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Bayesian Analysis of a 3-Component Mixture of Rayleigh Distributions under Type-I Right Censoring Scheme

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Abstract

Since the last few decades, constructing flexible parametric classes of probability distributions has been the most popular approach in the Bayesian analysis. As compared to simple probability models, a mixture model of some suitable lifetime distributions may be more capable of capturing the heterogeneity of the nature. In this study, a 3-component mixture of Rayleigh distributions is investigated by considering type-I right censoring scheme to obtain data from a heterogeneous population. The closed form expressions for the Bayes estimators and posterior risks assuming the non-informative (uniform and Jeffreys') priors under squared error loss function, precautionary loss function and DeGroot loss function are derived. The performance of the Bayes estimators for different sample sizes, test termination times and parametric values under different loss functions is investigated. The posterior predictive distribution for a future observation and the Bayesian predictive interval are constructed. In addition, the limiting expressions for the Bayes estimators and posterior risks are derived. Simulated data sets are used for the different comparisons and the model is finally illustrated using the real data.

Keywords: 3-Component mixture model; Loss function; Posterior risk; Predictive interval; Test termination time.

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1. Introduction

The Rayleigh distribution is a probability distribution with many applications in the field of communication engineering. There are many electro-vacuum and radio-wave devices whose failure rate depends upon their age, so the Rayleigh distribution (or a mixture of the Rayleigh) may be considered as a candidate model to

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study and analyze the life times of the objects. In this study, the data taken for analysis are assumed to be characterized by three different unknown members of Rayleigh distribution.

Hirai [1] derived the scale and location parameters for the two-parameter Rayleigh distribution using quadratic coefficients. Hirai [2] also estimated the scale-parameter of the Rayleigh distribution under type-II censoring scheme. Raqab [3] presented Type-II censored sample in the problems of prediction using Rayleigh distribution. Fernandez [4] described the Bayesian inference using type-II doubly censored Rayleigh data. Mostert et al. [5] discussed the Bayesian analysis of survival data with Rayleigh distribution and linex loss.

The use of mixture models in the situations when the data are given only for overall mixture distributions is known as direct application of the mixture models. In this study, the direct application of mixture models is considered. Li [6] and Li and Sedransk [7, 8] discussed different features of mixture models and define two types of mixture models. The mixture of the probability density functions from the same family is known as type-I mixture model and type-II mixture model is defined as a mixture of density functions from several families. We plan to study type-I mixture model of a 3-component mixture of Rayleigh distribution with unknown component parameters and unknown mixing proportions.

The Bayesian technique to analyze the 3-component mixture model has developed the interest between researchers. The posterior distribution, which is obtained when prior information is combined with likelihood, is the workbench of Bayesian inference. So, the prior information is very important and necessary for Bayesian inference and it is subjective assessment of an expert statistician before the data have been observed. In Bayesian analysis, the elicitation of hyperparameters of prior distribution is often very complicated, so we use non-informative priors. The most important and commonly used non-informative priors are uniform prior (UP) and Jeffreys' prior (JP).

There are many fields in which mixture models have been used, for example, engineering, biological sciences, physical sciences and social sciences etc. Most of the researchers worked on the Bayesian analysis of the 2-component mixture models. Sinha [9] used the Bayesian counterpart of the maximum likelihood estimates of the 2-component mixture model considered by Mendenhall and Hader [10]. Saleem and Aslam [11] discussed the use of the informative and the non-informative priors for Bayesian analysis of the 2-component mixture of the Rayleigh distribution and also Saleem et al. [12] presented the Bayesian analysis of the 2-component mixture of the Power distribution using the complete and censored sample. Kazmi et al. [13] described the Bayesian analysis for 2-component mixture of the Maxwell distribution.

Censoring is an important and valuable aspect of the lifetime applications. In real life most of the time, it is not suitable that the testing procedure continue until to get the last value of the data set. Censoring is a form of primary quality and missing data problems. A valuable account of censoring is given in Kalbfleisch and Prentice [14], Romeu [15] and Gijbels [16]. There are three main types of censoring, right censoring, left censoring and interval censoring. Right censoring is divided into type-I, type-II or random right censoring. The type-I censoring is further categorized as ordinary type-I, progressive type-I and generalized type-I censoring. Similarly, the type-II censoring is divided into ordinary type-II and progressive type-II censoring.

Motivated by the wider applications of Rayleigh distribution and their mixtures, in this study, the Bayesian analysis of a 3-component mixture of Rayleigh distributions using the UP and the JP under squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) is done. For our anticipated study, we assume ordinary type-I right censoring and fixed life-test termination time for all objects.

The rest of the article is organized as follows: A 3-component mixture of Rayleigh distributions is defined in Section 2. The expressions for likelihood function and joint posterior distributions using the non-informative priors are derived in Section 3. The Bayes estimators and posterior risks using the UP and the JP under SELF, PLF and DLF are presented in Sections 4, 5, and 6, respectively. The posterior predictive distribution and the Bayesian predictive intervals are given in Section 7. The limiting expressions of the Bayes estimators and their posterior risks are derived in Section 8. The simulation study and the real life application are presented in Sections 9 and 10, respectively. Finally, the conclusion of this study is given in Section 11.

2. 3-Component Mixture of Rayleigh Distributions

A random variable X is said to follow a finite mixture distribution with h components if the probability density function of X can be written in the form: $f(x)=\sum_{m=1}^{h} p_m f_m(x)$, where p_m $(m=1, 2,..., h)$ is m^{th} mixing proportion such that $p_h=1-\sum_{m=1}^{h-1} p_m$ and $f_m(x)$ is the m^{th} component density function. A finite 3-component mixture of Rayleigh distributions with mixing proportions p_1 and p_2 has its pdf as:

$$f(x;\ \boldsymbol{\psi})=p_1 f_1(x;\ \psi_1)+p_2 f_2(x;\ \psi_2)+(1-p_1-p_2)f_3(x;\ \psi_3), \quad p_1, p_2 \ge 0, p_1+p_2 \le 1, \tag{1}$$

where $\boldsymbol{\psi}=(\theta_1, \theta_2, \theta_3, p_1, p_2)$, $\psi_m=\theta_m$, $m=1, 2, 3$ and $f_m(x;\ \psi_m)$, pdf of the m^{th} component, is written as:

$$f_m(x;\ \psi_m)=\frac{x}{\theta_m^2}\exp\left(-\frac{x^2}{2\theta_m^2}\right), \quad 0<x<\infty,\ \theta_m>0,\ m=1, 2, 3. \tag{2}$$

The cdf of a 3-component mixture of Rayleigh distributions is:

$$F(x;\ \boldsymbol{\psi})=p_1 F_1(x;\ \psi_1)+p_2 F_2(x;\ \psi_2)+(1-p_1-p_2)F_3(x;\ \psi_3), \tag{3}$$

where $F_m(x;\ \psi_m)$, cdf of the m^{th} component, is given by:

$$F_m(x;\ \psi_m)=1-\exp\left(-\frac{x^2}{2\theta_m^2}\right), \quad 0<x<\infty,\ \theta_m>0,\ m=1, 2, 3. \tag{4}$$

3. The Joint Posterior Distribution using the Non-informative Priors

The joint posterior distributions of parameters given data $\mathbf{x}$ are derived using the non-informative (uniform and Jeffreys') priors. To derive joint posterior distributions, in the next subsection, first, we develop the likelihood function for the data $\mathbf{x}$ are obtained from a 3-component mixture of Rayleigh distributions under a type-I right censoring scheme.

