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A note on operations of hesitant fuzzy sets

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Abstract

In this paper, properties of operations and algebraic structures of hesitant fuzzy sets are investigated. Semilattices of hesitant fuzzy sets with union and intersection are discussed, respectively. By using $\oplus$ and $\otimes$ operators, the commutative monoid of hesitant fuzzy sets is provided, moreover, the lattice and distributive lattice of hesitant fuzzy sets are defined on the equivalence class of hesitant fuzzy sets. Based on the distributive lattice of hesitant fuzzy sets, the residuated lattices of hesitant fuzzy sets are constructed by residual implications, which are induced by intersection and $\otimes$, respectively. From the theoretical point of view, algebraic structures of hesitant fuzzy sets are useful for approximate reasoning and decision making to deal with hesitancy of information.

Keywords: hesitant fuzzy sets, operations, the lattice, the residuated lattice.

1. Introduction

Because of various types of uncertainties present in economics, engineering and decision making, theories of probability, fuzzy set [30] and rough set [13] as well-known and often useful mathematical tools have been proposed to describe and handle those uncertainties. Recently, hesitant fuzzy sets (HFSs) [19, 20] and its applications are progressing rapidly [18], as a generalization of fuzzy sets, HFSs are more suitable for dealing with the situations where decision makers have hesitancy in providing their preferences over objects, rather than a margin of error considered in intuitionistic fuzzy sets (IFSs) [1] or some possibility distribution on the possible values considered in type-2 fuzzy sets (T-2FS) and type-*n* fuzzy sets (T-*n*FSs) [5, 12]. HFSs permit the membership degree of an element to be a set, which is represented as several possible values between 0 and 1 [19, 20]. There are some conclusions of HFSs show that they are different to IFSs because the envelope of HFSs can be considered as an IFS characterized by a membership degree and a non-membership degree, different to T-2FSs [11,27] because all HFSs are T-2FSs in which the membership degree of a given element is defined as a fuzzy set, also different to fuzzy multisets (FMs) because HFSs and FMs are of the same form but have different operations [19, 20].

In decision making problems, experts are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement, such cases motivate experts to use hesitant fuzzy sets for decision makings [7, 35], *e.g.*, permit several membership values for a single thing in the reference set. For solving decision making problems based on hesitant fuzzy sets, many hesitant fuzzy distance measures and aggregation operators have been proposed, such as the entropy of hesitant fuzzy sets and interval-valued hesitant fuzzy sets [6], generalized hesitant fuzzy synergetic weighted distance measure [14] and hesitant normalized Hamming, hesitant normalized Hausdorff distance and their generalizations [25]; interval-valued hesitant fuzzy aggregation operators [4], operations of generalized hesitant fuzzy sets according to score function and consistency function [15], hesitant fuzzy prioritized operators and hesitant interval-valued fuzzy aggregation operators [21, 22], hesitant fuzzy ordered weighted averaging operator, hesitant fuzzy ordered weighted geometric operator and their generalization operators [24], TOPSIS and the maximizing deviation method with hesitant fuzzy information [26], the generalized hesitant fuzzy prioritized weighted average and generalized hesitant fuzzy prioritized weighted geometric operators [28], E-VIKOR method with hesitant fuzzy information for the multiple criteria decision making [31], hesitant fuzzy power aggregation operators [32], and hesitant fuzzy geometric Bonferroni means [33], *etc*. To deal with linguistic group decision making in hesitant situations, hesitant fuzzy linguistic term sets and corresponding with hesitant fuzzy linguistic aggregation have been proposed in [8–10, 16, 17, 23, 34].

