Doubt Atanassov's intuitionistic fuzzy Sub-implicative ideals in *BCI*-algebras

Tripti Bej [1] **Madhumangal Pal** [2]

[1] *Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721 102, India.*

E-mail: tapubej@gmail.com

[2] *Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721 102, India.*

E-mail: mmpalvu@gmail.com

Received 12 May 2014

Accepted 16 September 2014

Abstract

The notions of doubt Atanassov's intuitionistic fuzzy sub-implicative ideals and doubt Atanassov's intuitionistic fuzzy *P*-ideals of *BCI*-algebras are introduced. We show that an Atanassov's intuitionistic fuzzy subset of *BCI*-algebras is an Atanassov's intuitionistic fuzzy sub-implicative ideal if and only if the complement of this Atanassov's intuitionistic fuzzy subset is a doubt Atanassov's intuitionistic fuzzy sub-implicative-ideal. We prove that any doubt Atanassov's intuitionistic fuzzy *P*-ideal is always a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal. Moreover, some other properties about doubt Atanassov's intuitionistic fuzzy sub-implicative ideals and doubt Atanassov's intuitionistic fuzzy *P*-ideals of *BCI*-algebras are given.

Keywords: *BCI*-algebra, implicative *BCI*-algebra, doubt Atanassov's intuitionistic fuzzy ideal, fuzzy sub-implicative ideal, fuzzy *P*-ideal.

1. Introduction

After the introduction of fuzzy sets by Zadeh [1], there has been a number of generalisation of this fundamental concept. The notion of Atanassov's intuitionistic fuzzy sets introduced by Atanassov [2,3], is one among them. In 1966, Imai and Iśeki [4,5,6] introduced the notion of *BCI*-algebras. Xi [7] applied the concept of fuzzy set to *BCI*-algebras and gave some properties of it. After that Jun and Meng investigated the properties of fuzzy *BCI*-algebras and fuzzy ideals [8]. Liu and Meng [9] introduced the notion of sub-implicative ideals in *BCI*-algebras. In 2002, Jun [10] introduced fuzzy sub-implicative ideals in *BCI*-algebras. In 2012, Palaniappan et al. [11] introduced the notions of Atanassov's intuitionistic fuzzy sub-implicative ideals and Atanassov's intuitionistic fuzzy sub-commutative ideals of *BCI*-algebras. Solairaju [12] introduced the notion of Atanassov's intuitionistic fuzzy *P*-ideal of *BCI*-algebra and investigated some related properties.

In 2011, Mostafa [13] introduced anti fuzzy sub-implicative ideals of *BCI*-algebras and Jianming and Zhisong [14] defined doubt fuzzy *P*-ideals of a *BCI*-algebra. Senapati et al. have presented several results on BCK/BCI-algebras, *BG*-algebra and *B*-algebra [15,16,17,18,19].

In [20], the authors have studied doubt Atanassov's intuitionistic fuzzy *H*-ideals in BCK/BCI-algebras.

In this paper, we define doubt Atanassov's intuitionistic fuzzy sub-implicative ideals and doubt Atanassov's intuitionistic fuzzy *P*-ideals in *BCI*-algebras. We also investigate its properties. We give conditions for a doubt Atanassov's intuitionistic fuzzy ideal to be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideals in *BCI*-algebras. Relations among doubt Atanassov's intuitionistic fuzzy *p*-ideals and doubt Atanassov's intuitionistic fuzzy sub-implicative ideals are also investigated.

2. Preliminaries

Now here we give some already known results which will be used in this paper.

An algebra $(X;*,0)$ of type (2,0) is called a *BCI*-algebra if it satisfies the following axioms for all $x,y,z \in X$:
(A1) $((x*y)*(x*z))*(z*y)=0$
(A2) $(x*(x*y))*y=0$
(A3) $x*x=0$
(A4) $x*y=0$ and $y*x=0$ imply $x=y$
A partial ordering "$\leqslant$" on a *BCI*-algebra X can be defined by $x \leqslant y$ if and only if $x*y=0$.
Any *BCI*-algebra X satisfies the following properties for all $x,y,z \in X$:
(P1) $(x*y)*z=(x*z)*y$
(P2) $x*y \leqslant x$
(P3) $(x*z)*(y*z) \leqslant (x*y)$
(P4) $x \leqslant y \Rightarrow x*z \leqslant y*z, z*y \leqslant z*x$.
(P5) $x*0=x$
(P6) $0*(x*y)=(0*x)*(0*y)$
(P7) $x*(x*(x*y))=x*y$
Throughout this paper, X always means a *BCI*-algebra without any specification.

A *BCI*-algebra is said to be implicative if it satisfies: $(x*(x*y))*(y*x)=y*(y*x)$.

A non-empty subset I of a *BCI*-algebra X is called an ideal of X if
(i) $0 \in I$
(ii) $x*y \in I$ and $y \in I$ then $x \in I$, for all $x,y \in X$.

A non-empty subset I of a *BCI*-algebra X is said to be a sub-implicative ideal [9] of X if
(i) $0 \in I$
(ii) $((x*(x*y))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$, for all $x,y,z \in X$.

A fuzzy set $A=\{\langle x,\mu_A(x)\rangle : x \in X\}$ in X is called a doubt fuzzy sub-implicative ideal [13] of X if
(i) $\mu_A(0) \leqslant \mu_A(x)$
(ii) $\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$, for all $x,y,z \in X$.

The present paper is done on Atanassov's intuitionistic fuzzy set.

An Atanassov's intuitionistic fuzzy set A in a non-empty set X is an object having the form $A=\{x,\mu_A(x),\lambda_A(x)/x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively and $0 \leqslant \mu_A(x)+\lambda_A(x) \leqslant 1$, for all $x \in X$.