3.1. *The likelihood function*

Suppose n units are used in a life testing experiment from the 3-component mixture model. Let r units out of n units failed until fixed test termination time t and the remaining $n-r$ units are still working. According to Mendenhall and Hader [10], there are many practical situations where the failed objects can be pointed out easily as subset of either subpopulation-I or subpopulation-II or subpopulation-III. Out of r units, suppose r_1, r_2 and r_3 units belong to subpopulation-I, subpopulation-II and subpopulation-III, respectively, such that $r=r_1+r_2+r_3$. Define x_{lk}, $0<x_{lk}\le t$, be the failure time of k^{th} unit belonging to l^{th} subpopulation, where $l=1, 2, 3$ and $k=1, 2, \cdots, r_l$. The likelihood function for the random sample $\mathbf{x}$ from a 3-component mixture model is:

$$L(\boldsymbol{\psi}|\mathbf{x}) \propto \left\{\prod_{k=1}^{r_1} p_1 f_1(x_{1k})\right\}\left\{\prod_{k=1}^{r_2} p_2 f_2(x_{2k})\right\}\left\{\prod_{k=1}^{r_3}(1-p_1-p_2)f_3(x_{3k})\right\}\left\{1-F(t)\right\}^{n-r}, \tag{5}$$

$$L(\boldsymbol{\psi}|\mathbf{x}) \propto \left\{\prod_{k=1}^{r_1} p_1\frac{1}{\theta_1^2}\exp\left(-\frac{x_{1k}^2}{2\theta_1^2}\right)\right\}\left\{\prod_{k=1}^{r_2} p_2\frac{1}{\theta_2^2}\exp\left(-\frac{x_{2k}^2}{2\theta_2^2}\right)\right\}\left\{\prod_{k=1}^{r_3}(1-p_1-p_2)\frac{1}{\theta_3^2}\exp\left(-\frac{x_{3k}^2}{2\theta_3^2}\right)\right\}$$
$$\left\{p_1\exp\left(-\frac{t^2}{2\theta_1^2}\right)+p_2\exp\left(-\frac{t^2}{2\theta_2^2}\right)+(1-p_1-p_2)\exp\left(-\frac{t^2}{2\theta_3^2}\right)\right\}^{n-r}. \tag{6}$$

On substituting in the above expression and after simplification, the likelihood function of a 3-component mixture of Rayleigh distribution can be written as:

$$L(\boldsymbol{\psi}|\mathbf{x}) \propto \left[\sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \exp\left\{ -\frac{1}{\theta_1^2} \left((n-r-i)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_1} x_{1k}^2 \right) \right\} \exp\left\{ -\frac{1}{\theta_2^2} \left((i-j)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_2} x_{2k}^2 \right) \right\} \right.$$
$$\left. \exp\left\{ -\frac{1}{\theta_3^2} \left(j\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_3} x_{3k}^2 \right) \right\} \theta_1^{-2r_1} \theta_2^{-2r_2} \theta_3^{-2r_3} p_1^{n-r-i+r_1} p_2^{i-j+r_2} (1-p_1-p_2)^{j+r_3} \right], \tag{7}$$

where $\mathbf{x} = \left(x_{11}, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2}, x_{31}, x_{32}, ..., x_{3r_3} \right)$.

3.2. *The joint posterior distribution using the uniform prior*

When elicitation of hyperparameters is difficult or little prior information is given then usually the non-informative prior is assumed to be the uniform prior (UP). Bayes [17], Laplace [18] and Geisser [19] have used the uniform prior distribution for the unknown parameters. On the same lines, we assume the UP over the interval $(0, \infty)$ for the scale parameters θ_1, θ_2 and θ_3 and the UP over the interval $(0, 1)$ for the mixing proportions p_1 and p_2. Under these assumptions, the joint prior distribution of parameters θ_1, θ_2, θ_3, p_1 and p_2 is:

$$\pi_1(\boldsymbol{\psi}) \propto 1; \quad \theta_1, \theta_2, \theta_3 > 0, \; p_1, p_2 \geq 0, \; p_1 + p_2 \leq 1. \tag{8}$$

The joint posterior distribution of parameters θ_1, θ_2, θ_3, p_1 and p_2 given data $\mathbf{x}$ using the UP is:

$$q_1(\boldsymbol{\psi}|\mathbf{x}) = \frac{L(\boldsymbol{\psi}|\mathbf{x})\pi_1(\boldsymbol{\psi})}{\int_{\boldsymbol{\psi}} L(\boldsymbol{\psi}|\mathbf{x})\pi_1(\boldsymbol{\psi})\, d\boldsymbol{\psi}}, \tag{9}$$

$$q_1(\boldsymbol{\psi}|\mathbf{x}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \exp\left(-\frac{B_{11}}{\theta_1^2}\right) \exp\left(-\frac{B_{21}}{\theta_2^2}\right) \exp\left(-\frac{B_{31}}{\theta_3^2}\right) p_1^{A_{01}-1} p_2^{B_{01}-1} (1-p_1-p_2)^{C_{01}-1}}{D_1 \; \theta_1^{2A_{11}+1} \theta_2^{2A_{21}+1} \theta_3^{2A_{31}+1}}, \tag{10}$$

where $A_{11} = r_1 - \frac{1}{2}$, $A_{21} = r_2 - \frac{1}{2}$, $A_{31} = r_3 - \frac{1}{2}$, $B_{11} = (n-r-i)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_1} x_{1k}^2$, $B_{21} = (i-j)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_2} x_{2k}^2$,

$B_{31} = j\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_3} x_{3k}^2$, $A_{01} = n-r-i+r_1+1$, $B_{01} = i-j+r_2+1$, $C_{01} = j+r_3+1$,

$D_1 = \frac{1}{8}\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31}) \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} B(A_{01}, B_{01}, C_{01}) B_{11}^{-A_{11}} B_{21}^{-A_{21}} B_{31}^{-A_{31}}$.

3.3. *The joint posterior distribution using the Jeffreys' prior*

Jeffreys [20, 21], Bernardo [22] and Berger [23] discussed a rule for obtaining the Jeffreys' prior (JP) as $p(\theta) \propto \sqrt{|I(\theta)|}$ if θ is a *h*-vector valued component parameter, where $I(\theta) = (I_{uv})_{h\times h}$ is a $h \times h$ Fisher's information, in which the (*u*, *v*)-th element is $-E\left[\frac{\partial^2 \ln L(\theta|\mathbf{x})}{\partial\theta_u \partial\theta_v}\right]$; $u, v = 1, 2, ..., h$. The prior distributions of the mixing proportions p_1 and p_2 are again assumed as the uniform distributions over the interval $(0,\ 1)$. Assuming the independence of parameters, the joint prior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 is:

$$\pi_2(\boldsymbol{\psi}) \propto \frac{1}{\theta_1\theta_2\theta_3},\quad \theta_1,\theta_2,\theta_3 > 0,\ p_1,p_2 \ge 0,\ p_1+p_2 \le 1. \tag{11}$$

The joint posterior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 given data $\mathbf{x}$ using the JP is:

$$q_2(\boldsymbol{\psi}|\mathbf{x}) = \frac{L(\boldsymbol{\psi}|\mathbf{x})\pi_2(\boldsymbol{\psi})}{\int_{\boldsymbol{\psi}} L(\boldsymbol{\psi}|\mathbf{x})\pi_2(\boldsymbol{\psi})\,d\boldsymbol{\psi}}, \tag{12}$$

$$q_2(\boldsymbol{\psi}|\mathbf{x}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left(-\frac{B_{12}}{\theta_1^2}\right)\exp\left(-\frac{B_{22}}{\theta_2^2}\right)\exp\left(-\frac{B_{32}}{\theta_3^2}\right)p_1^{A_{02}-1}p_2^{B_{02}-1}\left(1-p_1-p_2\right)^{C_{02}-1}}{D_2\ \theta_1^{2A_{12}+1}\theta_2^{2A_{22}+1}\theta_3^{2A_{32}+1}}, \tag{13}$$

where $A_{12}=r_1$, $A_{22}=r_2$, $A_{32}=r_3$, $B_{12}=(n-r-i)\frac{t^2}{2}+\frac{1}{2}\sum_{k=1}^{r_1}x_{1k}^2$, $B_{22}=(i-j)\frac{t^2}{2}+\frac{1}{2}\sum_{k=1}^{r_2}x_{2k}^2$,

$B_{32}=j\frac{t^2}{2}+\frac{1}{2}\sum_{k=1}^{r_3}x_{3k}^2$, $A_{02}=n-r-i+r_1+1$, $B_{02}=i-j+r_2+1$, $C_{02}=j+r_3+1$,

$D_2=\frac{1}{8}\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B(A_{02},B_{02},C_{02})B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}$.