In many practical decision making problems, the evaluation experts are requested to provide the performance of the evaluation objects and the familiarity with the evaluation areas, which are called confidence levels of decision making, the corresponding decision makings are called decision making with confidence levels. The concept of confidence levels is also used in fuzzy set theory, *i.e.*, $\alpha-$level sets, formally, a fuzzy set can be expressed by its all $\alpha-$level sets, and operations and algebraic properties of fuzzy sets with $\alpha-$level are widely discussed [3, 30]. Confidence levels (or degrees) are used in all extension of fuzzy sets, such as, in [29], many intuitionistic fuzzy aggregation operators with confidences levels of aggregated arguments are proposed and utilized in multiple attribute group decision making problems with intuitionistic fuzzy information. In hesitant fuzzy sets, Torra [19] defined $\alpha-$upper and $\alpha-$lower bounds of a hesitant fuzzy set to help explanations of union and intersection of hesitant fuzzy sets, formally, $\alpha-$upper and $\alpha-$lower bounds of a hesitant fuzzy set can be considered as a hesitant fuzzy set with $\alpha-$confidence level (or degree). Inspired by existed interesting conclusions of fuzzy sets and fuzzy decision making with confidence levels, we investigate properties of operations on HFSs with $\alpha-$confidence level and algebraic structures of hesitant fuzzy sets in this paper. The rest of this paper is arranged as follows: In Section 2, we introduce HFSs and its'operations, analyze $\alpha-$upper and $\alpha-$lower bounds of a hesitant fuzzy set and discuss some properties of operations. In Section 3, we prove that union, intersection, $\oplus$ and $\otimes$ on hesitant fuzzy sets with confidence levels satisfy commutativity, associativity, idempotency, absorption and boundary, *etc*. In Section 4, we construct semilattices of hesitant fuzzy sets based on union and intersection with $\alpha-$confidence level, and commutative monoids of hesitant fuzzy sets based on $\oplus$ and $\otimes$ with $\alpha-$confidence level. Lattices and distributive lattices of hesitant fuzzy sets are constructed on the equivalence class of hesitant fuzzy sets, which is induced by special hesitant fuzzy sets, *i.e.*, $h(x)$ is a closed interval of $[0,1]$ for any $x \in X$. Based on distributive lattices of hesitant fuzzy sets, residuated lattices of hesitant fuzzy sets are constructed by residual implications, which are induced by intersection or $\otimes$ with $\alpha-$confidence level, respectively. We conclude the paper in Section 5.

2. Preliminaries

This section starts with the definition of hesitant fuzzy sets introduced in [19, 20], and views some operations on hesitant fuzzy sets. Throughout this paper, $X = \{x_1, x_2, \cdots, x_n\}$ is used frequently to denote the discourse set.

Definition 1. [19] Let X be a reference set, then we define a hesitant fuzzy set on X in terms of a function h that when applied to X returns a subset of [0, 1]. A hesitant fuzzy set M on X is also denoted as $M = \{\langle x, h(x)\rangle | \forall x \in X\}$, where $h(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to M. The set of all HFSs on X is denoted by $H(X) = \{\{\langle x, h(x)\rangle | \forall x \in X\}|$for any x in X and any function h, $h(x)$ is a set of some different values in [0, 1]$\}$. In $H(X)$, as special cases, $h^0 = \{\langle x, \{0\}\rangle | \forall x \in X\}$ is the empty hesitant set, $h^1 = \{\langle x, \{1\}\rangle | \forall x \in X\}$ is the full hesitant set, $h^{[0,1]} = \{\langle x, [0,1]\rangle | \forall x \in X\}$ is the set to represent complete ignorance for $x \in X$ and $h^{\emptyset} = \{\langle x, \emptyset\rangle | \forall x \in X\}$ is the nonsense set. For any h, h_1 and h_2 in $H(X)$, some operations on them can be described as follows [19,20,24]:

1. lower bound: $h^-(x) = min\ h(x)$;
2. $\alpha-$lower bound: $h^-_\alpha(x) = \{\gamma \in h(x) | \gamma \leqslant \alpha\}$;
3. upper bound: $h^+(x) = max\ h(x)$;
4. $\alpha-$upper bound: $h^+_\alpha(x) = \{\gamma \in h(x) | \gamma \geqslant \alpha\}$;
5. complement: $h^c(x) = \{1-\gamma | \gamma \in h(x)\}$;
6. union: $(h_1 \cup h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \geqslant max\{(h_1)^-(x), (h_2)^-(x)\}\}$;
7. intersection: $(h_1 \cap h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \leqslant min\{(h_1)^+(x), (h_2)^+(x)\}\}$;
8. $h^\lambda(x) = \{\gamma^\lambda | \gamma \in h(x)\}$;
9. $\lambda h(x) = \{1-(1-\gamma)^\lambda | \gamma \in h(x)\}$;
10. $(h_1 \oplus h_2)(x) = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 | \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$;
11. $(h_1 \otimes h_2)(x) = \{\gamma_1\gamma_2 | \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$.

$\alpha-$lower bound $h^-_\alpha(x)$ and $\alpha-$upper bound $h^+_\alpha(x)$ can be used to well explain union and intersection of hesitant fuzzy sets [19], *i.e.*, $(h_1 \cup h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \geqslant max\{(h_1)^-(x), (h_2)^-(x)\}\} = (h_1 \cup h_2)^+_\alpha(x)\ for\ \alpha = max\{(h_1)^-(x), (h_2)^-(x)$,

$(h_1 \cap h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \leqslant min\{(h_1)^+(x), (h_2)^+(x)\}\} = (h_1 \cap h_2)^-_\alpha(x)\ for\ \alpha = min\{(h_1)^+(x), (h_2)^+(x)\}$.

As a special case, we have the following results of $\alpha-$lower bound $h^-_\alpha(x)$ and $\alpha-$upper bound $h^+_\alpha(x)$,

1. if $\alpha = h^-(x) = min\ h(x)$, then $h^+_\alpha(x) = h(x)$;
2. if $\alpha = h^+(x) = max\ h(x)$, then $h^-_\alpha(x) = h(x)$.

Accordingly, $\alpha-$lower bound and $\alpha-$upper bound can be considered as a hesitant fuzzy set with $\alpha-$confidence level (or degree), that is, $\alpha-$lower bound and $\alpha-$upper bound are $\alpha-$lower confidence level and $\alpha-$upper confidence level of hesitancy in a hesitant fuzzy set, respectively. h^-_α and h^+_α are called the hesitant fuzzy set h with $\alpha-$confidence level (or degree) in this paper.

Example 1. Let $X = \{x_1, x_2, x_3\}$ be the discourse set, HFSs h_1 and h_2 on X be $h_1 = \{\langle x_1, \{0.3, 0.4\}\rangle, \langle x_2, \{0.6, 0.8\}\rangle, \langle x_3, \{0.3, 0.4, 0.5, 0.7\}\rangle\}$ and $h_2 = \{\langle x_1, \{0.5, 0.6\}\rangle, \langle x_2, \{0.4, 0.5\}\rangle, \langle x_3, \{0.2, 0.3, 0.4, 0.6\}\rangle\}$, respectively. Then we have