For the sake of simplicity, we represent the Atanassov's intuitionistic fuzzy set $A=\{\langle x,\mu_A(x),\lambda_A(x)\rangle : x \in X\}$ by $A=(X,\mu_A,\lambda_A)$ or (μ_A,λ_A).
The two operators used in this paper are defined as:

If $A=(\mu_A,\lambda_A)$ is an Atanassov's intuitionistic fuzzy set then,

$$\Pi A=\{(x,\mu_A(x),\overline{\mu}_A(x))/x \in X\}$$

$$\Diamond A=\{(x,\overline{\lambda}_A(x),\lambda_A(x))/x \in X\}.$$

For the sake of simplicity, we also use $x \bigvee y$ for $max(x,y)$, and $x \bigwedge y$ for $min(x,y)$.
An Atanassov's intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in X is called an Atanassov's intuitionistic fuzzy ideal [21] of X, if it satisfies the following axioms:
(i) $\mu_A(0) \geqslant \mu_A(x), \lambda_A(0) \leqslant \lambda_A(x)$,
(ii) $\mu_A(x) \geqslant \mu_A(x*y) \bigwedge \mu_A(y)$,
(iii) $\lambda_A(x) \leqslant \lambda_A(x*y) \bigvee \lambda_A(y)$, for all $x,y \in X$.

Let $A=(\mu_A,\lambda_A)$ be an Atanassov's intuitionistic fuzzy subset of a *BCI*-algebra X, then A is called a doubt Atanassov's intuitionistic fuzzy subalgebra [20] of X if
(i) $\mu_A(x*y) \leqslant \mu_A(x) \bigvee \mu_A(y)$,
(ii) $\lambda_A(x*y) \geqslant \lambda_A(x) \bigwedge \lambda_A(y)$, for all $x,y \in X$.

An Atanassov's intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in a *BCI*-algebra X is called a doubt Atanassov's intuitionistic fuzzy ideal [20] if
(i) $\mu_A(0) \leqslant \mu_A(x); \lambda_A(0) \geqslant \lambda_A(x)$
(ii) $\mu_A(x) \leqslant \mu_A(x*y) \bigvee \mu_A(y)$
(iii) $\lambda_A(x) \geqslant \lambda_A(x*y) \bigwedge \lambda_A(y)$, for all $x,y \in X$.

A non-empty subset I of a *BCI*-algebra X is said to be a *P*-ideal [22] of X if

(i) $0 \in I$
(ii) $(x*z)*(y*z) \in I$ and $y \in I$ then $x \in I$, for all $x,y,z \in X$.

A fuzzy set $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$ in X is called a doubt fuzzy P-ideal [14] of X if
(i) $\mu_A(0) \leqslant \mu_A(x)$
(ii) $\mu_A(x) \leqslant \mu_A((x*z)*(y*z)) \bigvee \mu(y)$, for all $x,y,z \in X$.

3. Doubt Atanassov's intuitionistic fuzzy sub-implicative ideals in *BCI*-algebras

In this section, we define doubt Atanassov's intuitionistic fuzzy sub-implicative ideals in *BCI*-algebras and investigate its properties.

Definition 1. Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy subset of a *BCI*-algebra X, then A is called a **doubt Atanassov's intuitionistic fuzzy sub-implicative ideal** of X (briefly, *DAIFSI*-ideal) if
(i) $\mu_A(0) \leqslant \mu_A(x), \lambda_A(0) \geqslant \lambda_A(x)$
(ii) $\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$
(iii) $\lambda_A(y*(y*x)) \geqslant \lambda_A(((x*(x*y))*(y*x))*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$.

Theorem 1. *Let $A = (\mu_A, \lambda_A)$ in X be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of a BCI-algebra X. Then, if the inequility $x \leqslant z$ holds in X, then*
(i) $\mu_A(x) \leqslant \mu_A(z)$ and (ii) $\lambda_A(x) \geqslant \lambda_A(z)$.

Proof: Let $x,z \in X$ be such that $x \leqslant z$ then $x*z = 0$ and since A is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, so

$\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$. when $y = x$, then using (A3) and (P5), we get $\mu_A(x) \leqslant \mu_A(x*z) \bigvee \mu_A(z) = \mu_A(0) \bigvee \mu_A(z) = \mu_A(z)$. Therefore, $\mu_A(x) \leqslant \mu_A(z)$.

Again, $\lambda_A(y*(y*x)) \geqslant \lambda_A(((x*(x*y))*(y*x))*z) \bigwedge \lambda_A(z)$. when $y = x$, then using (A3) and (P5), we get $\lambda_A(x) \geqslant \lambda_A(x*z) \bigwedge \lambda_A(z) = \lambda_A(0) \bigwedge \lambda_A(z) = \lambda_A(z)$. Therefore, $\lambda_A(x) \geqslant \lambda_A(z)$. This completes the proof. ■

Proposition 2. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of a BCI-algebra X. Then $\mu_A(0*(0*x)) \leqslant \mu_A(x)$ and $\lambda_A(0*(0*x)) \geqslant \lambda_A(x)$, for all $x \in X$.*

Proof: $\mu_A(0*(0*x)) \leqslant \mu_A(((x*(x*0))*(0*x))*z) \bigvee \mu_A(z) = \mu_A(((x*x)*(0*x))*z) \bigvee \mu_A(z) = \mu_A((0*(0*x))*z) \bigvee \mu_A(z)$. When $z = x$ we get, $\mu_A(0*(0*x)) \leqslant \mu_A((0*(0*x))*x) \bigvee \mu_A(x)$ or, $\mu_A(0*(0*x)) \leqslant \mu_A(0) \bigvee \mu_A(x)$ [by using (A2)].

Therefore, $\mu_A(0*(0*x)) \leqslant \mu_A(x)$, for all $x \in X$.