4. The Bayes Estimators and Posterior Risks using the UP and the JP under SELF

If $L(\theta,\ d)$ is a loss function then the expected value of the loss function for a given decision with respect to the posterior distribution is termed as posterior risk function and if $\hat{d}$ is a Bayes estimator then $\rho(\hat{d})$ is called posterior risk and is defined as: $\rho(\hat{d}) = E_{\theta|\mathbf{x}}\left\{L(\theta,\ \hat{d})\right\}$. The SELF defined as $L(\theta,\ d)=(\theta-d)^2$ was suggested by Legendre [24]. By using SELF, the Bayes estimators and posterior risk are $\hat{d}=E_{\theta|\mathbf{x}}(\theta)$ and $\rho(\hat{d})=E_{\theta|\mathbf{x}}(\theta^2)-\left\{E_{\theta|\mathbf{x}}(\theta)\right\}^2$, respectively. So, the expressions for Bayes estimators and posterior risks assuming the UP and the JP for parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 under SELF are obtained with respective marginal posterior distribution as:

$$\hat{\theta}_{1v} = \frac{\Gamma\left(A_{1v}-\frac{1}{2}\right)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-\left(A_{1v}-\frac{1}{2}\right)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) \tag{14}$$

$$\hat{\theta}_{2v} = \frac{\Gamma(A_{1v})\Gamma\left(A_{2v}-\frac{1}{2}\right)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-\left(A_{2v}-\frac{1}{2}\right)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) \tag{15}$$

$$\hat{\theta}_{3v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma\left(A_{3v}-\frac{1}{2}\right)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-\left(A_{3v}-\frac{1}{2}\right)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) \tag{16}$$

$$\hat{p}_{1v}=\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v}) \quad (17)$$

$$\hat{p}_{2v}=\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v}) \quad (18)$$

$$\begin{aligned}\rho(\hat{\theta}_{1v})=&\frac{\Gamma(A_{1v}-1)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&-\left\{\frac{\Gamma(A_{1v}-0.5)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-0.5)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2\end{aligned} \quad (19)$$

$$\begin{aligned}\rho(\hat{\theta}_{2v})=&\frac{\Gamma(A_{1v})\Gamma(A_{2v}-1)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v}-0.5)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-0.5)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2\end{aligned} \quad (20)$$

$$\begin{aligned}\rho(\hat{\theta}_{3v})=&\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}-1)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}-0.5)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-0.5)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2\end{aligned} \quad (21)$$

$$\begin{aligned}\rho(\hat{p}_{1v})=&\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\\&-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\right\}^2\end{aligned} \quad (22)$$

$$\begin{aligned}\rho(\hat{p}_{2v})=&\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})\\&-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^2,\end{aligned} \quad (23)$$

where $v=1$ for the UP and $v=2$ for the JP.

5. The Bayes Estimators and Posterior Risks using the UP and the JP under PLF

Norstrom [25] discussed an asymmetric PLF and a special case of general class of PLFs defined as $L(\theta,\, d)=(\theta-d)^2/d$. Under PLF, the Bayes estimators and posterior risk are $\hat{d}=\left\{E_{\theta|\mathbf{x}}(\theta^2)\right\}^{\frac{1}{2}}$ and $\rho(\hat{d})=2\left\{E_{\theta|\mathbf{x}}(\theta^2)\right\}^{\frac{1}{2}}-2E_{\theta|\mathbf{x}}(\theta)$, respectively. The expressions for Bayes estimators and posterior risks assuming the UP and the JP for parameters θ_1, θ_2, θ_3, p_1 and p_2 under PLF are derived with respective marginal posterior distribution as:

$$\hat{\theta}_{1v}=\left\{\frac{\Gamma(A_{1v}-1)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{24}$$

$$\hat{\theta}_{2v}=\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v}-1)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{25}$$

$$\hat{\theta}_{3v}=\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}-1)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{26}$$

$$\hat{p}_{1v}=\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{27}$$

$$\hat{p}_{2v}=\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{28}$$

$$\begin{aligned}\rho(\hat{\theta}_{1v})=2&\left\{\frac{\Gamma(A_{1v}-1)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}}\\&-2\left\{\frac{\Gamma(A_{1v}-0.5)\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-0.5)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}\end{aligned} \tag{29}$$

$$\begin{aligned}\rho(\hat{\theta}_{2v})=2&\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v}-1)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}}\\&-2\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v}-0.5)\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-0.5)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}\end{aligned} \tag{30}$$

$$\begin{aligned}\rho(\hat{\theta}_{3v})=2&\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}-1)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}}\\&-2\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}-0.5)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-0.5)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}\end{aligned} \tag{31}$$

$$\begin{aligned}\rho(\hat{p}_{1v})=2&\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\right\}^{\frac{1}{2}}\\&-2\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\right\}\end{aligned} \tag{32}$$

$$\rho\left(\hat{p}_{2v}\right)=2\left\{\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)\right\}^{\frac{1}{2}}$$
$$-2\left\{\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)}{8D_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)\right\}. \tag{33}$$

6. The Bayes Estimators and Posterior Risks using the UP and the JP under DLF

DeGroot [26] introduced the asymmetric loss function as $L(\theta,\, d)=(\theta-d)^2/d^2$, known as DLF. The Bayes estimators and their posterior risk under DLF are $\hat{d}=\frac{E_{\theta|\mathbf{x}}\left(\theta^2\right)}{E_{\theta|\mathbf{x}}\left(\theta\right)}$ and $\rho\left(\hat{d}\right)=1-\frac{\left\{E_{\theta|\mathbf{x}}\left(\theta\right)\right\}^2}{E_{\theta|\mathbf{x}}\left(\theta^2\right)}$, respectively. The expressions for Bayes estimators and posterior risks assuming the UP and the JP for parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 under DLF are derived with respective marginal posterior distribution as:

$$\hat{\theta}_{1v}=\frac{\Gamma\left(A_{1v}-1\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-\left(A_{1v}-1\right)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)}{\Gamma\left(A_{1v}-0.5\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-\left(A_{1v}-\frac{1}{2}\right)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{34}$$

$$\hat{\theta}_{2v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}-1\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-\left(A_{2v}-1\right)}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)}{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}-0.5\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-\left(A_{2v}-\frac{1}{2}\right)}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{35}$$

$$\hat{\theta}_{3v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}-1\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-\left(A_{3v}-1\right)}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)}{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}-0.5\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-\left(A_{3v}-\frac{1}{2}\right)}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{36}$$

$$\hat{p}_{1v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+2,B_{0v}+C_{0v}\right)}{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+1,B_{0v}+C_{0v}\right)} \tag{37}$$

$$\hat{p}_{2v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)}{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)} \tag{38}$$

$$\rho\left(\hat{\theta}_{1v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}-0.5\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-0.5)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\right\}^2}{8D_v\Gamma\left(A_{1v}-1\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-(A_{1v}-1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{39}$$

$$\rho\left(\hat{\theta}_{2v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}-0.5\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-0.5)}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\right\}^2}{8D_v\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}-1\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}-1)}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{40}$$

$$\rho\left(\hat{\theta}_{3v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}-0.5\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-0.5)}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\right\}^2}{8D_v\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}-1\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}-1)}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)} \tag{41}$$

$$\rho\left(\hat{p}_{1v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+1,B_{0v}+C_{02}\right)\right\}^2}{8D_v\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+2,B_{0v}+C_{02}\right)} \tag{42}$$

$$\rho\left(\hat{p}_{2v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)\right\}^2}{8D_v\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)}. \tag{43}$$