1. $(h_1)^-(x_1) = min\{0.3, 0.4\} = 0.3$, $(h_1)^+(x_1) = max\{0.3, 0.4\} = 0.4$;
2. $(h_1)^+_{0.3}(x_1) = \{\gamma \in h_1(x_1) | \gamma \geqslant 0.3\} = \{0.3, 0.4\} = h_1(x_1)$;
3. $(h_1)^-_{0.4}(x_1) = \{\gamma \in h_1(x_1) | \gamma \leqslant 0.4\} = \{0.3, 0.4\} = h_1(x_1)$;
4. $(h_1)^-_{0.45}(x_3) = \{\gamma \in h_1(x_3) | \gamma \leqslant 0.45\} = \{0.3, 0.4\}$;
5. $(h_1)^+_{0.45}(x_3) = \{\gamma \in h_1(x_3) | \gamma \geqslant 0.45\} = \{0.5, 0.7\}$;
6. $(h_1)^c(x_2) = \cup_{\gamma \in h_1(x_2)}\{1-\gamma\} = \{1-0.6, 1-0.8\} = \{0.4, 0.2\}$;
7. $(h_1 \cup h_2)(x_3) = \{\gamma \in h_1(x_3) \cup h_2(x_3) | \gamma \geqslant max\{(h_1)^-(x_3), (h_2)^-(x_3)\}\} = \{\gamma \in h_1(x_3) \cup h_2(x_3) | \gamma \geqslant max\{0.3, 0.2\}\} = \{0.3, 0.4, 0.5, 0.6, 0.7\}$;
8. $(h_1 \cap h_2)(x_3) = \{\gamma \in h_1(x_3) \cup h_2(x_3) | \gamma \leqslant min\{(h_1)^+(x_3), (h_2)^+(x)\}\} = \{\gamma \in h_1(x_3) \cup h_2(x_3) | \gamma \leqslant min\{0.7, 0.6\}\} = \{0.2, 0.3, 0.4, 0.5, 0.6\}$;
9. $(h_1 \oplus h_2)(x_1) = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 | \gamma_1 \in h_1(x_1), \gamma_2 \in h_2(x_1)\} = \{0.65, 0.72, 0.7, 0.76\}$;
10. $(h_1 \otimes h_2)(x_1) = \{\gamma_1\gamma_2 | \gamma_1 \in h_1(x_1), \gamma_2 \in h_2(x_1)\} = \{0.15, 0.18, 0.2, 0.24\}$.

The following proposition shows that union and intersection of HFSs satisfy commutative, associative, idempotence and absorption laws in the limited condition.

Proposition 1. *For any h_1, h_2, h_3 in $H(X)$ and $x \in X$, we have*

1. *Commutativity: $(h_1 \cup h_2)(x) = (h_2 \cup h_1)(x)$ and $(h_1 \cap h_2)(x) = (h_2 \cap h_1)(x)$;*
2. *Associativity: $((h_1 \cup h_2) \cup h_3)(x) = (h_1 \cup (h_2 \cup h_3))(x)$ and $((h_1 \cap h_2) \cap h_3)(x) = (h_1 \cap (h_2 \cap h_3))(x)$;*
3. *Idempotency: $(h_1 \cup h_1)(x) = h_1(x)$ and $(h_1 \cap h_1)(x) = h_1(x)$;*
4. *Absorption: if $(h_1)^+(x) \leqslant (h_2)^-(x)$, then $(h_2 \cap (h_1 \cup h_2))(x) = h_2(x)$ and $(h_1 \cup (h_1 \cap h_2))(x) = h_1(x)$.*

Proof. (1), (2) and (3) are obvious.

If $(h_1)^+(x) < (h_2)^-(x)$, then $(h_1)^-(x) \leqslant (h_1)^+(x) < (h_2)^-(x) \leqslant (h_2)^+(x)$. Hence, $(h_1 \cup h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \geqslant max\{(h_1)^-(x), (h_2)^-(x)\}\} = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \geqslant (h_2)^-(x)\} = h_2(x)$ and $(h_1 \cap h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \leqslant min\{(h_1)^+(x), (h_2)^+(x)\}\} = \{\gamma \in h_1(x) \cup h_2(x) | \gamma \leqslant (h_1)^+(x)\} = h_1(x)$. Accordingly, $(h_2 \cap (h_1 \cup h_2))(x) = (h_2 \cap h_2)(x) = h_2(x)$ and $(h_1 \cup (h_1 \cap h_2))(x) = (h_1 \cup h_1)(x) = h_1(x)$. □

Example 2. (Continues Example 1) Due to $(h_1)^+(x_1) = 0.4 < (h_2)^-(x_1) = 0.5$, we have $(h_1 \cup h_2)(x_1) = \{\gamma \in h_1(x_1) \cup h_2(x_1) | \gamma \geqslant max\{(h_1)^-(x_1), (h_2)^-(x_1)\}\} = \{\gamma \in h_1(x_1) \cup h_2(x_1) | \gamma \geqslant max\{0.3, 0.5\}\} = \{0.5, 0.6\} = h_2(x_1)$ and $(h_1 \cap h_2)(x_1) = \{\gamma \in h_1(x_1) \cup h_2(x_1) | \gamma \leqslant min\{(h_1)^+(x_1), (h_2)^+(x_1)\}\} = \{\gamma \in h_1(x_1) \cup h_2(x_1) | \gamma \leqslant min\{0.4, 0.6\}\} = \{0.3, 0.4\} = h_1(x_1)$.