Again, $\lambda_A(0*(0*x)) \geqslant \lambda_A(((x*(x*0))*(0*x))*z) \bigwedge \lambda_A(z) = \lambda_A(((x*x)*(0*x))*z) \bigwedge \lambda_A(z) = \lambda_A((0*(0*x))*z) \bigwedge \lambda_A(z)$. When $z = x$ we get, $\lambda_A(0*(0*x)) \leqslant \lambda_A((0*(0*x))*x) \bigwedge \lambda_A(x)$ or, $\lambda_A(0*(0*x)) \leqslant \lambda_A(0) \bigwedge \lambda_A(x)$ [by using A2]. Therefore, $\lambda_A(0*(0*x)) \geqslant \lambda_A(x)$, for all $x \in X$. ■

Example 1. Let $X = \{0,1,2,3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2	3
μ_A	0.1	0.4	0.5	0.6
λ_A	0.8	0.6	0.5	0.4

Then A is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X .

Theorem 3. *Every doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X is a doubt Atanassov's intuitionistic fuzzy subalgebra of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, then (i) $\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$, and (ii) $\lambda_A(y*(y*x)) \geqslant \lambda_A(((x*(x*y))*(y*x))*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$.
If $y = x$, then from (i) and (ii), $\mu_A(x) \leqslant \mu_A(x*z) \bigvee \mu_A(z)$ and $\lambda_A(x) \geqslant \lambda_A(x*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$. This also implies that, $\mu_A(x*z) \leqslant \mu_A((x*z)*z) \bigvee \mu_A(z)$ and $\lambda_A(x*z) \geqslant \lambda_A((x*z)*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$. Again, $((x*z)*z) \leqslant$

$x*z \leqslant x$, [by using (P2) and (P4)]. Hence by Theorem 3.2, $\mu_A((x*z)*z) \leqslant \mu_A(x)$. Thus, $\mu_A(x*z) \leqslant \mu_A(x) \bigvee \mu_A(z)$ and also, $\lambda_A((x*z)*z) \geqslant \lambda_A(x)$. So, $\lambda_A(x*z) \geqslant \lambda_A(x) \bigwedge \lambda_A(z)$. Hence, A is a doubt Atanassov's intuitionistic fuzzy subalgebra of X. ■

Theorem 4. *Every doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X is a doubt Atanassov's intuitionistic fuzzy ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, then (i) $\mu_A(0) \leqslant \mu_A(x); \lambda_A(0) \geqslant \lambda_A(x)$, (ii) $\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$, and (iii) $\lambda_A(y*(y*x)) \geqslant \lambda_A(((x*(x*y))*(y*x))*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$.
If $y = x$, then from (ii) and (iii), $\mu_A(x) \leqslant \mu_A(x*z) \bigvee \mu_A(z)$ and $\lambda_A(x) \geqslant \lambda_A(x*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$.

Hence, A is a doubt Atanassov's intuitionistic fuzzy ideal of X. ■

But the converse does not hold in general. There is some doubt Atanassov's intuitionistic fuzzy ideals of X, which are not a doubt Atanassov's intuitionistic fuzzy sub-implicative ideals of X. It can be verified by the following example:

Example 2. Let $X = \{0,1,2,3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2	3
μ_A	0	0.5	0.5	0.6
λ_A	1	0.5	0.5	0.4

which is a doubt Atanassov's intuitionistic fuzzy ideal of X. But, A is not a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, as $\mu_A(2*(2*1)) \nleq max\{\mu_A(((1*(1*2))*(2*1))*0), \mu_A(0)\}$. Because it implies that, $\mu_A(1) \leqslant \mu_A(0)$, which is a contradiction.

We now give a condition for a doubt Atanassov's intuitionistic fuzzy ideal of X to be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X.

Theorem 5. *If a doubt Atanassov's intuitionistic fuzzy ideal of X satisfies the inequalities,* $\mu_A(y*(y*x)) \leqslant \mu_A((x*(x*y))*(y*x))$, *and* $\lambda_A(y*(y*x)) \geqslant \lambda_A((x*(x*y))*(y*x))$, *for all* $x,y \in X$, *then it becomes a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy ideal of X satisfying the inequalities, $\mu_A(y*(y*x)) \leqslant \mu_A((x*(x*y))*(y*x))$, and $\lambda_A(y*(y*x)) \geqslant \lambda_A((x*(x*y))*(y*x))$, for all $x,y \in X$. Now, $\mu_A(y*(y*x)) \leqslant \mu_A((x*(x*y))*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z)$, and $\lambda_A(y*(y*x)) \geqslant \lambda_A((x*(x*y))*(y*x)) \geqslant \lambda_A(((x*(x*y))*(y*x))*z) \bigwedge \lambda_A(z)$, for all $x,y,z \in X$, [because A is a doubt Atanassov's intuitionistic fuzzy ideal]. Hence, A is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. This completes the proof. ■

Lemma 6. *If X is an implicative BCI-algebra, then every doubt Atanassov's intuitionistic fuzzy ideal of X is an DAIFSI-ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy ideal of an implicative *BCI*-algebra X, then $\mu_A(x) \leqslant max\{\mu_A(x*z), \mu_A(z)\}$, and $\lambda_A(x) \geqslant min\{\lambda_A(x*z), \lambda_A(z)\}$, for all $x,z \in X$. So, $\mu_A(y*(y*x)) \leqslant max\{\mu_A(y*((y*x))*z), \mu_A(z)\}$, and $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(y*((y*x))*z), \lambda_A(z)\}$. But X is implicative *BCI*-algebra, then $((x*(x*y))*(y*x)) = (y*(y*x))$. Hence $\mu_A(y*(y*x)) \leqslant max\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$ and $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(((x*(x*y))*(y*x))*z), \lambda_A(z)\}$, for all $x,y,z \in X$. This completes the proof. ■

Theorem 7. *If X is an implicative BCI-algebra, then an Atanassov's intuitionistic fuzzy set A of X is a doubt Atanassov's intuitionistic fuzzy ideal of X if and only if it is an DAIFSI-ideal of X.*

Proof: It can be proved by applying Lemma 6 and Theorem 4 . ■

Let us illustrate the Theorem 5, Lemma 6 and

Theorem 7 using following example.