7. The Posterior Predictive Distribution and Bayesian Predictive Interval

The posterior predictive distribution contains the information about the future observation $Y=X_{n+1}$ of a random variable given data $\mathbf{x}$. Arnold and Press [27], Al-Hussaini *et al*. [28], Al-Hussaini and Ahmad [29], Bolstad [30] and Bansal [31] have given a detailed discussion on prediction and predictive distribution under the Bayesian framework. We, now, present the derivation of posterior predictive distribution and Bayesian predictive interval.

The posterior predictive distribution of a future observation $Y=X_{n+1}$ given data $\mathbf{x}$ assuming the UP and the JP is written as:

$$f\left(y|\mathbf{x}\right)=\int_{p_2}\int_{p_1}\int_{\theta_3}\int_{\theta_2}\int_{\theta_1}f\left(y|\boldsymbol{\psi}\right)q_v\left(\boldsymbol{\psi}|\mathbf{x}\right)\,d\theta_1 d\theta_2 d\theta_3 dp_1 dp_2\,, \tag{44}$$

where $f\left(y|\boldsymbol{\psi}\right)=p_1 f_1\left(y;\ \psi_1\right)+p_2 f_2\left(y;\ \psi_2\right)+\left(1-p_1-p_2\right)f_3\left(y;\ \psi_3\right)$,

$$f_m\left(y;\ \psi_m\right)=\frac{y}{\theta_m^2}\exp\left(-\frac{y^2}{2\theta_m^2}\right),\quad 0<y<\infty,\ \theta_m>0,\ m=1,\ 2,\ 3$$

and $$q_v(\boldsymbol{\psi}|\mathbf{x}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left(-\frac{B_{1v}}{\theta_1^2}\right)\exp\left(-\frac{B_{2v}}{\theta_2^2}\right)\exp\left(-\frac{B_{3v}}{\theta_3^2}\right)p_1^{A_{0v}-1}p_2^{B_{0v}-1}\left(1-p_1-p_2\right)^{C_{0v}-1}}{D_v\;\theta_1^{2A_{1v}+1}\theta_2^{2A_{2v}+1}\theta_3^{2A_{3v}+1}}$$

So, the posterior predictive distribution given in (44) assuming the UP and the JP of a future observation $Y = X_{n+1}$ given data $\mathbf{x}$ is given by:

$$f(y|\mathbf{x}) = \frac{y}{8D_v}\left[\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\left(B_{1v}+\frac{y^2}{2}\right)^{-(A_{1v}+1)} B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v}+1,C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{\left\{\Gamma\left(A_{1v}+1\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+\right.$$
$$\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}\left(B_{2v}+\frac{y^2}{2}\right)^{-(A_{2v}+1)} B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)}{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}+1\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+$$
$$\left.\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}\left(B_{3v}+\frac{y^2}{2}\right)^{-(A_{3v}+1)} B\left(A_{0v},C_{0v}+1\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}+1\right)\right\}^{-1}}\right]. \tag{45}$$

In order to construct a Bayesian predictive interval, suppose L and U be the two endpoints of the Bayesian predictive interval. These two endpoints can be obtained using the posterior predictive distribution defined in (45). A $100(1-\alpha)\%$ Bayesian predictive interval (L, U) can be obtained by solving the following equations:

$$\int_0^L f(y|\mathbf{x})\,dy = \frac{\alpha}{2} = \int_U^\infty f(y|\mathbf{x})\,dy,$$

or

$$\frac{1}{8D_v}\left[\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\left\{B_{1v}^{-A_{1v}}-\left(B_{1v}+\frac{L^2}{2}\right)^{-A_{1v}}\right\}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v}+1,C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{A_{1v}\left\{\Gamma\left(A_{1v}+1\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+\right.$$
$$\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}\left\{B_{2v}^{-A_{2v}}-\left(B_{2v}+\frac{L^2}{2}\right)^{-A_{2v}}\right\}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)}{A_{2v}\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}+1\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+$$
$$\left.\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}\left\{B_{3v}^{-A_{3v}}-\left(B_{3v}+\frac{L^2}{2}\right)^{-A_{3v}}\right\}B\left(A_{0v},C_{0v}+1\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{A_{3v}\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}+1\right)\right\}^{-1}}\right]=\frac{\alpha}{2}$$

and

$$\frac{1}{8D_v}\left[\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\left(B_{1v}+\frac{U^2}{2}\right)^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v}+1,C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{A_{1v}\left\{\Gamma\left(A_{1v}+1\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+\right.$$

$$\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}\left(B_{2v}+\frac{U^2}{2}\right)^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)}{A_{2v}\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}+1\right)\Gamma\left(A_{3v}\right)\right\}^{-1}}+$$

$$\left.\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}\left(B_{3v}+\frac{U^2}{2}\right)^{-A_{3v}}B\left(A_{0v},C_{0v}+1\right)B\left(B_{0v},A_{0v}+C_{0v}+1\right)}{A_{3v}\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}+1\right)\right\}^{-1}}\right]=\frac{\alpha}{2}$$

8. Limiting Expressions for Bayes Estimators and Posterior Risks

When t tends to ∞, r tends to n and r_l tends to n_l, $l=1,2,3$ then all the values which are censored become uncensored in our analysis. So the information contained in the sample is increased and consequently the posterior risks of the Bayes estimates diminish. The efficiency of the Bayes estimates is increased because all the values are incorporated in our sample. The limiting expressions for the Bayes estimators and posterior risks using the UP and the JP under SELF, PLF and DLF are given in Tables 1-6.

Table 1. Limiting expressions for the Bayes estimators as $t\to\infty$ using the UP and the JP under SELF.

Parameters	Bayes Estimators	
	UP	JP
θ_1	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}\Gamma\left(n_1-1\right)}{\Gamma\left(n_1-0.5\right)}$	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}\Gamma\left(n_1-0.5\right)}{\Gamma\left(n_1\right)}$
θ_2	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}\Gamma\left(n_2-1\right)}{\Gamma\left(n_2-0.5\right)}$	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}\Gamma\left(n_2-0.5\right)}{\Gamma\left(n_2\right)}$
θ_3	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}\Gamma\left(n_3-1\right)}{\Gamma\left(n_3-0.5\right)}$	$\dfrac{\left(\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}\Gamma\left(n_3-0.5\right)}{\Gamma\left(n_3\right)}$
p_1	$\left(n_1+1\right)\left(n+3\right)^{-1}$	$\left(n_1+1\right)\left(n+3\right)^{-1}$
p_2	$\left(n_2+1\right)\left(n+3\right)^{-1}$	$\left(n_2+1\right)\left(n+3\right)^{-1}$

Table 2. Limiting expressions for the posterior risks as $t\to\infty$ using the UP and the JP under SELF.

