Dynamics of High-order Fuzzy Cellular Neural Networks with Time-varying Delays

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Abstract

In this paper, dynamic behavior of a class of high-order fuzzy cellular neural networks (HFCNNs) with time-varying delays is investigated. Compared with the previous results in the literature, the restrictions are loosed, since we do not assume the boundedness and monotonicity on the activation functions, and the differentiability of time-varying delay functions. Some sufficient conditions are derived for ascertaining the existence, uniqueness and exponential stability of equilibrium point and uniform boundedness of solutions of the HFCNNs. Finally, two examples are given to show the effectiveness of the proposed criteria, which complement some previously known results.

Keywords: High-order neural networks; Equilibrium point; Boundedness; Time-varying delays; Exponential stability.

1. Introduction

The fuzzy cellular neural networks (FCNNs), which combines fuzzy logic with the traditional CNNs, has been recognized as an appealing and efficient tool in classification, image processing and pattern recognition, since it was introduced by Yang et al. in 1996 [1,2]. Analysis of stability has been the focus of scientific community, because it is required that the neural networks be stable in applications. Various techniques, such as M-matrix, algebraic inequality method, and LMI technique, are employed to derive the stability results of CNNs or FCNNs (see [3] - [12] and reference therein).

Recently, the high-order neural networks have attracted considerable attention, as high-order neural networks have stronger approximation property,

faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks (see [13]- [32] and reference therein). Many results on dynamic behavior for various high-order neural networks were reported in the literature [13]-[32]. By using the generalized Halanay inequality and the fixed point theorem, Guo et al. [15] discussed the global exponential stability and existence of periodic solutions for neural networks. In [17]-[21], by using Lyapunov functional method, Razumikhin method and LMI technique, the authors have studied the stability of HBAM. Employing nonsingular M-matrix and Lyapunov functional method, Xu et al. [18] studied Hopfields NNs without time-delay, and Jiang et al. [23] studied the exponential stability of HFCNNs with time-varying delays. By using Lyapunov stability theory, LMI and free-

weighting matrix method, Zheng et al. [25] derived some delay-independent and delay-dependent sufficient conditions for a class of high-order Hopfield-type neural networks with time-varying delays. By employing a Lyapunov-Krasovskii functional and linear matrix inequality, Wang et al. [30] got some criteria of global exponential stability in the mean square for the reaction-diffusion high-order neural networks. However, the boundedness of the activation functions is assumed in the previous results [16]- [28], and the differentiability of time-varying delays is needed in [23] and [27]- [30]. It is obvious that the assumption on the boundedness of activation functions and the differentiability of time-varying delays would make the results not applicable to some important engineering and physical problems. For example, the Lurie-type activation functions are unbounded. To facilitate the design of

neural networks, it is worthwhile to continue to investigate the dynamic behavior of system with more general activation functions and time-varying delays [34]- [37].

Motivated by the above discussion, our objective in this paper is to study the dynamics of high-order fuzzy cellular neural networks (HFCNNs). Without assuming the boundedness on the activation functions and the differentiability of time-varying delays, we will give some criteria on existence, uniqueness and exponential stability of equilibrium point, and prove the uniform boundedness of all solutions of this HFCNNs.

2. Model Description and Preliminaries

In this paper, we consider the following delayed H-FCNNs:

$$
\begin{aligned}
\frac{dx_i(t)}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t - \tau_{ij}(t))) + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} g_j(x_j(t - \tau_{ij}(t))) g_l(x_l(t - \tau_{ij}(t))) \\
&+ \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t - \tau_{ij}(t))) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t - \tau_{ij}(t))) \\
&+ \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j(t - \tau_{ij}(t))) g_l(x_l(t - \tau_{ij}(t)))) \\
&+ \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j(t - \tau_{ij}(t))) g_l(x_l(t - \tau_{ij}(t)))) \\
&+ \bigwedge_{j=1}^{n} T_{ij} \mu_j + \bigvee_{j=1}^{n} H_{ij} \mu_j + I_i,
\end{aligned}
\tag{1}
$$

with the initial condition

$$
x_i(t) = \varphi_i(t), \qquad -h \leqslant t \leqslant 0,
$$

where $i, j, l = 1, 2, ..., n$. $x_i(t)$ denotes the potential (or voltage) of the cell i at time t; d_i is positive constant, which denotes the rate with which the i-th cell resets its potential to the resting state when isolated from the other cells and inputs; $\tau_{ij}(t)$ represents the transmission time-varying delay, which satisfies $0 \leqslant \tau_{ij}(t) \leqslant h$; a_{ij} and b_{ijl} are the first-order and second-order connection weights of neural network, respectively; $\alpha_{ij}, \beta_{ij}, T_{ij}$ and H_{ij} are elements of the first-order fuzzy feedback MIN template, first-order fuzzy feedback MAX template, first-order fuzzy feed-forward MIN template and first-order fuzzy feed-forward MAX template, respectively; γ_{ijl} and σ_{ijl} are elements of the second-order fuzzy feedback MIN template, second-order fuzzy feedback MAX template,

respectively; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively; μ_i and I_i denote input and bias of the i-th neurons, respectively; f_i, g_i are the activation functions.