Example 3. Let $X = \{0, 1, 2\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Here X is an implicative *BCI*-algebra. Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2
μ_A	0	0.8	0.8
λ_A	1	0.2	0.2

Hence, A is a doubt Atanassov's intuitionistic fuzzy ideal as well as doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X.

Theorem 8. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. Then, so is $\Pi A = \{\langle x, \mu_A(x), \overline{\mu}_A(x)\rangle / x \in X\}$.*

Proof: Since $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, then $\mu_A(0) \leqslant \mu_A(x)$ and $\mu_A(y*(y*x)) \leqslant max\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$. Now, $\mu_A(0) \leqslant \mu_A(x)$, or $1 - \overline{\mu}_A(0) \leqslant 1 - \overline{\mu}_A(x)$, or $\overline{\mu}_A(0) \geqslant \overline{\mu}_A(x)$, for any $x \in X$. Now for any $x, y, z \in X$, $\mu_A(y*(y*x)) \leqslant max\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$. This gives, $1 - \overline{\mu}_A(y*(y*x)) \leqslant max\{1 - \overline{\mu}_A(((x*(x*y))*(y*x))*z), 1 - \overline{\mu}_A(z)\}$ or, $\overline{\mu}_A(y*(y*x)) \geqslant 1 - max\{1 - \overline{\mu}_A(((x*(x*y))*(y*x))*z), 1 - \overline{\mu}_A(z)\}$. Finally, $\overline{\mu}_A(y*(y*x)) \geqslant min\{\overline{\mu}_A(((x*(x*y))*(y*x))*z), \overline{\mu}_A(z)\}$. Hence, $\Pi A = \{(x, \mu_A(x), \overline{\mu}_A(x))/x \in X\}$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. ■

Theorem 9. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. Then so is $\Diamond A = \{\langle x, \overline{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$.*

Proof: Since $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, then $\lambda_A(0) \geqslant \lambda_A(x)$. Also, $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(((x*(x*y))*(y*x))*z), \lambda_A(z)\}$.

Again, we have, $\lambda_A(0) \geqslant \lambda_A(x)$, or $1 - \overline{\lambda}_A(0) \geqslant 1 - \overline{\lambda}_A(x)$, or $\overline{\lambda}_A(0) \leqslant \overline{\lambda}_A(x)$, for any $x \in X$. Also for any $x, y, z \in X$, $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(((x*(x*y))*(y*x))*z), \lambda_A(z)\}$.

This implies, $1 - \overline{\lambda}_A(y*(y*x) \geqslant min\{1 - \overline{\lambda}_A(((x*(x*y))*(y*x))*z), 1 - \overline{\lambda}_A(z)\}$. That is, $\overline{\lambda}_A(y*(y*x) \leqslant 1 - min\{1 - \overline{\lambda}_A(((x*(x*y))*(y*x))*z), 1 - \overline{\lambda}_A(z)\}$ or, $\overline{\lambda}_A(y*(y*x) \leqslant max\{\overline{\lambda}_A(((x*(x*y))*(y*x))*z), \overline{\lambda}_A(z)\}$. Hence, $\Diamond A = \{\langle x, \overline{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. ■

Theorem 10. *Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set in X. Then $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X if and only if $\Pi A = \{\langle x, \mu_A(x), \overline{\mu}_A(x)\rangle / x \in X\}$ and $\Diamond A = \{\langle x, \overline{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$ are doubt Atanassov's intuitionistic fuzzy sub-implicative ideals of X.*

Proof: The proof is similar to those of Theorem 8 and Theorem 9. ■

Let us illustrate the Theorem 8, Theorem 9 and Theorem 10 using the following example.

Example 4. Let $X = \{0, 1, 2, 3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X defined by

X	0	1	2	3
μ_A	0	0.3	0.5	0.6
λ_A	0.8	0.6	0.5	0.4

Then $\Pi A = \{\langle x, \mu_A(x), \overline{\mu}_A(x)\rangle / x \in X\}$, where $\mu_A(x)$ and $\overline{\mu}_A(x)$ are defined as follows:

X	0	1	2	3
μ_A	0	0.3	0.5	0.6
$\overline{\mu}_A$	1	0.7	0.5	0.4

Also $\Diamond A = \{\langle x, \overline{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$,whose $\lambda_A(x)$ and $\overline{\lambda}_A(x)$ are defined by

X	0	1	2	3
$\overline{\lambda}_A$	0.2	0.4	0.5	0.6
λ_A	0.8	0.6	0.5	0.4

So, it can be verified that ΠA and $\Diamond A$ are doubt Atanassov's intuitionistic fuzzy sub-implicative ideals of X.

Theorem 11. *An Atanassov's intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of a BCI-algebra X if and only if the fuzzy sets $\mu_A$and $\overline{\lambda}_A$ are doubt fuzzy sub-implicative ideals of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. Then it is obvious that μ_A is a doubt fuzzy sub-implicative ideal of X, and from Theorem 9, we can prove that $\overline{\lambda}_A$ is a doubt fuzzy sub-implicative ideal of X.

Conversely, let μ_A be a doubt fuzzy sub-implicative ideal of X. Therefore $\mu_A(0) \leqslant \mu_A(x)$ and $\mu_A(y*(y*x)) \leqslant max\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$, for all $x,y,z \in X$. Again, let $\overline{\lambda}_A$ is a doubt fuzzy sub-implicative ideal of X, so, $\overline{\lambda}_A(0) \leqslant \overline{\lambda}_A(x)$, gives $1-\lambda_A(0) \leqslant 1-\lambda_A(x)$, implies $\lambda_A(0) \geqslant \lambda_A(x)$.