The following proposition shows that complement of HFSs is combined with lower bound, upper bound, $\alpha-$lower bound and $\alpha-$upper bound of HFSs.

Proposition 2. *For any $h \in H(X)$ and $x \in X$,*

1. *$(h^c)^-(x) = 1 - h^+(x)$ and $(h^c)^+(x) = 1 - h^-(x)$;*
2. *$(h^+_\alpha)^c(x) = (h^c)^-_{1-\alpha}(x)$ and $(h^-_\alpha)^c(x) = (h^c)^+_{1-\alpha}(x)$.*

Proof. (1) $(h^c)^-(x) = min\ h^c(x) = min\{1 - r | r \in h(x)\} = 1 - max\{r | r \in h(x)\} = 1 - h^+(x)$ and $(h^c)^+(x) = max\ h^c(x) = max\{1 - r | r \in h(x)\} = 1 - min\{r | r \in h(x)\} = 1 - h^-(x)$.

(2) $(h^+_\alpha)^c(x) = \{\gamma \in h(x) | \gamma \geqslant \alpha\}^c = \{1 - \gamma | \gamma \in h(x), \gamma \geqslant \alpha\} = \{1 - \gamma | 1 - \gamma \in h^c(x), 1 - \gamma \leqslant 1 - \alpha\} = \{\gamma' | \gamma' \in h^c(x), \gamma' \leqslant 1 - \alpha\} = (h^c)^-_{1-\alpha}(x)$ and $(h^-_\alpha)^c(x) = \{\gamma \in h(x) | \gamma \leqslant \alpha\}^c = \{1 - \gamma | \gamma \in h(x), \gamma \leqslant \alpha\} = \{1 - \gamma | 1 - \gamma \in h^c(x), 1 - \gamma \geqslant 1 - \alpha\} = \{\gamma' | \gamma' \in h^c(x), \gamma' \geqslant 1 - \alpha\} = (h^c)^+_{1-\alpha}(x)$. □

Example 3. (Continues Example 1) $((h_1)^c)^-(x_1) = min\{1 - 0.3, 1 - 0.4\} = 0.6 = 1 - (h_1)^+(x_1)$, $((h_1)^c)^+(x_1) = max\{1 - 0.3, 1 - 0.4\} = 0.7 = 1 - (h_1)^-(x_1)$. $((h_2)^+_{0.4})^c(x_2) = \{1 - 0.4, 1 - 0.5\} = \{0.6, 0.5\}$ and $((h_2)^c)^-_{1-0.4}(x_2) = (\{1 - 0.4, 1 - 0.5\})^-_{0.6} = \{0.6, 0.5\}$, *i.e.*, $((h_2)^+_{0.4})^c(x_2) = ((h_2)^c)^-_{1-0.4}(x_2)$. $((h_2)^-_{0.6})^c(x_2) = \{1 - 0.4, 1 - 0.5\} = \{0.6, 0.5\}$ and $((h_2)^c)^+_{1-0.6}(x_2) = (\{1 - 0.4, 1 - 0.5\})^+_{0.4} = \{0.6, 0.5\}$, *i.e.*, $((h_2)^-_{0.6})^c(x_2) = ((h_2)^c)^+_{0.4}(x_2)$.

The following proposition discusses boundary of HFSs.