For convenience, we introduce the following notations. $R^+ = [0, +\infty)$. $C[X,Y]$ is a continuous mapping set from the topological space X to the topological space Y. For $x(t) : R^+ \to R^n$, we define $\|x(t)\| = \max\limits_{1 \leqslant i \leqslant n} \{q_i \mid x_i(t) \mid\}$, and for the initial value $\varphi \in C$ we define $\| \varphi \|_h = \sup\limits_{-h \leqslant s \leqslant 0} \{\| \varphi(s) \|\}$, where $\| \varphi(s) \| = \max\limits_{1 \leqslant i \leqslant n} \{q_i \mid \varphi_i(s) \mid\}$.

$$M = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \frac{(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|)q_i q_j^{-1} L_j}{d_i},$$

$$L = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|)q_i q_j^{-1} q_l^{-1} K_j K_l}{d_i},$$

$$P = \max_{1 \leqslant i \leqslant n} \frac{q_i}{d_i}\left[\sum_{j=1}^{n} |a_{ij}f_j(0)| + \bigwedge_{j=1}^{n} |\alpha_{ij}f_j(0)| + \bigvee_{j=1}^{n} |\beta_{ij}f_j(0)| + \bigwedge_{j=1}^{n} |T_{ij}\mu_j| + \bigvee_{j=1}^{n} |H_{ij}\mu_j| + |I_i| \right],$$

$$D_1 = \frac{1 - M - \sqrt{(1-M)^2 - 4LP}}{2L},$$

$$D_2 = \frac{1 - M + \sqrt{(1-M)^2 - 4LP}}{2L}.$$

For system (1), we make the following assumptions:
(H1) There are some positive constants L_i, K_i such that

$$| f_i(\mu) - f_i(\nu) | \leqslant L_i \mid \mu - \nu \mid, \qquad | g_i(\mu) - g_i(\nu) | \leqslant K_i \mid \mu - \nu \mid,$$

$\mu, \nu \in R$, $g_i(0) = 0$, $i = 1, 2, ..., n$.

(H2) There exist constants $q_i > 0, i = 1, 2, ..., n$, such that $M < 1$ and

$$(M - 1)^2 - 4LP > 0.$$

To obtain our main results, we need the following lemma.
Lemma 1 [2]   Suppose $x = (x_1, x_2, ... x_n)$ and $y = (y_1, y_2, ... y_n)$ are two states of system (1), then we have

$$| \bigwedge_{j=1}^{n} \alpha_{ij}f_i(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij}f_i(y_j) | \leqslant \sum_{j=1}^{n} | \alpha_{ij} || f_i(x_j) - f_i(y_j) |,$$

$$| \bigvee_{j=1}^{n} \alpha_{ij}f_i(x_j) - \bigvee_{j=1}^{n} \alpha_{ij}f_i(y_j) | \leqslant \sum_{j=1}^{n} | \alpha_{ij} || f_i(x_j) - f_i(y_j) | .$$

3. Main results

Theorem 1 Under the assumptions (H1)-(H2), system (1) has a unique equilibrium point x^* in the region $S = \{\varphi \in C \mid \quad \| \varphi \|_h \leqslant d\}$, where $d \in [D_1, D_2)$ is a constant.

Proof: If $x^* = (x_1^*, x_1^*, \cdots, x_n^*)$ is an equilibrium of system (1), x^* satisfies the following equation:

$$
\begin{aligned}
x_i^* =\ & \frac{1}{d_i} \Bigg[\sum_{j=1}^n a_{ij} f_j(x_j^*) + \bigwedge_{j=1}^n \alpha_{ij} f_j(x_j^*) + \bigvee_{j=1}^n \beta_{ij} f_j(x_j^*) + \sum_{j=1}^n \sum_{l=1}^n b_{ijl} g_j(x_j^*) g_l(x_l^*) \\
& + \bigwedge_{j=1}^n (\bigwedge_{l=1}^n \gamma_{ijl} g_j(x_j^*) g_l(x_l^*)) + \bigvee_{j=1}^n (\bigvee_{l=1}^n \sigma_{ijl} g_j(x_j^*) g_l(x_l^*)) \\
& + \bigwedge_{j=1}^n T_{ij} \mu_j + \bigvee_{j=1}^n H_{ij} \mu_j + I_i \Bigg], \quad i = 1, 2, ..., n.
\end{aligned}
\tag{2}
$$