Also, $\overline{\lambda}_A(y*(y*x)) \leqslant max\{\overline{\lambda}_A(((x*(x*y))*(y*x))*z), \overline{\lambda}_A(z)\}$ or, $1-\lambda_A(y*(y*x)) \leqslant max\{1-\lambda_A(((x*(x*y))*(y*x))*z), 1-\lambda_A(z)\}$ or, $\lambda_A(y*(y*x)) \geqslant 1-max\{1-\lambda_A(((x*(x*y))*(y*x))*z), 1-\lambda_A(z)\}$. Finally, $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(((x*(x*y))*(y*x))*z), \lambda_A(z)\}$, for all $x,y,z \in X$. Hence, $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. ■

Corollary 12. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of a BCI-algebra X. Then the sets, $D_{\mu_A} = \{x \in X / \mu_A(x) = \mu_A(0)\}$ and $D_{\lambda_A} = \{x \in X / \lambda_A(x) = \lambda_A(0)\}$ are sub-implicative ideals of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. Obviously, $0 \in D_{\mu_A}$ and $0 \in D_{\lambda_A}$. Now, let $x,y,z \in X$, such that $(((x*(x*y))*(y*x))*z) \in D_{\mu_A}$, $z \in D_{\mu_A}$. Then $\mu_A(((x*(x*y))*(y*x))*z) = \mu_A(0) = \mu_A(z)$. Now, $\mu_A(y*(y*x)) \leqslant max\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\} = \mu_A(0)$.

Again, since μ_A is a doubt fuzzy sub-implicative ideal of X, $\mu_A(0) \leqslant \mu_A(y*(y*x))$. Therefore, $\mu_A(0) = \mu_A(y*(y*x))$. It follows that, $(y*(y*x)) \in D_{\mu_A}$, for all $x,y,z \in X$. Therefore, D_{μ_A} is an sub-implicative ideal of X.

Also, let $x,y,z \in X$, such that $(((x*(x*y))*(y*x))*z) \in D_{\lambda_A}$, $z \in D_{\lambda_A}$. Then $\lambda_A(((x*(x*y))*(y*x))*z) = \lambda_A(0) = \lambda_A(z)$. Now, $\lambda_A(y*(y*x)) \geqslant min\{\lambda_A(((x*(x*y))*(y*x))*z), \lambda_A(z)\} = \lambda_A(0)$.

Again, since $\overline{\lambda}_A$ is a doubt fuzzy sub-implicative ideal of X, $\lambda_A(0) \geqslant \lambda_A(y*(y*x))$. Therefore, $\lambda_A(0) = \lambda_A(y*(y*x))$. It follows that, $(y*(y*x)) \in D_{\lambda_A}$, for all $x,y,z \in X$. Therefore, D_{λ_A} is an sub-implicative ideal of X. ■

Definition 2. Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set of X, and $t,s \in [0,1]$, then μ level t-cut and λ level s-cut of A, is as followes:

$$\mu_{A,t}^{\leqslant} = \{x \in X / \mu_A(x) \leqslant t\}$$

and $\lambda_{A,s}^{\geqslant} = \{x \in X / \lambda_A(x) \geqslant s\}$.

Theorem 13. *If $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, then $\mu_{A,t}^{\leqslant}$ and $\lambda_{A,s}^{\geqslant}$ are sub-implicative ideals of X for any $t,s \in [0,1]$.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X, and let $t \in [0,1]$ with $\mu_A(0) \leqslant t$. So, $0 \in \mu_{A,t}^{\leqslant}$. Let $x,y,z \in X$ be such that $(((x*(x*y))*(y*x))*z) \in \mu_{A,t}^{\leqslant}$ and $z \in \mu_{A,t}^{\leqslant}$. Therefore, $\mu_A(((x*(x*y))*(y*x))*z) \leqslant t$ and $\mu_A(z) \leqslant t$. Since μ_A is a doubt fuzzy sub-implicative ideal of X, it follows that, $\mu_A(y*(y*x)) \leqslant \mu_A(((x*(x*y))*(y*x))*z) \bigvee \mu_A(z) \leqslant t$ and hence $(y*(y*x)) \in \mu_{A,t}^{\leqslant}$, for all $x,y,z \in X$. Therefore, $\mu_{A,t}^{\leqslant}$ is an sub-implicative ideal of X for $t \in [0,1]$. Similarly, we can prove that $\lambda_{A,s}^{\geqslant}$ is an sub-implicative ideal of X for $s \in [0,1]$. ■

Theorem 14. *If $\mu_{A,t}^{\leqslant}$ and $\lambda_{A,s}^{\geqslant}$ are either empty or sub-implicative ideals of X for all $t,s \in [0,1]$, then $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X.*