Parameters	Posterior Risks	
	UP	JP
θ_1	$\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\left[\frac{1}{(n_1-1.5)}-\left\{\frac{\Gamma(n_1-1)}{\Gamma(n_1-0.5)}\right\}^2\right]$	$\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\left[\frac{1}{(n_1-1)}-\left\{\frac{\Gamma(n_1-0.5)}{\Gamma(n_1)}\right\}^2\right]$
θ_2	$\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\left[\frac{1}{(n_2-1.5)}-\left\{\frac{\Gamma(n_2-1)}{\Gamma(n_2-0.5)}\right\}^2\right]$	$\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\left[\frac{1}{(n_2-1)}-\left\{\frac{\Gamma(n_2-0.5)}{\Gamma(n_2)}\right\}^2\right]$
θ_3	$\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\left[\frac{1}{(n_3-1.5)}-\left\{\frac{\Gamma(n_3-1)}{\Gamma(n_3-0.5)}\right\}^2\right]$	$\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\left[\frac{1}{(n_3-1)}-\left\{\frac{\Gamma(n_3-0.5)}{\Gamma(n_3)}\right\}^2\right]$
p_1	$(n_1+1)(n_2+n_3+2)(n+3)^{-2}(n+4)^{-1}$	$(n_1+1)(n_2+n_3+2)(n+3)^{-2}(n+4)^{-1}$
p_2	$(n_2+1)(n_1+n_3+2)(n+3)^{-2}(n+4)^{-1}$	$(n_2+1)(n_1+n_3+2)(n+3)^{-2}(n+4)^{-1}$

Table 3. Limiting expressions for the Bayes estimators as $t\to\infty$ using the UP and the JP under PLF.

Parameters	Bayes Estimators	
	UP	JP
θ_1	$\left(\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}(n_1-1.5)^{-1/2}$	$\left(\frac{1}{2}\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}(n_1-1)^{-1/2}$
θ_2	$\left(\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}(n_2-1.5)^{-1/2}$	$\left(\frac{1}{2}\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}(n_2-1)^{-1/2}$
θ_3	$\left(\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}(n_3-1.5)^{-1/2}$	$\left(\frac{1}{2}\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}(n_3-1)^{-1/2}$
p_1	$(n_1+1)^{1/2}(n_1+2)^{1/2}(n+3)^{-1/2}(n+4)^{-1/2}$	$(n_1+1)^{1/2}(n_1+2)^{1/2}(n+3)^{-1/2}(n+4)^{-1/2}$
p_2	$(n_2+1)^{1/2}(n_2+2)^{1/2}(n+3)^{-1/2}(n+4)^{-1/2}$	$(n_2+1)^{1/2}(n_2+2)^{1/2}(n+3)^{-1/2}(n+4)^{-1/2}$

Table 4: Limiting expressions for the posterior risks as $t\to\infty$ using the UP and the JP under PLF

Parameters	Posterior Risks	
	UP	JP
θ_1	$\left(2\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}\left\{\frac{1}{(n_1-1.5)^{1/2}}-\frac{\Gamma(n_1-1)}{\Gamma(n_1-0.5)}\right\}$	$\left(2\sum_{k=1}^{n_1}x_{1k}^2\right)^{1/2}\left\{\frac{1}{(n_1-1)^{1/2}}-\frac{\Gamma(n_1-0.5)}{\Gamma(n_1)}\right\}$
θ_2	$\left(2\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}\left\{\frac{1}{(n_2-1.5)^{1/2}}-\frac{\Gamma(n_2-1)}{\Gamma(n_2-0.5)}\right\}$	$\left(2\sum_{k=1}^{n_2}x_{2k}^2\right)^{1/2}\left\{\frac{1}{(n_2-1)^{1/2}}-\frac{\Gamma(n_2-0.5)}{\Gamma(n_2)}\right\}$
θ_3	$\left(2\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}\left\{\frac{1}{(n_3-1.5)^{1/2}}-\frac{\Gamma(n_3-1)}{\Gamma(n_3-0.5)}\right\}$	$\left(2\sum_{k=1}^{n_3}x_{3k}^2\right)^{1/2}\left\{\frac{1}{(n_3-1)^{1/2}}-\frac{\Gamma(n_3-0.5)}{\Gamma(n_3)}\right\}$
p_1	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{(n_1+2)^{1/2}(n_1+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{(n_1+2)^{1/2}(n_1+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$
p_2	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{(n_2+2)^{1/2}(n_2+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{(n_2+2)^{1/2}(n_2+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$

Table 5. Limiting expressions for the Bayes estimators as $t \to \infty$ using the UP and the JP under DLF.

Parameters	Bayes Estimators	
	UP	JP
θ_1	$\left(\frac{1}{2}\sum_{k=1}^{n_1} x_{1k}^2\right)^{1/2} \frac{\Gamma(n_1-1.5)}{\Gamma(n_1-1)}$	$\left(\frac{1}{2}\sum_{k=1}^{n_1} x_{1k}^2\right)^{1/2} \frac{\Gamma(n_1-1)}{\Gamma(n_1-0.5)}$
θ_2	$\left(\frac{1}{2}\sum_{k=1}^{n_2} x_{2k}^2\right)^{1/2} \frac{\Gamma(n_2-1.5)}{\Gamma(n_2-1)}$	$\left(\frac{1}{2}\sum_{k=1}^{n_2} x_{2k}^2\right)^{1/2} \frac{\Gamma(n_2-1)}{\Gamma(n_2-0.5)}$
θ_3	$\left(\frac{1}{2}\sum_{k=1}^{n_3} x_{3k}^2\right)^{1/2} \frac{\Gamma(n_3-1.5)}{\Gamma(n_3-1)}$	$\left(\frac{1}{2}\sum_{k=1}^{n_3} x_{3k}^2\right)^{1/2} \frac{\Gamma(n_3-1)}{\Gamma(n_3-0.5)}$
p_1	$(n_1+2)(n+4)^{-1}$	$(n_1+2)(n+4)^{-1}$
p_2	$(n_2+2)(n+4)^{-1}$	$(n_2+2)(n+4)^{-1}$

Table 6. Limiting expressions for the posterior risks as $t \to \infty$ using the UP and the JP under DLF.

Parameters	Posterior Risks	
	UP	JP
θ_1	$1-\frac{\{\Gamma(n_1-1)\}^2}{\Gamma(n_1-0.5)\Gamma(n_1-1.5)}$	$1-\frac{\{\Gamma(n_1-0.5)\}^2}{\Gamma(n_1)\Gamma(n_1-1)}$
θ_2	$1-\frac{\{\Gamma(n_2-1)\}^2}{\Gamma(n_2-0.5)\Gamma(n_2-1.5)}$	$1-\frac{\{\Gamma(n_2-0.5)\}^2}{\Gamma(n_2)\Gamma(n_2-1)}$
θ_3	$1-\frac{\{\Gamma(n_3-1)\}^2}{\Gamma(n_3-0.5)\Gamma(n_3-1.5)}$	$1-\frac{\{\Gamma(n_3-0.5)\}^2}{\Gamma(n_3)\Gamma(n_3-1)}$
p_1	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$
p_2	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$

9. Simulation Study

For a comparative study of the Bayes estimators (under different priors and loss functions) a simulation study is conducted. The performance of Bayes estimators has been scrutinized under different priors, loss functions, parametric values, sample sizes and test termination times. We calculate the Bayes estimates and posterior risks of five parameters θ_1, θ_2, θ_3, p_1 and p_2 of a 3-component mixture of Rayleigh distributions given in (1) and (4) through a simulation using the following steps.