Let $\Omega = \{x \in C \mid \|x\| \leqslant d\}$. It is obvious that $\Omega \subset S$ holds. Consider a mapping $F : \Omega \to R^n$ defined by

$$
\begin{aligned}
F_i(x) =\ & \frac{1}{d_i} \Bigg[\sum_{j=1}^n a_{ij} f_j(x_j) + \bigwedge_{j=1}^n \alpha_{ij} f_j(x_j) + \bigvee_{j=1}^n \beta_{ij} f_j(x_j) + \sum_{j=1}^n \sum_{l=1}^n b_{ijl} g_j(x_j) g_l(x_l) \\
& + \bigwedge_{j=1}^n (\bigwedge_{l=1}^n \gamma_{ijl} g_j(x_j) g_l(x_l)) + \bigvee_{j=1}^n (\bigvee_{l=1}^n \sigma_{ijl} g_j(x_j) g_l(x_l)) \\
& + \bigwedge_{j=1}^n T_{ij} \mu_j + \bigvee_{j=1}^n H_{ij} \mu_j + I_i \Bigg], \quad i = 1, 2, ..., n,
\end{aligned}
\tag{3}
$$

where $F(x) = (F_1(x), F_2(x), ... F_n(x))^T$.

Without loss of generality, we assume that $P \neq 0$. From the assumption (H2), we know the equation $Ld^2 + (M-1)d + P = 0$ has two solutions D_1 and D_2. For any $d \in [D_1, D_2)$, we have

$$
Ld^2 + (M-1)d + P \leqslant 0 \quad \text{and} \quad M + Ld < 1.
\tag{4}
$$

From assumption (H1), we obtain $|f_j(x_j) - f_j(0)| \leqslant L_j |x_j|$ and

$$
\begin{aligned}
& \sum_{j=1}^n \sum_{l=1}^n b_{ijl} g_j(x_j) g_l(x_l) + \bigwedge_{j=1}^n (\bigwedge_{l=1}^n \gamma_{ijl} g_j(x_j) g_l(x_l)) + \bigvee_{j=1}^n (\bigvee_{l=1}^n \sigma_{ijl} g_j(x_j) g_l(x_l)) \\
& \leqslant \sum_{j=1}^n \sum_{l=1}^n (|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|) K_j K_l |x_j| |x_l|.
\end{aligned}
\tag{5}
$$

For any $x \in \Omega$, and $i = 1, 2, ..., n$, we can get

$$
\begin{aligned}
q_i \mid F_i(x) \mid \; = \; & \frac{q_i}{d_i} \left| \sum_{j=1}^{n} a_{ij} f_j(x_j) + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} g_j(x_j) g_l(x_l) \right. \\
& + \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j) g_l(x_l)) + \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j) g_l(x_l)) + \bigwedge_{j=1}^{n} T_{ij} \mu_j + \left. \bigvee_{j=1}^{n} H_{ij} \mu_j + I_i \right| \\
= \; & \frac{q_i}{d_i} \left| \sum_{j=1}^{n} a_{ij} f_j(x_j) - \sum_{j=1}^{n} a_{ij} f_j(0) + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(0) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(0) \right. \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} g_j(x_j) g_l(x_l) + \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j) g_l(x_l)) + \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j) g_l(x_l)) + \sum_{j=1}^{n} a_{ij} f_j(0) \\
& + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(0) + \bigvee_{j=1}^{n} \beta_{ij} f_j(0) + \bigwedge_{j=1}^{n} T_{ij} \mu_j + \left. \bigvee_{j=1}^{n} H_{ij} \mu_j + I_i \right| \\
\leqslant \; & \frac{q_i}{d_i} \left[\sum_{j=1}^{n} |(a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|) L_j |x_j| + \sum_{j=1}^{n} \sum_{l=1}^{n} 2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|) K_j K_l |x_j| |x_l| \right. \\
& + \sum_{j=1}^{n} |a_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |\alpha_{ij} f_j(0)| + \bigvee_{j=1}^{n} |\beta_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |T_{ij} \mu_j| + \left. \bigvee_{j=1}^{n} |H_{ij} \mu_j| + |I_i| \right] . \\
= \; & \sum_{j=1}^{n} \frac{(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|) q_i q_j^{-1} L_j}{d_i} q_j |x_j| \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|) q_i q_j^{-1} q_l^{-1} K_j K_l}{d_i} q_j |x_j| q_l |x_l| \\
& + \frac{q_i}{d_i} \left[\sum_{j=1}^{n} |a_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |\alpha_{ij} f_j(0)| \bigvee_{j=1}^{n} |\beta_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |T_{ij} \mu_j| + \bigvee_{j=1}^{n} |H_{ij} \mu_j| + |I_i| \right] \\
\leqslant \; & \sum_{j=1}^{n} \frac{(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|) q_i q_j^{-1} L_j}{d_j} \|x\| \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|) q_i q_j^{-1} q_l^{-1} K_j K_l}{d_i} \|x\|^2 \\
& + \frac{q_i}{d_i} \left[\sum_{j=1}^{n} |a_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |\alpha_{ij} f_j(0)| \bigvee_{j=1}^{n} |\beta_{ij} f_j(0)| + \bigwedge_{j=1}^{n} |T_{ij} \mu_j| + \bigvee_{j=1}^{n} |H_{ij} \mu_j| + |I_i| \right]
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant\ \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\frac{(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)q_i q_j^{-1}L_j}{d_j}d \\
&+\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{2(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)q_i q_j^{-1}q_l^{-1}K_j K_l}{d_i}d^2 \\
&+\max_{1\leqslant i\leqslant n}\frac{q_i}{d_i}\left[\sum_{j=1}^{n}|a_{ij}f_j(0)|+\bigwedge_{j=1}^{n}|\alpha_{ij}f_j(0)|\bigvee_{j=1}^{n}|\beta_{ij}f_j(0)|+\bigwedge_{j=1}^{n}|T_{ij}\mu_j|+\bigvee_{j=1}^{n}|H_{ij}\mu_j|+|I_i|\right] \\
&=\ Md+Ld^2+P \\
&\leqslant\ d.
\end{aligned}
\tag{6}
$$