Proof: Let $\mu_{A,t}^{\leqslant}$ and $\lambda_{A,s}^{\geqslant}$ be either empty or sub-implicative ideals of X for all $t,s \in [0,1]$. For any $x \in X$, let $\mu_A(x) = t$ and $\lambda_A(x) = s$. Then $x \in \mu^{\leqslant}_{A,t} \cap \lambda^{\geqslant}_{A,s}$, so $\mu^{\leqslant}_{A,t} \neq \phi \neq \lambda^{\geqslant}_{A,s}$. Since $\mu^{\leqslant}_{A,t}$ and $\lambda^{\geqslant}_{A,s}$ are sub-implicative ideals of X, therefore $0 \in \mu^{\leqslant}_{A,t} \cap \lambda^{\geqslant}_{A,s}$. Hence, $\mu_A(0) \leqslant t = \mu_A(x)$ and $\lambda_A(0) \geqslant s = \lambda_A(x)$, where $x \in X$. If there exist $x', y', z' \in X$ such that $\mu_A(y' * (y' * x')) > max\{\mu_A(((x' * (x' * y')) * (y' * x')) * z'), \mu_A(z')\}$, then by taking, $t_0 = \frac{1}{2}(\mu_A(y' * (y' * x')) + max\{\mu_A(((x' * (x' * y')) * (y' * x')) * z'), \mu_A(z')\})$. We have, $\mu_A(y' * (y' * x') > t_0 > max\{\mu_A(((x' * (x' * y')) * (y' * x')) * z'), \mu_A(z')\}$. Hence, $y' * (y' * x') \notin \mu^{\leqslant}_{A,t_0}$, $(((x' * (x' * y')) * (y' * x')) * z') \in \mu^{\leqslant}_{A,t_0}$ and $z' \in \mu^{\leqslant}_{A,t_0}$, that is $\mu^{\leqslant}_{A,t_0}$ is not a sub-implicative ideal of X, which is a contradiction. Therefore, $\mu_A(y * (y * x)) \leqslant \mu_A(((x * (x * y)) * (y * x)) * z) \bigvee \mu_A(z)$, for any $x,y,z \in X$.

Finally, assume that there exist $p,q,r \in X$ such that $\lambda_A(q * (q * p) < min\{\lambda_A(((p * (p * q)) * (q * p)) * r), \lambda_A(r)\}$. Taking $s_0 = \frac{1}{2}(\lambda_A(q * (q * p) + min\{\lambda_A(((p * (p * q)) * (q * p)) * r), \lambda_A(r)\})$, then $min\{\lambda_A(((p * (p * q)) * (q * p)) * r), \lambda_A(r)\} > s_0 > \lambda_A(q * (q * p)$. Therefore, $((p * (p * q)) * (q * p)) * r) \in \lambda^{\geqslant}_{A,s}$ and $r \in \lambda^{\geqslant}_{A,s}$ but $(q * (q * p) \notin \lambda^{\geqslant}_{A,s}$. Again a contradiction. This completes the proof. ■

4. Doubt Atanassov's intuitionistic fuzzy *P*-ideals in *BCI*-algebras

In this section, we define doubt Atanassov's intuitionistic fuzzy *P*-ideals in *BCI*-algebras and investigate its properties.

Definition 3. Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy subset of a *BCI*-algebra X, then A is called a **doubt Atanassov's intuitionistic fuzzy *P*-ideal** of X (briefly, *DAIFP*-ideal) if

(i) $\mu_A(0) \leqslant \mu_A(x), \lambda_A(0) \geqslant \lambda_A(x)$
(ii) $\mu_A(x) \leqslant max\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$
(iii) $\lambda_A(x) \geqslant min\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$, for all $x,y,z \in X$

Example 5. Let $X = \{0,1,2,3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2	3
μ_A	0	0.5	0.6	0.6
λ_A	1	0.5	0.4	0.4

Then $A = (\mu_A, \lambda_A)$ be a *DAIFP*-ideal of X.

Theorem 15. *Every doubt Atanassov's intuitionistic fuzzy P-ideal of X is a doubt Atanassov's intuitionistic fuzzy ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy P-ideal of X, then (i) $\mu_A(0) \leqslant \mu_A(x)$; $\lambda_A(0) \geqslant \lambda_A(x)$, (ii) $\mu_A(x) \leqslant max\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$
and (iii) $\lambda_A(x) \geqslant min\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\}$, for all $x,y,z \in X$. If we put $z = 0$, then from (ii) and (iii), we get, $\mu_A(x) \leqslant \mu_A((x*0)*(y*0)) \bigvee \mu_A(y)$ and $\lambda_A(x) \geqslant \lambda_A((x*0)*(y*0)) \bigwedge \lambda_A(y)$, for all $x,y \in X$. Hence, every *DAIFP*-ideal in X satisfies the inequalities: $\mu_A(x) \leqslant \mu_A(x*y) \bigvee \mu_A(y)$ and $\lambda_A(x) \geqslant \lambda_A(x*y) \bigwedge \lambda_A(y)$, for all $x,y \in X$. Hence, A is a doubt Atanassov's intuitionistic fuzzy ideal of X. ■

But the converse does not hold in general. It can be verified by the following example:

Example 6. Let $X = \{0,1,2,3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy ideal of X defined by

X	0	1	2	3
μ_A	0	0.3	0.4	0.5
λ_A	1	0.7	0.6	0.5

But $A = (\mu_A, \lambda_A)$ is not a doubt Atanassov's intuitionistic fuzzy P-ideal of X, since $\mu_A(2) = 0.4$ and $max(\mu_A((2*3)*(1*3)), \mu_A(1)) = \mu_A(1) = 0.3$, which implies $\mu_A(2) \nleqslant max(\mu_A((2*3)*(1*3)), \mu_A(1))$

Now we give a condition for the Atanassov's intuitionistic fuzzy ideal $A = (\mu_A, \lambda_A)$ of X to be a doubt Atanassov's intuitionistic fuzzy P-ideal of X.