Step 1. A sample of the mixtures is generated as follows:

(i) The $p_1 n$ observations were taken randomly from first component density $f_1(x;\theta_1)$.

(ii) The $p_2 n$ observations were chosen randomly from second component density $f_2(x;\theta_2)$.

(iii) Remaining $(1-p_1-p_2)n$ observations were selected randomly from third component density $f_3(x;\theta_3)$.

Step 2. A sample censored at a fixed test termination time t is selected. The observations which are greater than a fixed test termination time t are taken as censored ones.

Step 3. Using the steps 1-2 for the fixed values of parameters, test termination time and sample size, 1000 samples are generated.

Step 4. The Bayes estimates and posterior risks of parameters θ_1, θ_2, θ_3, p_1 and p_2 are calculated based on 1000 repetitions by solving (14)-(43).

The above steps 1-4 are used for each of the sample sizes $n = 50,\ 100,\ 200$, each choice of the vector of the parameters $(\theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2) = \{(14,\ 12,\ 10,\ 0.5,\ 0.3),\ (16,\ 14,\ 12, 0.5, 0.3)\}$ taking $t = 25$ and 30. The choice of the test termination time is made in such a way that the censoring rate in resulting sample remains in between approximately 8% to 27%.

From Tables 7-14, it can be seen that the first component parameter θ_1 is under estimated and second and third component parameters, θ_2 and θ_3, are over estimated using the UP and the JP under SELF, PLF and DLF for different sample sizes at different test termination times and same is the case with increasing test termination time at different sample sizes. On the other hand, the first mixing proportion parameter p_1 is under estimated but second mixing proportion parameter p_2 is over estimated for different test termination times and different sample sizes. It is also observed that first component parameter θ_1 and first mixing proportion parameter p_1 are highly under estimated but second component parameter θ_2, third component parameter θ_3 and second mixing proportion parameter p_2 are highly over estimated for small test termination time and different sample sizes. Similarly, parameters θ_1 and p_1 are under estimated and parameters θ_2, θ_3 and p_2 are over estimated, with a lesser degree, for smaller values of component parameter at different test termination times and sample sizes. The degree of under or over estimation is reduced with an increase in sample size and/or test termination times.

Table 7. Bayes estimates (BE) and posterior risks (PR) using the UP under SELF, PLF and DLF with parameters $\theta_1 = 14,\ \theta_2 = 12,\ \theta_3 = 10,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 25$.

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	50	SELF	BE	13.61660	12.92240	11.76780	0.461779	0.312864
			PR	**4.532860**	**7.302360**	**9.800650**	**0.006105**	**0.005259**
		PLF	BE	13.65180	13.21880	12.15650	0.468459	0.321551
			PR	**0.317153**	**0.535797**	**0.766400**	**0.013103**	**0.016509**
		DLF	BE	13.74200	13.44650	12.59640	0.475416	0.328461
			PR	**0.022405**	**0.038845**	**0.059318**	**0.027628**	**0.050744**
	100	SELF	BE	13.61890	12.65150	11.04510	0.473665	0.311142
			PR	**2.267110**	**3.672770**	**4.200150**	**0.003369**	**0.002903**
		PLF	BE	13.69180	12.75010	11.24570	0.478501	0.314372
			PR	**0.162659**	**0.281566**	**0.360224**	**0.007079**	**0.009209**
		DLF	BE	13.75820	12.87130	11.39920	0.482014	0.319356
			PR	**0.011877**	**0.021755**	**0.031074**	**0.014863**	**0.029082**
	200	SELF	BE	13.66825	12.38945	10.50893	0.483701	0.308232
			PR	**1.126665**	**1.852856**	**1.769164**	**0.001778**	**0.001526**
		PLF	BE	13.72018	12.46556	10.63072	0.485754	0.310656
			PR	**0.082782**	**0.148116**	**0.165702**	**0.003699**	**0.004931**
		DLF	BE	13.80017	12.56750	10.68891	0.487667	0.313123
			PR	**0.006050**	**0.011771**	**0.014995**	**0.007670**	**0.015856**

Table 8. Bayes estimates (BE) and posterior risks (PR) using the UP under SELF, PLF and DLF with parameters $\theta_1 = 14,\ \theta_2 = 12,\ \theta_3 = 10,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 30$.

t	n	Loss Functions		UP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
30	50	SELF	BE	13.84720	12.66850	11.07980	0.479142	0.307347
			PR	**3.077080**	**4.824410**	**5.991890**	**0.005019**	**0.004279**
		PLF	BE	13.85340	13.05130	11.56060	0.482435	0.314924
			PR	**0.223193**	**0.381843**	**0.533656**	**0.010552**	**0.013869**
		DLF	BE	14.13800	13.14360	11.66740	0.489073	0.321534
			PR	**0.015905**	**0.028881**	**0.044356**	**0.021750**	**0.043761**
	100	SELF	BE	13.89461	12.49881	10.62590	0.486137	0.305910
			PR	**1.551918**	**2.404741**	**2.548607**	**0.002664**	**0.002265**
		PLF	BE	13.92163	12.55621	10.81606	0.489434	0.308882
			PR	**0.110118**	**0.187273**	**0.230604**	**0.005472**	**0.007352**
		DLF	BE	13.92788	12.65676	10.87924	0.492466	0.312998
			PR	**0.007898**	**0.014698**	**0.020735**	**0.011140**	**0.023621**
	200	SELF	BE	13.92148	12.26676	10.25838	0.492629	0.303763
			PR	**0.748405**	**1.170295**	**1.036255**	**0.001359**	**0.001156**
		PLF	BE	13.93816	12.35343	10.33408	0.493925	0.305617
			PR	**0.053378**	**0.093849**	**0.098372**	**0.002757**	**0.003790**
		DLF	BE	13.94119	12.43274	10.44372	0.494862	0.307757
			PR	**0.003918**	**0.007662**	**0.009570**	**0.005627**	**0.012416**

Table 9. Bayes estimates (BE) and posterior risks (PR) using the JP under SELF, PLF and DLF with parameters $\theta_1 = 14,\ \theta_2 = 12,\ \theta_3 = 10,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 25$.

t	n	Loss Functions		JP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	50	SELF	BE	13.39350	12.67030	11.41530	0.464110	0.312778
			PR	**4.174710**	**6.609760**	**8.508760**	**0.006033**	**0.005202**
		PLF	BE	13.47350	12.89260	11.67010	0.470394	0.320900
			PR	**0.305905**	**0.509247**	**0.69107**	**0.013004**	**0.016507**
		DLF	BE	13.63820	13.11700	12.02360	0.479237	0.327643
			PR	**0.022167**	**0.038726**	**0.056323**	**0.027492**	**0.051343**
	100	SELF	BE	13.54333	12.44052	10.74275	0.476556	0.310346
			PR	**2.166230**	**3.46152**	**3.773216**	**0.003338**	**0.002869**
		PLF	BE	13.62400	12.61430	10.93540	0.481282	0.313582
			PR	**0.15926**	**0.274854**	**0.338587**	**0.007033**	**0.009197**
		DLF	BE	13.70310	12.70860	11.08050	0.483959	0.318937
			PR	**0.011750**	**0.021391**	**0.029551**	**0.014752**	**0.029040**
	200	SELF	BE	13.72104	12.31254	10.41913	0.485220	0.307310
			PR	**1.117253**	**1.825202**	**1.694527**	**0.001774**	**0.001522**
		PLF	BE	13.73624	12.35744	10.54723	0.486537	0.309606
			PR	**0.081844**	**0.145002**	**0.157900**	**0.003682**	**0.004905**
		DLF	BE	13.77086	12.54591	10.73238	0.488126	0.312520
			PR	**0.005989**	**0.011709**	**0.014974**	**0.007612**	**0.015808**

Table 10. Bayes estimates (BE) and posterior risks (PR) using the JP under SELF, PLF and DLF with parameters $\theta_1 = 14,\ \theta_2 = 12,\ \theta_3 = 10,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 30$.