Accordingly, we have $\|f(x)\|=\max_{1\leqslant i\leqslant n}\{q_i|F_i(x_i)|\}\leqslant d$, which implies F maps the set Ω into itself. By the Brower's fixed point theorem [33], we get $F(x^*)=x^*, x^*\in\Omega$, i.e., the system (1) has a solution.

Assume that $x=(x_1,x_2,...x_n)$ and $y=(y_1,y_2,...y_n)$ are two solutions of system (1), then

$$
\begin{aligned}
x_i\ =\ \frac{1}{d_i}\Bigg[&\sum_{j=1}^{n}a_{ij}f_j(x_j)+\bigwedge_{j=1}^{n}\alpha_{ij}f_j(x_j)+\bigvee_{j=1}^{n}\beta_{ij}f_j(x_j)+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(x_j)g_l(x_l) \\
&+\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(x_j)g_l(x_l))+\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(x_j)g_l(x_l))+\bigwedge_{j=1}^{n}T_{ij}\mu_j+\bigvee_{j=1}^{n}H_{ij}\mu_j+I_i\Bigg],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
y_i\ =\ \frac{1}{d_i}\Bigg[&\sum_{j=1}^{n}a_{ij}f_j(y_j)+\bigwedge_{j=1}^{n}\alpha_{ij}f_j(y_j)+\bigvee_{j=1}^{n}\beta_{ij}f_j(y_j)+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(y_j)g_l(y_l) \\
&+\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(y_j)g_l(y_l))+\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(y_j)g_l(y_l))+\bigwedge_{j=1}^{n}T_{ij}\mu_j+\bigvee_{j=1}^{n}H_{ij}\mu_j+I_i\Bigg].
\end{aligned}
\tag{8}
$$

Thus,

$$
\begin{aligned}
|x_i-y_i|\ =\ \frac{1}{d_i}\Bigg|&\sum_{j=1}^{n}a_{ij}f_j(x_j)-\sum_{j=1}^{n}a_{ij}f_j(y_j)+\bigwedge_{j=1}^{n}\alpha_{ij}f_j(x_j)-\bigwedge_{j=1}^{n}\alpha_{ij}f_j(y_j) \\
&+\bigvee_{j=1}^{n}\beta_{ij}f_j(x_j)-\bigvee_{j=1}^{n}\beta_{ij}f_j(y_j)+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(x_j)g_l(x_l)-\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(y_j)g_l(y_l) \\
&+\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(x_j)g_l(x_l))-\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(y_j)g_l(y_l)) \\
&+\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(x_j)g_l(x_l))-\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(y_j)g_l(y_l))\Bigg|,
\end{aligned}
\tag{9}
$$

$i=1,2,...,n$. From (9), we have

$$
\begin{aligned}
q_i|x_i - y_i| \;\leqslant\; & \frac{q_i}{d_i}\Bigg[\sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)L_j|x_j-y_j| \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)\,|\,g_j(x_j)g_l(x_l)-g_j(y_j)g_l(y_l)\,|\Bigg] \\
\;\leqslant\; & \frac{q_i}{d_i}\Bigg[\sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)L_j|x_j-y_j| \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)K_jK_l(|x_j||x_l-y_l|+|y_l||x_j-y_j|)\Bigg] \\
\;\leqslant\; & \frac{q_i}{d_i}\Bigg[\sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)L_j q_j^{-1}\|x-y\| \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)K_jK_l q_j^{-1}q_l^{-1}(\|x\|\|x-y\|+\|y\|\|x-y\|)\Bigg] \\
\;\leqslant\; & M\|x-y\|+Ld\|x-y\| \\
\;=\; & (M+Ld)\|x-y\|, \quad i=1,2,...,n.
\end{aligned}
\tag{10}
$$

Thus $\|x-y\| = \max\limits_{1\leqslant i\leqslant n}\{q_i|x_i-y_i|\} \leqslant (M+Ld)\|x-y\|$. If $\|x-y\| \neq 0$, we have $M+Ld \geqslant 1$, which contradicts $M+Ld < 1$. Hence $x = y$. The proof is complete.

Theorem 2   Under the assumptions (H1)-(H2), all the solutions of system (1) are uniformly bounded in the region $S = \{\varphi \in C \mid \;\; \|\varphi\|_h \leqslant d\}$, where $d \in [D_1, D_2)$ is a constant.