Proposition 16. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy ideal of a BCI-algebra X. If (i) $\mu_A(x*y) \leqslant \mu_A((x*z)*(y*z))$ and (ii) $\lambda_A(x*y) \geqslant \lambda_A((x*z)*(y*z))$, for all $x,y,z \in X$, then A is a DAIFP-ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy ideal of X satisfying (i) $\mu_A(x*y) \leqslant \mu_A((x*z)*(y*z))$ and (ii) $\lambda_A(x*y) \geqslant \lambda_A((x*z)*(y*z))$, for all $x,y,z \in X$. Then $\mu_A((x*z)*(y*z)) \bigvee \mu_A(y) \geqslant \mu_A(x*y) \bigvee \mu_A(y) \geqslant \mu_A(x)$. Again, $\lambda_A((x*z)*(y*z)) \bigwedge \lambda_A(y) \leqslant \lambda_A(x*y) \bigwedge \lambda_A(y) \leqslant \lambda_A(x)$. This completes the proof. ■

Proposition 17. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy P-ideal of a BCI-algebra X. Then $\mu_A(x) \leqslant \mu_A(0*(0*x))$ and $\lambda_A(x) \geqslant \lambda_A(0*(0*x))$, for all $x \in X$.*

Proof: Since A is a $DAIFP$-ideal of X, then $\mu_A(x) \leqslant \mu_A((x*z)*(y*z)) \bigvee \mu_A(y)$ and $\lambda_A(x) \geqslant \lambda_A((x*z)*(y*z)) \bigwedge \lambda_A(y)$, for all $x,y,z \in X$. Now putting $z = x$ and $y = 0$ we get, $\mu_A(x) \leqslant \mu_A((x*x)*(0*x)) \bigvee \mu_A(0) = \mu_A(0*(0*x)) \bigvee \mu_A(0) = \mu_A(0*(0*x))$. And $\lambda_A(x) \geqslant \lambda_A((x*x)*(0*x)) \bigwedge \lambda_A(0) = \lambda_A(0*(0*x)) \bigwedge \lambda_A(0) = \lambda_A(0*(0*x))$.

Therefore, $\mu_A(x) \leqslant \mu_A(0*(0*x))$, and $\lambda_A(x) \geqslant \lambda_A(0*(0*x))$, for all $x \in X$. ■

Corollary 18. *Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy P-ideal of a BCI-algebra X. Then the sets, $D_{\mu_A} = \{x \in X/\mu_A(x) = \mu_A(0)\}$ and $D_{\lambda_A} = \{x \in X/\lambda_A(x) = \lambda_A(0)\}$ are P-ideals of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy P-ideal of X. Obviously, $0 \in D_{\mu_A}$ and D_{λ_A}. Now, let $x,y,z \in X$, such that $(x*z)*(y*z) \in D_{\mu_A}$, and $y \in D_{\mu_A}$. Then $\mu_A(x) \leqslant max\{\mu_A((x*z)*(y*z)), \mu_A(y)\} = \mu_A(0)$. But $\mu_A(0) \leqslant \mu_A(x)$, for all $x \in X$. Therefore, $\mu_A(0) = \mu_A(x)$. It follows that, $x \in D_{\mu_A}$, for all $x,y,z \in X$. Therefore, D_{μ_A} is a P-ideal of X.

Also, let $x,y,z \in X$, such that $(x*z)*(y*z) \in D_{\lambda_A}$, and $y \in D_{\lambda_A}$. Then $\lambda_A(x) \geqslant max\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\} = \lambda_A(0)$. But, $\lambda_A(0) \geqslant \lambda_A(x)$, for all $x \in X$. Therefore, $\lambda_A(0) = \lambda_A(x)$. It follows that, $x \in D_{\lambda_A}$, for all $x,y,z \in X$. Therefore, D_{λ_A} is a P-ideal of X. ■

Theorem 19. *Every doubt Atanassov's intuitionistic fuzzy P-ideal of X is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ be a doubt Atanassov's intuitionistic fuzzy P-ideal of X, then A is a doubt Atanassov's intuitionistic fuzzy ideal of X by Theorem 15 .

Now, $(0*(0*(y*(y*x))))*((x*(x*y))*(y*x))$

$$\begin{aligned}
&= (0*((x*(x*y))*(y*x)))*(0*(y*(y*x))) \quad \text{[by P1]} \\
&= ((0*(x*(x*y)))*(0*(y*x)))*((0*y)*(0*(y*x)))\text{[by P6]} \\
&= (((0*x)*(0*(x*y)))*(0*(y*x)))*((0*y)*(0*(y*x))) \\
&\leqslant ((0*x)*(0*(x*y)))*(0*y)\text{[by P3]} \\
&= ((0*x)*(0*y))*(0*(x*y))\text{[by P1]} \\
&= (0*(x*y))*(0*(x*y))\text{[by P6]} \\
&= 0\text{[by A3]}
\end{aligned}$$

Hence, $(0*(0*(y*(y*x)))) \leqslant ((x*(x*y))*(y*x))$.

Since, $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy ideal, then μ_A is order preserving and λ_A is order reversing so, $\mu_A(0*(0*(y*(y*x)))) \leqslant \mu_A((x*(x*y))*(y*x))$ and $\lambda_A(0*(0*(y*(y*x)))) \geqslant \lambda_A((x*(x*y))*(y*x))$.
But by Proposition 17, we have $\mu_A(y*(y*x)) \leqslant \mu_A(0*(0*(y*(y*x))))$ and $\lambda_A(y*(y*x)) \geqslant \lambda_A(0*(0*(y*(y*x))))$. Hence, $\mu_A(y*(y*x)) \leqslant \mu_A((x*(x*y))*(y*x))$ and $\lambda_A(y*(y*x)) \geqslant \lambda_A((x*(x*y))*(y*x))$. By Theorem 5, we see that $A =$

(μ_A, λ_A) is a doubt Atanassov's intuitionistic fuzzy sub-implicative ideal of X. ■

But, the converse of Theorem 19 does not hold in general, which is established by Example 4. As, $\mu_A(2) \nleq \mu_A((2*3)*(1*3)) \bigvee \mu_A(1)$.

