained by some modifications.

7.2. *Progressive Type II Censoring*

The progressive Type II censoring, after starting the life-testing experiment with n units, arises as follows. Suppose that n units are placed on a life-testing experiment and only $m(< n)$ units are completely observed until failure. Immediately following the first failure, R_1 surviving units are removed from the test at random and the experiment continues with $(n-R_1-1)$ units. Then, immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the m-th failure, all the remaining $R_m = n-R_1-R_2-\cdots-R_{m-1}-m$ units are removed from the experiment. Here, the values of R_i, $i=1,2,\ldots,m$, are fixed prior to study.

Let $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denote a progressively Type II censored sample from the Lindley distribution with pdf in Eq. (1.1), with $(R_1,\cdots,R_m)$ being the progressive censoring scheme. Based on the observed censored sample $x_{1:m},\cdots,x_{m:m}$, the likelihood function is

$$\begin{aligned} L(\theta|\mathbf{x}) &= A\prod_{i=1}^{m} f(x_{i:m},\theta)[1-F(x_{i:m},\theta)]^{R_i} \\ &= A\ \theta^{2m}(1+\theta)^{-n}\prod_{i=1}^{m}\left\{(1+x_{i:m})(1+\theta+\theta x_{i:m})^{R_i}\right\} e^{-\theta\sum_{i=1}^{m}(R_i+1)x_{i:m}}, \end{aligned}$$

where $A = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1\cdots-R_{m-1})$. Therefore, the MLE of θ can be obtained from solving the likelihood equation

$$\frac{d\ln L(\theta|\mathbf{x})}{d\theta} = \frac{2m}{\theta} - \frac{n}{1+\theta} + \sum_{i=1}^{m} R_i \frac{1+x_{i:m}}{1+\theta+\theta x_{i:m}} - \sum_{i=1}^{m}(R_i+1)x_{i:m} = 0.$$

Considering a gamma prior for θ as described in Eq. (3.1), the posterior distribution of θ given the data can be written as

$$\pi(\theta|\mathbf{x}) \propto \pi\left(\theta; 2m+a, \sum_{i=1}^{m}(R_i+1)x_{i:m}+b\right) g_1(\theta|\mathbf{x}),$$

where

$$g_1(\theta|\mathbf{x}) = \frac{\prod_{i=1}^{m}(1+\theta+\theta x_{i:m})^{R_i}}{(1+\theta)^n}.$$

Therefore, the Bayes estimator and credible interval of θ can be computed again as described in Section 3.

For the prediction of the future lifetimes, $Y = X_{s:R_i}(s=1,2,\ldots,R_i;\ i=1,2,\ldots,m)$, let us first consider the conditional density of $Y = X_{s:R_i}$ given $\mathbf{x} = (x_{1:m:n},\cdots,x_{m:m:n})$. The conditional density of Y, given $X_{m:m:n} = x_{m:m:n}$, is

$$\begin{aligned}
f(y|\mathbf{x},\theta) &= s\binom{R_i}{s} f(y;\theta)\left[F(y;\theta) - F(x_{i:m:n};\theta)\right]^{s-1} \\
&\quad \times \left[1 - F(y;\theta)\right]^{R_i - s}\left[1 - F(x_{i:m:n};\theta)\right]^{-R_i} \\
&= s\binom{R_i}{s}\theta^2(1+y)\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^{s-k-1}(1+\theta+\theta y)^{R_i-k-1} \\
&\quad \times \exp\left\{-\theta[(R_i-k-1)y + (k-R_i)x_{i:m:n}]\right\}(1+\theta+\theta x_{i:m:n})^{k-R_i}.
\end{aligned}$$

Now, the Bayes predictive density function of Y given $\mathbf{X} = \mathbf{x}$ is

$$\begin{aligned}
f_s^*(y|\mathbf{x}) &= \int_0^\infty f(y|\mathbf{x},\theta)\pi(\theta|\mathbf{x})d\theta \\
&= \int_0^\infty f(y|\mathbf{x},\theta)\pi\left(\theta; 2m+a, \sum_{i=1}^m (R_i+1)x_{i:m} + b\right) g_1(\theta|\mathbf{x})d\theta,
\end{aligned}$$

which has the similar form as Eq. (4.6). Therefore, Bayesian point and interval predictors can be obtained as described in Section 4.

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