Proof: We will show that $\|x(t)\| \leqslant d, \quad d \in [D_1, D_2)$, for $t \geqslant 0$, where $x(t)$ is a solution of the system (1) with the initial $\varphi \in S$.

To prove $\|x(t)\| \leqslant d, \quad d \in [D_1, D_2)$, we first show that, for any $d \in [D_1, D_2)$, there exists $\beta > 1$, such that $\|x(t)\| < \beta d, \quad \beta d \in (D_1, D_2), t \geqslant 0$.

If $\|x(t)\| < \beta d, \quad \beta d \in (D_1, D_2), t \geqslant 0$ is not true, there must be some $t_1 > 0$ such that

$$\|x(t_1)\| = \beta d, \qquad D^-(\|x(t_1)\|) \geqslant 0$$

and

$$\|x(t)\| < \beta d, \qquad 0 \leqslant t < t_1$$

hold.

By the definition of the vector norm, there exists $i \in \{1,2,...,n\}$, so that

$$\|x(t_1)\| = q_i|x_i(t_1)| = \beta d, \qquad D^-(\|x(t_1)\|) \geqslant 0 \tag{11}$$

and

$$q_j|x_j(t_1)| < \beta d, \qquad 0 \leqslant t < t_1, \qquad j=1,2,...,n. \tag{12}$$

From the system (1), we have

$$
\begin{aligned}
D^{-}(q_i|x_i(t)|) &= q_i\dot{x}_i(t)sgn(x_i(t)) \\
&= \left[-q_i d_i x_i(t) + q_i \sum_{j=1}^{n} a_{ij} f(x_j(t-\tau_{ij}(t))) + q_i \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau_{ij}(t))) \right. \\
&\quad + q_i \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau_{ij}(t))) + q_i \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl} g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t))) \\
&\quad + q_i \bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t)))) \\
&\quad + q_i \bigvee_{j=1}^{n}(\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t)))) \\
&\quad \left. + q_i \bigwedge_{j=1}^{n} T_{ij}\mu_j + q_i \bigvee_{j=1}^{n} H_{ij}\mu_j + q_i I_i \right] sgn(x_i(t)).
\end{aligned}
\tag{13}
$$

From (13), we get

$$
\begin{aligned}
D^{-}(q_i|x_i(t_1)|) &\leqslant -q_i d_i |x_i(t_1)| + q_i \sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)L_j|x_j(t_1-\tau_{ij}(t_1))| \\
&\quad + q_i \sum_{j=1}^{n}\sum_{l=1}^{n} 2(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)K_j K_l |x_j(t_1-\tau_{ij}(t_1))||x_l(t_1-\tau_{ij}(t_1))| \\
&\quad + q_i\left[\sum_{j=1}^{n}|a_{ij}f_j(0)| + \bigwedge_{j=1}^{n}|\alpha_{ij}f_j(0)| + \bigvee_{j=1}^{n}|\beta_{ij}f_j(0)| \right. \\
&\quad \left. + \bigwedge_{j=1}^{n}|T_{ij}\mu_j| + \bigvee_{j=1}^{n}|H_{ij}\mu_j| + |I_i| \right] \\
&\leqslant d_i\left\{ -\beta d + \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n} \frac{(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)q_i q_j^{-1}L_j}{d_i}\beta d \right. \\
&\quad + \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\sum_{l=1}^{n} \frac{2(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)q_i q_j^{-1}q_l^{-1}K_j K_l}{d_i}(\beta d)^2 \\
&\quad + \max_{1\leqslant i\leqslant n}\frac{q_i}{d_i}\left[\sum_{j=1}^{n}|a_{ij}f_j(0)| + \bigwedge_{j=1}^{n}|\alpha_{ij}f_j(0)| + \bigvee_{j=1}^{n}|\beta_{ij}f_j(0)| \right. \\
&\quad \left.\left. + \bigwedge_{j=1}^{n}|T_{ij}\mu_j| + \bigvee_{j=1}^{n}|H_{ij}\mu_j| + |I_i| \right]\right\} \\
&= d_i[-\beta d + M(\beta d) + L(\beta d)^2 + P] \\
&= d_i[L(\beta d)^2 + (M-1)(\beta d) + P] < 0,
\end{aligned}
\tag{14}
$$

which contradicts the inequality (9). So $\|x(t)\| < \beta d$, $\beta d \in (D_1, D_2)$ holds. Let $\beta \to 1$, we have $\|x(t)\| < d$, $d \in [D_1, D_2)$. The proof is complete.

Theorem 3 Under the assumptions (H1)-(H2), the equilibrium of the system (1) is exponentially stable in the region $S = \{\varphi \in C \mid \ \| \varphi \|_h \leqslant d\}$, where $d \in [D_1, D_2)$ is a constant.