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
30	50	SELF	BE	13.68670	12.52110	10.72590	0.479873	0.306706
			PR	**2.96808**	**4.58495**	**5.334450**	**0.005016**	**0.004279**
		PLF	BE	13.77510	12.73580	10.93250	0.483974	0.315034
			PR	**0.214693**	**0.358819**	**0.461467**	**0.010488**	**0.013831**
		DLF	BE	13.87720	12.91540	11.30450	0.488657	0.322428
			PR	**0.015545**	**0.027395**	**0.040838**	**0.021727**	**0.043464**
	100	SELF	BE	13.78005	12.31759	10.46270	0.487439	0.305450
			PR	**1.474216**	**2.252780**	**2.341053**	**0.002641**	**0.002246**
		PLF	BE	13.84483	12.49764	10.42876	0.490466	0.309097
			PR	**0.107058**	**0.182651**	**0.208072**	**0.005430**	**0.007339**
		DLF	BE	13.90681	12.49266	10.61947	0.493060	0.312634
			PR	**0.007697**	**0.014331**	**0.019433**	**0.011066**	**0.023612**
	200	SELF	BE	13.86261	12.24702	10.21321	0.492538	0.303907
			PR	**0.737520**	**1.149594**	**1.000934**	**0.001358**	**0.001155**
		PLF	BE	13.92914	12.24655	10.25669	0.494271	0.305575
			PR	**0.053137**	**0.092307**	**0.096036**	**0.002755**	**0.003785**
		DLF	BE	13.96508	12.25574	10.30146	0.496752	0.306433
			PR	**0.003795**	**0.007506**	**0.009232**	**0.005541**	**0.012402**

Table 11. Bayes estimates (BE) and posterior risks (PR) using the UP under SELF, PLF and DLF with parameters $\theta_1 = 16,\ \theta_2 = 14,\ \theta_3 = 12,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 25$.

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	50	SELF	BE	15.11570	15.21360	14.70180	0.446887	0.315727
			PR	**7.668310**	**13.42780**	**19.90600**	**0.007654**	**0.006813**
		PLF	BE	15.31340	15.61380	15.54320	0.455036	0.325516
			PR	**0.501003**	**0.848322**	**1.283060**	**0.017251**	**0.021270**
		DLF	BE	15.51450	15.98220	16.42600	0.461612	0.335228
			PR	**0.032234**	**0.053767**	**0.078823**	**0.038399**	**0.065239**
	100	SELF	BE	15.11800	14.75620	13.76680	0.460287	0.312265
			PR	**4.118250**	**7.027940**	**9.440140**	**0.004614**	**0.004037**
		PLF	BE	15.32560	14.88800	14.12520	0.464883	0.318137
			PR	**0.270099**	**0.458719**	**0.646914**	**0.010067**	**0.012600**
		DLF	BE	15.48550	15.08730	14.29960	0.472795	0.323096
			PR	**0.017351**	**0.030657**	**0.044418**	**0.021502**	**0.039785**
	200	SELF	BE	15.43531	14.46959	13.09508	0.473544	0.309107
			PR	**2.267797**	**3.794180**	**4.576478**	**0.002684**	**0.002312**
		PLF	BE	15.44052	14.62811	13.25725	0.475648	0.313620
			PR	**0.144151**	**0.253375**	**0.331562**	**0.005624**	**0.007216**
		DLF	BE	15.50953	14.85048	13.32168	0.479002	0.318504
			PR	**0.009597**	**0.017746**	**0.024852**	**0.012124**	**0.023497**

Table 12. Bayes estimates (BE) and posterior risks (PR) using the UP under SELF, PLF and DLF with parameters $\theta_1 = 16,\ \theta_2 = 14,\ \theta_3 = 12,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 30$.

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
30	50	SELF	BE	15.56400	14.80440	14.07450	0.467505	0.308815
			PR	**5.27490**	**8.80986**	**12.80430**	**0.005739**	**0.004918**
		PLF	BE	15.71080	15.24940	14.24750	0.474324	0.318034
			PR	**0.332132**	**0.572637**	**0.845632**	**0.012296**	**0.015742**
		DLF	BE	15.75580	15.43460	15.09480	0.478758	0.325394
			PR	**0.020762**	**0.036651**	**0.056445**	**0.025952**	**0.048931**
	100	SELF	BE	15.61538	14.58070	13.13655	0.479123	0.307469
			PR	**2.649712**	**4.402702**	**5.530214**	**0.003132**	**0.002671**
		PLF	BE	15.71521	14.83954	13.39674	0.480205	0.313373
			PR	**0.171548**	**0.296408**	**0.401634**	**0.006620**	**0.008647**
		DLF	BE	15.76687	14.88723	13.47946	0.485096	0.316351
			PR	**0.010754**	**0.019744**	**0.028809**	**0.013690**	**0.027485**
	200	SELF	BE	15.70247	14.31163	12.63754	0.486731	0.305901
			PR	**1.353830**	**2.262099**	**2.464204**	**0.001660**	**0.001406**
		PLF	BE	15.74906	14.37846	12.69776	0.488738	0.307630
			PR	**0.085556**	**0.154499**	**0.186951**	**0.003398**	**0.004561**
		DLF	BE	15.78090	14.49420	12.94743	0.489275	0.310224
			PR	**0.005491**	**0.010591**	**0.014468**	**0.007025**	**0.014771**

Table 13. Bayes estimates (BE) and posterior risks (PR) using the JP under SELF, PLF and DLF with parameters $\theta_1 = 16,\ \theta_2 = 14,\ \theta_3 = 12,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 25$.

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	50	SELF	BE	14.96780	14.57550	13.93930	0.453393	0.313068
			PR	**7.306700**	**12.00010**	**16.88400**	**0.007623**	**0.006667**
		PLF	BE	15.14660	15.11190	14.43220	0.459733	0.325297
			PR	**0.487670**	**0.814612**	**1.152940**	**0.017248**	**0.021235**
		DLF	BE	15.41070	15.50230	15.13280	0.468276	0.335345
			PR	**0.031401**	**0.052208**	**0.075760**	**0.037618**	**0.064932**
	100	SELF	BE	15.13860	14.48150	13.41110	0.462403	0.312755
			PR	**4.075790**	**6.598950**	**8.646480**	**0.004606**	**0.004010**
		PLF	BE	15.25440	14.59210	13.68450	0.470232	0.317048
			PR	**0.261286**	**0.440179**	**0.615404**	**0.009932**	**0.012494**
		DLF	BE	15.50490	14.84930	13.99560	0.476549	0.322732
			PR	**0.017062**	**0.030355**	**0.043842**	**0.021345**	**0.039716**
	200	SELF	BE	15.30680	14.46934	12.84924	0.473443	0.311552
			PR	**2.217586**	**3.762853**	**4.306267**	**0.002664**	**0.002309**
		PLF	BE	15.42961	14.51248	13.01358	0.477681	0.313408
			PR	**0.143310**	**0.251185**	**0.316505**	**0.005621**	**0.007213**
		DLF	BE	15.53817	14.65853	13.10938	0.480496	0.317891
			PR	**0.009418**	**0.017384**	**0.023887**	**0.011969**	**0.023147**

Table 14. Bayes estimates (BE) and posterior risks (PR) using the JP under SELF, PLF and DLF with parameters $\theta_1 = 16,\ \theta_2 = 14,\ \theta_3 = 12,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 30$.