Theorem 20. *Union of any two doubt Atanassov's intuitionistic fuzzy P-ideals of X, is also a doubt Atanassov's intuitionistic fuzzy P-ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two doubt Atanassov's intuitionistic fuzzy P-ideals of X. Again let, $C = A \cup B = (\mu_C, \lambda_C)$, where $\mu_C = \mu_A \vee \mu_B$ and $\lambda_C = \lambda_A \wedge \lambda_B$. Let $x, y, z \in X$, then, $\mu_C(0) = max\{\mu_A(0), \mu_B(0)\} \leqslant max\{\mu_A(x), \mu_B(x)\} = \mu_C(x)$ and $\lambda_C(0) = min\{\lambda_A(0), \lambda_B(0)\} \geqslant min\{\lambda_A(x), \lambda_B(x)\} = \lambda_C(x)$, for all $x \in X$.

Also,

$$\begin{aligned}\mu_C(x) &= max\{\mu_A(x), \mu_B(x)\}\\ &\leqslant max\{max[\mu_A((x*z)*(y*z)), \mu_A(y)],\\ &\qquad max[\mu_B((x*z)*(y*z)), \mu_B(y)]\}\\ &= max\{max[\mu_A((x*z)*(y*z)), \mu_B((x*z)\\ &\qquad *(y*z))], max[\mu_A(y), \mu_B(y)]\}\\ &= max[\mu_C((x*z)*(y*z)), \mu_C(y)].\end{aligned}$$

Similarly, we can prove that, $\lambda_C(x) \geqslant min[\lambda_C((x*z)*(y*z)), \lambda_C(y)]$.
This completes the proof. ■

Theorem 21. *Let A and B be two Atanassov's intuitionistic fuzzy subsets of X, such that one is contained into another. Also A and B are two doubt Atanassov's intuitionistic fuzzy P-ideals of X. Then the intersection of A and B is also a doubt Atanassov's intuitionistic fuzzy P-ideal of X.*

Proof: Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two doubt Atanassov's intuitionistic fuzzy P-ideals of X. Again let, $D = A \cap B = (\mu_D, \lambda_D)$, where $\mu_D = min\{\mu_A, \mu_B\}$ and $\lambda_D = max\{\lambda_A, \lambda_B\}$. Let $x \in X$, then $\mu_D(0) = min\{\mu_A(0), \mu_B(0)\} \leqslant min\{\mu_A(x), \mu_B(x)\} = \mu_D(x)$ and $\lambda_D(0) = max\{\lambda_A(0), \lambda_B(0)\} \geqslant max\{\lambda_A(x), \lambda_B(x)\} = \lambda_D(x)$.

Also, for $x, y, z \in X$

$$\begin{aligned}\mu_D(x) &= min\{\mu_A(x), \mu_B(x)\}\\ &\leqslant min[max\{\mu_A((x*z)*(y*z)), \mu_A(y)\},\\ &\qquad max\{\mu_B((x*z)*(y*z)), \mu_B(y)\}]\\ &= max[\min\{\mu_A((x*z)*(y*z)), \mu_B((x*z)*\\ &\qquad (y*z))\}, min\{\mu_A(y), \mu_B(y)\}],\\ &\qquad \text{[because one is contained into another]}\\ &= max[\mu_D((x*z)*(y*z)), \mu_D(y)].\end{aligned}$$

Again,

$$\begin{aligned}\lambda_D(x) &= max\{\lambda_A(x), \lambda_B(x)\}\\ &\geqslant max[min\{\lambda_A((x*z)*(y*z)), \lambda_A(y)\},\\ &\qquad min\{\lambda_B((x*z)*(y*z)), \lambda_B(y)\}]\\ &= min[\max\{\lambda_A((x*z)*(y*z)), \lambda_B((x*z)*\\ &\qquad (y*z))\}, max\{\lambda_A(y), \lambda_B(y)\}],\\ &\qquad \text{[because one is contained into another]}\\ &= min[\lambda_D((x*z)*(y*z)), \lambda_D(y)].\end{aligned}$$

This completes the proof. ■

We now proof the Theorem 20 and Theorem 21 by the following example.

Example 7. Let $X = \{0,1,2,3\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Let $A = (\mu_A, \lambda_A)$ be an Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2	3
μ_A	0	0.7	0.7	0.8
λ_A	1	0.3	0.3	0.2

Then $A = (\mu_A, \lambda_A)$ is a doubt Atanassov's intuitionistic fuzzy P-ideal of X.
Again, let $B = (\mu_B, \lambda_B)$ be an Atanassov's intuitionistic fuzzy set of X defined by

X	0	1	2	3
μ_B	0	0.4	0.4	0.5
λ_B	1	0.6	0.6	0.5

Then $B = (\mu_B, \lambda_B)$ is a doubt Atanassov's intuitionistic fuzzy P-ideal of X.
We also assume that $P = A \cup B = (\mu_P, \lambda_P)$, where $\mu_P = \mu_A \vee \mu_B$ and $\lambda_P = \lambda_A \wedge \lambda_B$ and P is defined as:

X	0	1	2	3
μ_P	0	0.7	0.7	0.8
λ_P	1	0.3	0.3	0.2

Then $P = (\mu_P, \lambda_P)$ is a doubt Atanassov's intuitionistic fuzzy P-ideal of X.

Now let, $Q = A \cap B = (\mu_Q, \lambda_Q)$ where $\mu_Q = \mu_A \wedge \mu_B$ and $\lambda_Q = \lambda_A \vee \lambda_B$.
Then Q is an Atanassov's intuitionistic fuzzy set of X which can be defined as:

X	0	1	2	3
μ_Q	0	0.4	0.4	0.5
λ_Q	1	0.6	0.6	0.5

Then it is clear that $Q = (\mu_Q, \lambda_Q)$ is a doubt Atanassov's intuitionistic fuzzy P-ideal of X.

5. Conclusions

Hope that this work will develop a deep impact on the upcoming research works in this particular field and at the same time, it will prove to be very helpful in the scholastic study of other concerned fields to open up new horizons of interest, erudition and innovations.

Acknowledgments

We are very grateful and would like to express our sincere thanks to anonymous editor and referees for their valuable comments and suggestions which help us to improving this paper a lot.

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