Proof: Suppose x^* is the equilibrium of system (1). Let $y(t) = x(t) - x^*$, then system (1) can be transformed into the following system:

$$
\begin{aligned}
\frac{dy_i}{dt} = {} & -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j) - \sum_{j=1}^{n} a_{ij} f_j(x_j^*) \\
& + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j^*) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j^*) \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} g_j(x_j) g_l(x_l) - \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} g_j(x_j^*) g_l(x_l^*) \\
& + \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j) g_l(x_l)) - \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl} g_j(x_j^*) g_l(x_l^*)) \\
& + \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j) g_l(x_l)) - \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl} g_j(x_j^*) g_l(x_l^*)),
\end{aligned}
\tag{15}
$$

with the initial condition

$$
y_i(t) = \phi_i(t), \qquad -h \leqslant t \leqslant 0
$$

where $i, j, l = 1, 2, ..., n$, $\phi_i(t) = \varphi_i(t) - x_i^*$. Obviously, to prove equilibrium x^* of the the system (1) is exponentially stable, we will only consider the exponential stability of the equilibrium point O of the system (15).

From the inequality $M + Ld < 1$, there exists λ, $0 < \lambda \ll \min\limits_{1 \leqslant i \leqslant n} \{d_i\}$, such that $\eta + \dfrac{\lambda}{\min\limits_{1 \leqslant i \leqslant n} \{d_i\}} < 1$, where

$$
\eta = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \frac{(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|) q_i q_j^{-1} L_j e^{\lambda h}}{d_i} + \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|) q_i q_j^{-1} q_l^{-1} K_j K_l d e^{\lambda h}}{d_i}.
$$

Let $V(t) = \|y(t)\| e^{\lambda t}$, for any $\varphi \in S$, we will show that

$$
V(t) \leqslant \frac{1}{1-\eta} \|\phi\|_h, \qquad \forall \ t \geqslant 0
\tag{16}
$$

where $\phi = \varphi - x^*$.

To prove (16), we first show that for any given $\xi > 1$,

$$
V(t) \leqslant \frac{\xi}{1-\eta} \|\phi\|_h
\tag{17}
$$

holds for all $t \geqslant 0$.

Similar to the discussion of Theorem 2, if the inequality (17) is not true, there must be exists $t_2 \geqslant 0$, such that

$$
V(t_2) = \frac{\xi}{1-\eta} \|\phi\|_h, \qquad D^-(V(t_2)) \geqslant 0
\tag{18}
$$

and

$$V(t_2) < \frac{\xi}{1-\eta}\|\phi\|_h, \qquad 0 \leqslant t < t_2. \tag{19}$$

By the definition of the vector norm, there exists $i \in \{1,2,...,n\}$, such that

$$V(t_2) = \|y(t_2)\|e^{\lambda t_2} = q_i|y_i(t_2)|e^{\lambda t_2} = \frac{\xi}{1-\eta}\|\phi\|_h, \qquad D^-(V(t_2)) \geqslant 0. \tag{20}$$

Obviously,

$$
\begin{aligned}
D^-(V(t)) &= D^-(q_i|y_i(t)|e^{\lambda t}) \\
&= \lambda q_i|y_i(t)|e^{\lambda t} + q_i\dot{y}_i(t)e^{\lambda t}sgn(y_i(t)) \\
&= \lambda q_i|y_i(t)|e^{\lambda t} + q_ie^{\lambda t}sgn(y_i(t))\left[-d_iy_i(t) + \sum_{j=1}^{n}a_{ij}f_j(x_j(t-\tau_{ij}(t)))\right. \\
&\quad - \sum_{j=1}^{n}a_{ij}f_j(x_j^*) + \bigwedge_{j=1}^{n}\alpha_{ij}f_j(x_j(t-\tau_{ij}(t))) - \bigwedge_{j=1}^{n}\alpha_{ij}f_j(x_j^*) \\
&\quad + \bigvee_{j=1}^{n}\beta_{ij}f_j(x_j(t-\tau_{ij}(t))) - \bigvee_{j=1}^{n}\beta_{ij}f_j(x_j^*) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t))) - \sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}g_j(x_j^*)g_l(x_l^*) \\
&\quad + \bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t)))) - \bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}g_j(x_j^*)g_l(x_l^*)) \\
&\quad \left. + \bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(x_j(t-\tau_{ij}(t)))g_l(x_l(t-\tau_{ij}(t)))) - \bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}g_j(x_j^*)g_l(x_l^*))\right]. \tag{21}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
D^-(V(t_2)) &\leqslant \lambda q_i|y_i(t_2)|e^{\lambda t_2} \\
&\quad + q_ie^{\lambda t_2}\left[-d_i|y_i(t_2)| + \sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)|F_j(y_j(t_2-\tau_{ij}(t_2)))|\right. \\
&\quad \left. + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}|+|\gamma_{ijl}|+|\sigma_{ijl}|)|G_j(y_j(t_2-\tau_{ij}(t_2)))G_l(y_l(t_2-\tau_{ij}(t_2)))|\right], \tag{22}
\end{aligned}
$$

where

$$F_j(y_j(t_2-\tau_{ij}(t_2))) = f_j(x_j(t_2-\tau_{ij}(t_2))) - f_j(x_j^*),$$
$$G_j(y_j(t_2-\tau_{ij}(t_2)))G_l(y_l(t_2-\tau_{ij}(t_2))) = g_j(x_j(t_2-\tau_{ij}(t_2)))g_l(x_l(t_2-\tau_{ij}(t_2))) - g_j(x_j^*)g_l(x_l^*).$$