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
30	50	SELF	BE	15.35800	14.57660	13.31560	0.471060	0.308452
			PR	**4.947810**	**8.266080**	**10.78650**	**0.005713**	**0.004898**
		PLF	BE	15.59180	14.89550	13.75830	0.475835	0.318052
			PR	**0.320650**	**0.542823**	**0.769462**	**0.012210**	**0.015686**
		DLF	BE	15.62690	15.21680	14.11540	0.480833	0.326659
			PR	**0.020461**	**0.035675**	**0.053454**	**0.025755**	**0.048920**
	100	SELF	BE	15.52212	14.37243	12.73206	0.481181	0.307124
			PR	**2.565359**	**4.211739**	**4.986836**	**0.003115**	**0.002657**
		PLF	BE	15.61134	14.54063	12.96967	0.483012	0.312483
			PR	**0.165003**	**0.282848**	**0.373641**	**0.006510**	**0.008535**
		DLF	BE	15.68469	14.75707	13.26403	0.485645	0.317044
			PR	**0.010688**	**0.019446**	**0.028039**	**0.013630**	**0.027442**
	200	SELF	BE	15.68846	14.29900	12.46872	0.487551	0.305483
			PR	**1.333162**	**2.232128**	**2.316557**	**0.001650**	**0.001403**
		PLF	BE	15.66988	14.32500	12.59988	0.489181	0.307299
			PR	**0.084804**	**0.152388**	**0.182305**	**0.003390**	**0.004560**
		DLF	BE	15.74674	14.40692	12.73768	0.490581	0.309792
			PR	**0.005413**	**0.010447**	**0.013925**	**0.006959**	**0.014748**

As far as posterior risks are concerned, it is observed from the Tables 7-14 that the posterior risks of Bayes estimators using the UP and the JP under SELF, PLF and DLF reduce with an increase in sample size and/or test termination times. For larger (smaller) test termination time, the posterior risks of Bayes estimators assuming the UP and the JP under SELF, PLF and DLF are smaller (larger) at any fixed sample size. An important feature about selection of the UP and the JP under SELF, PLF and DLF based on posterior risk is observed as follows. For different sample sizes and test termination times, the JP, due to less posterior risk under SELF, PLF and DLF, yields efficient and preferable results as compared to the UP. Through a comparison between the three different loss functions it may be observed that the posterior risks of Bayes estimators of component parameters θ_1, θ_2 and θ_3 using the UP and the JP under PLF are smaller than SELF but larger than DLF at different sample sizes and test termination times. Thus, in our study, the DLF is more preferable. However, assuming the UP and the JP, the posterior risks of Bayes estimators of proportion parameters p_1 and p_2 under PLF at different sample sizes and test termination times are higher (lower) than that under the SELF (DLF). Hence, SELF is a preferable loss function.

10. Real Life Application

The real mixture data, $\mathbf{y} = \left(y_{11}, y_{12}, ..., y_{1r_1}, y_{21}, y_{22}, ..., y_{2r_2}, y_{31}, y_{32}, ..., y_{3r_3}\right)$, are taken from Davis [32]. These data represent hours to failure of a V805 Transmitter Tube**,** a Transmitter Tube and a V600 Indicator Tube used in aircraft radar sets. Davis [32] showed that the data $\mathbf{y}$ can be modeled by a mixture of exponential distributions. The transformation $x = \sqrt{2y}$ of an exponential random data $(\mathbf{y})$ yields the Rayleigh random data $(\mathbf{x})$. This transformation allows us to use the Davis mixture data for applying the proposed Bayesian analysis. To have a type-I right censored data we fix $t = 800$ hours. The tests are conducted 1340 times. Thus, we have a type-I right censored data at $t = 800$ hours on $n = 1340$ radar sets. The data summary required to evaluate the Bayes estimates and posterior risks is:

$$n=1340,\ r_1=891,\ r_2=337,\ r_3=92,\ r=1320,\ n-r=20,\ \sum_{k=1}^{r_1}x_{1k}^2=2\sum_{k=1}^{r_1}y_{1k}=302260,$$
$$\sum_{k=1}^{r_2}x_{2k}^2=2\sum_{k=1}^{r_2}y_{2k}=100750,\ \sum_{k=1}^{r_3}x_{3k}^2=2\sum_{k=1}^{r_3}y_{3k}=45100.$$

The Bayes estimates and posterior risks using the UP and the JP under SELF, PLF and DLF are showcased in Table 15.

Table 15. Bayes estimates (BE) and posterior risks (PR) using the UP and the JP under SELF, PLF and DLF with real life mixture data.

Prior	Loss Function		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
UP	SELF	BE	13.3794157	12.4368133	17.8110600	0.6717713	0.2531304
		PR	**0.0593967**	**0.1349736**	**1.3915633**	**0.0001685**	**0.0001419**
	PLF	BE	13.3816352	12.4422385	17.8500818	0.6718968	0.2534104
		PR	**0.0044390**	**0.0108504**	**0.0780437**	**0.0002508**	**0.0005601**
	DLF	BE	13.3838551	12.4476661	17.8891892	0.6720222	0.2536908
		PR	**0.0003317**	**0.0008719**	**0.0043674**	**0.0003733**	**0.0022091**
JP	SELF	BE	13.3785388	12.4278616	17.7402374	0.6718352	0.2531327
		PR	**0.0593650**	**0.1346396**	**1.3770762**	**0.0001685**	**0.0001418**
	PLF	BE	13.3807573	12.4332773	17.7790072	0.6719606	0.2534128
		PR	**0.0044370**	**0.0108313**	**0.0775397**	**0.0002507**	**0.0005601**
	DLF	BE	13.3829762	12.4386953	17.8178618	0.6720860	0.2536931
		PR	**0.0003316**	**0.0008710**	**0.0043566**	**0.0003732**	**0.0022091**

From the Table 15, it is observed that the results based on the real data are compatible with simulated results. The results about selecting the best prior and the best loss function are also the same as we have discussed in the Section 9.

Table 16. Bayesian predictive interval (L, U) using the UP and the JP with real life mixture data.

n	UP		JP	
	L	U	L	U
1340	4.26752	33.3881	4.26565	33.3601

The results in the Table 16 are the 90% Bayesian predictive intervals assuming the UP and the JP. It is observed that the Bayesian predictive intervals using the JP are narrower than the predictive intervals using the UP.

11. Concluding Remarks

The importance and application of mixture models in real life problems is un-deniable. An extensive simulation study is conducted to compare and illustrate some interesting properties of the BEs of 3-component mixture of Rayleigh distribution using the UP and the JP under SELF, PLF and DLF. From Tables 7-14, we conclude that an increase in sample size or an increase in test termination time provides us improved Bayes estimates. For different sample sizes at different test termination times, first component and mixing proportion parameters are under estimated but second and third component and second mixing proportion parameters are over estimated using the UP and the JP under SELF, PLF and DLF and same is the case with different test termination times at different sample sizes. Furthermore, it is clear from the Tables 7-14 that as sample size increases the posterior risks of Bayes estimators decrease. The posterior risks of Bayes estimators are smaller for smaller values of component parameters. Also, the DLF (SELF) appears as a suitable loss function for estimating component (proportion). Moreover, due to smaller posterior risk under SELF, PLF and DLF, the JP is more preferable than UP. Finally, we conclude that the more efficient and suitable prior is the JP under DLF (SELF) for estimating component (proportion) parameters.

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