From the assumption (H1), we have

$$|F_j(y_j(t_2-\tau_{ij}(t_2)))| \leqslant L_j \, | \, y_j(t_2-\tau_{ij}(t_2)) \, | \tag{23}$$

and

$$
\begin{aligned}
&|G_j(y_j(t_2 - \tau_{ij}(t_2)))G_l(y_l(t_2 - \tau_{ij}(t_2)))| \\
=\ &|g_j(x_j(t_2 - \tau_{ij}(t_2)))g_l(x_l(t_2 - \tau_{ij}(t_2))) - g_j(x_j(t_2 - \tau_{ij}(t_2)))g_l(x_l^*) \\
&+ g_j(x_j(t_2 - \tau_{ij}(t_2)))g_l(x_l^*) - g_j(x_j^*)g_l(x_l^*)| \\
\leqslant\ &K_j|x_j(t_2 - \tau_{ij}(t_2))|K_l|x_l(t_2 - \tau_{ij}(t_2)) - x_l^*| + K_l|x_l^*|K_j|x_j(t_2 - \tau_{ij}(t_2)) - x_j^*| \\
=\ &K_jK_l|y_j(t_2 - \tau_{ij}(t_2)) + x_l^*||y_l(t_2 - \tau_{ij}(t_2))| + K_jK_l|x_l^*||y_j(t_2 - \tau_{ij}(t_2))|.
\end{aligned}
\tag{24}
$$

Thus,

$$
\begin{aligned}
D^-(V(t_2)) \leqslant\ &\lambda q_i|y_i(t_2)|e^{\lambda t_2} + q_i e^{\lambda t_2}\Big[-d_i|y_i(t_2)| + \sum_{j=1}^n (|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|)L_j|y_j(t_2 - \tau_{ij}(t_2))| \\
&+ \sum_{j=1}^n \sum_{l=1}^n (|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|)K_jK_l \\
&\times (|y_j(t_2 - \tau_{ij}(t_2)) + x_l^*||y_l(t_2 - \tau_{ij}(t_2))| + |x_l^*||y_j(t_2 - \tau_{ij}(t_2))|)\Big] \\
=\ &\lambda q_i|y_i(t_2)|e^{\lambda t_2} - d_i q_i|y_i(t_2)|e^{\lambda t_2} + \sum_{j=1}^n (|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|)L_j q_i q_j^{-1} q_j |y_j(t_2 - \tau_{ij}(t_2))|e^{\lambda t_2} \\
&+ \sum_{j=1}^n \sum_{l=1}^n (|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|)K_jK_l q_i q_j^{-1} q_l^{-1} q_j q_l \\
&\times \big[|y_j(t_2 - \tau_{ij}(t_2)) + x_l^*||y_l(t_2 - \tau_{ij}(t_2))| + |x_l^*||y_j(t_2 - \tau_{ij}(t_2))|\big]e^{\lambda t_2} \\
\leqslant\ &\lambda \frac{\xi}{1-\eta}\|\Phi\|_h - d_i \frac{\xi}{1-\eta}\|\Phi\|_h + \sum_{j=1}^n (|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|)L_j q_i q_j^{-1} \frac{\xi}{1-\eta}\|\Phi\|_h e^{\lambda h} \\
&+ \sum_{j=1}^n \sum_{l=1}^n 2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|)K_jK_l q_i q_j^{-1} q_l^{-1} \frac{\xi}{1-\eta}d\|\Phi\|_h e^{\lambda h} \\
\leqslant\ &\xi\|\Phi\|_h \Bigg[\frac{\lambda}{1-\eta} + \frac{d_i}{1-\eta}\Bigg(-1 + \max_{1\leqslant i\leqslant n}\sum_{j=1}^n \frac{(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|)q_i q_j^{-1}L_j e^{\lambda h}}{d_i} \\
&+ \max_{1\leqslant i\leqslant n}\sum_{j=1}^n\sum_{l=1}^n \frac{2(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|)q_i q_j^{-1}q_l^{-1}K_jK_l d e^{\lambda h}}{d_i}\Bigg)\Bigg] \\
=\ &\xi\|\Phi\|_h \times \Bigg[\frac{\lambda}{1-\eta} + \frac{d_i}{1-\eta}(\eta - 1)\Bigg] \\
=\ &\xi\|\Phi\|_h \times \Bigg(\frac{\lambda}{1-\eta} - d_i\Bigg) < 0,
\end{aligned}
\tag{25}
$$

which contradicts $D^-(V(t_2)) \geqslant 0$. So $V(t) = \|y(t)\|e^{\lambda t} < \frac{\xi}{1-\eta}\|\Phi\|_h$ holds. Let $\xi \to 1$, then (16) holds . That is

$$
\|x(t) - x^*\| = \|y(t)\| \leqslant \frac{1}{1-\eta}\|\Phi\|_h e^{-\lambda t} = \frac{1}{1-\eta}\|\varphi - x^*\|e^{-\lambda t}.
\tag{26}
$$

The proof is complete.

Remark 1. The HFCNNs in [23] is a special case of system (1) with $f_i = g_i$. In [23] the activation functions f_i and g_i in system (1) are assumed to be bounded, and these restrictions are not needed in this paper.

Remark 2. Let $\alpha_{ij} = 0, \beta_{ij} = 0, \gamma_{ijl} = 0, \sigma_{ijl} = 0, T_{ij} = 0, H_{ij} = 0$ in the system (1), then it degenerates into special cases of the systems which were studied in [20] [22] [24] [25] [27] [28] [31]. Even specially, if $\tau_{ij}(t) = 0$ holds, system (1) becomes the model in [18]. However, the boundedness of activation functions is necessary in these mentioned works, the derivative of the time delays is required to be less than one in [27], and the monotonicity of activation functions must be satisfied in [31], while these limitations are removed in our results.

4. Illustrative Example

Example 1. Consider the 2-dimension HFCNNs with time delays. For system (1), we take activation functions $f_i(x) = \frac{1}{2}(|1+x| + |1-x|)$, $g_i(x) = \frac{2}{5}(|x+1| - |x-1|)$ $(i = 1,2)$. Obviously, $f_i(x), g_i(x)$ satisfies Hypotheses (H1), where $L_i = 1, K_i = 0.8$ $(i = 1,2)$, we take $q_1 = q_2 = 1, \tau_{ij}(t) = 2 + |\sin(t)|(i, j = 1,2)$, $I_1 = 1, I_2 = -1, \mu_1 = \mu_2 = 1$,

$$D = diag\{d_1, d_2\} = \begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix},$$

$$A = \{a_{ij}\}_{2\times2} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{1}{10} & \frac{1}{5} \end{bmatrix},$$

$$\alpha = \{\alpha_{ij}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & -\frac{1}{10} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix},$$

$$\beta = \{\beta_{ij}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix},$$

$$B_1 = \{b_{1jl}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & -\frac{1}{10} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix},$$

$$B_2 = \{b_{2jl}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{10} \end{bmatrix},$$

$$R_1 = \{\gamma_{1jl}\}_{2\times2} = \begin{bmatrix} -\frac{1}{10} & \frac{1}{8} \\ \frac{1}{5} & -\frac{1}{4} \end{bmatrix},$$

$$R_2 = \{\gamma_{2jl}\}_{2\times2} = \begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{8} & \frac{1}{10} \end{bmatrix},$$

$$Q_1 = \{\sigma_{1jl}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix},$$

$$Q_2 = \{\sigma_{2jl}\}_{2\times2} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix},$$

$$T = \{T_{ij}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & -\frac{1}{8} \\ \frac{1}{5} & \frac{1}{8} \end{bmatrix},$$

$$H = \{H_{ij}\}_{2\times2} = \begin{bmatrix} \frac{1}{10} & -\frac{1}{5} \\ \frac{1}{8} & \frac{1}{5} \end{bmatrix}.$$

By simple calculation, we have $M = 0.1 < 1, L = 0.296, P = 0.2375$, and $(1 - M)^2 - 4LP > 0$ which implies the hypotheses (H1)-(H2) hold. Therefore, it follows from Theorems 1-3 that the system has a unique equilibrium point x^* which is exponentially stable in the region $S = \{\varphi \in C| \quad \|\varphi\|_h \leqslant d\}$, where $d \in [0.2920, 2.7486)$ is a constant.

Since the activation functions are unbounded and the time-delays are not differentiable in Example 1, the results in [23] is not applicable to this system.

Example 2. If $\alpha_{ij} = 0, \beta_{ij} = 0, \gamma_{ijl} = 0, \sigma_{ijl} = 0, T_{ij} = 0, H_{ij} = 0$ in the system (1), and the other parameters are the same as that in Example 1, then the system (1) becomes a Hopfield type network. Let $f_i(x) = g_i(x) = |x|$ and choose $L_i = K_i = 1$ and $q_i = 1$ $(i = 1,2)$. We get $M = 0.0375, L = 0.125$ and $P = 0.1625$. It is easy to know that the conditions in Theorems 1-3 are satisfied, so the system has a unique exponentially stable equilibrium point. However the results that are mentioned in Remark 2 cannot be used to ascertain the stability of this neural networks.

5. Conclusions

In this paper, the dynamic behavior of HFCNNs model with time-varying delays has been investigated. By using Brouwer's fixed point theorem and employing inequality technique, some sufficient criteria ensuring the existence, uniqueness and exponential stability of equilibrium point and uniform boundedness of HFCNNs have been derived. Two examples are used to show that our results are new and complement some previously known results, since the activation functions are not necessarily bounded or monotonic, and the time-varying delays are not necessarily differentiable.

Acknowledgments

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