Proposition 3. *For any $h \in H(X)$ and $x \in X$,*

1. *$(h \cup h^1)(x) = h^1(x)$ and $(h \cap h^1)(x) = h(x)$;*
2. *$(h \cup h^0)(x) = h(x)$ and $(h \cap h^0)(x) = h^0(x)$;*
3. *$(h \oplus h^1)(x) = h^1(x)$ and $(h \otimes h^1)(x) = h(x)$*
4. *$(h \oplus h^0)(x) = h(x)$ and $(h \otimes h^0)(x) = h^0(x)$.*

Proof. (1) $(h \cup h^1)(x) = \{\gamma \in h(x) \cup \{1\} | \gamma \geqslant max\{h^-(x), 1\}\} = \{\gamma \in h(x) \cup \{1\} | \gamma \geqslant 1\} = h^1(x)$ and $(h \cap h^1)(x) = \{\gamma \in h(x) \cup \{1\} | \gamma \leqslant min\{h^+(x), 1\}\} = \{\gamma \in h(x) \cup \{1\} | \gamma \leqslant h^+(x)\} = h(x)$.

(2) $(h \cup h^0)(x) = \{\gamma \in h(x) \cup \{0\} | \gamma \geqslant max\{h^-(x), 0\}\} = \{\gamma \in h(x) \cup \{0\} | \gamma \geqslant h^-(x)\} = h(x)$ and $(h \cap h^0)(x) = \{\gamma \in h(x) \cup \{0\} | \gamma \leqslant min\{h^+(x), 0\}\} = \{\gamma \in h(x) \cup \{0\} | \gamma \leqslant 0\} = h^0(x)$.

(3) $(h \oplus h^1)(x) = \cup_{\gamma_1 \in h(x), \gamma_2 \in \{1\}}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} = \cup_{\gamma_1 \in h(x), \gamma_2 \in \{1\}}\{\gamma_1 + 1 - \gamma_1 \times 1\} = \cup_{\gamma_1 \in h(x), \gamma_2 \in \{1\}}\{1\} = h^1(x)$ and $(h \otimes h^1)(x) =$

$\cup_{\gamma_1\in h(x),\gamma_2\in\{1\}}\{\gamma_1\gamma_2\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{1\}}\{\gamma_1 \times 1\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{1\}}\{\gamma_1\} = h(x)$.

(4) $(h \oplus h^0)(x) = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{\gamma_1 + 0 - \gamma_1 \times 0\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{\gamma_1\} = h(x)$ and $(h \otimes h^0)(x) = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{\gamma_1\gamma_2\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{\gamma_1 \times 0\} = \cup_{\gamma_1\in h(x),\gamma_2\in\{0\}}\{0\} = h^0(x)$. □

3. Operations on HFSs with $\alpha-$confidence level

In this section, properties of operations on HFSs with $\alpha-$lower confidence level and $\alpha-$upper confidence level are discussed, respectively. For convenience, for any h_1 and h_2 in $H(X)$, $x \in X$ and $\alpha \in [0,1]$, we denote

1. $(h_1 \cup^+_\alpha h_2)(x) = ((h_1)^+_\alpha \cup (h_2)^+_\alpha)(x)$;
2. $(h_1 \cap^+_\alpha h_2)(x) = ((h_1)^+_\alpha \cap (h_2)^+_\alpha)(x)$;
3. $(h_1 \cup^-_\alpha h_2)(x) = ((h_1)^-_\alpha \cup (h_2)^-_\alpha)(x)$;
4. $(h_1 \cap^-_\alpha h_2)(x) = ((h_1)^-_\alpha \cap (h_2)^-_\alpha)(x)$.

Example 4. For hesitant fuzzy sets $h_A(x) = \{0.1,0.3,0.6\}$ and $h_B(x) = \{0.4,0.5,0.8\}$, we have $(h_A \cup h_B)(x) = \{\gamma \in h_A(x) \cup h_B(x)|\gamma \geqslant max\{h^-_A(x),h^-_B(x)\}\} = \{\gamma \in h_A(x) \cup h_B(x)|\gamma \geqslant max\{0.1,0.4\}\} = \{\gamma \in h_A(x) \cup h_B(x)|\gamma \geqslant 0.4\} = \{0.4,0.5,0.6,0.8\}$ and $(h_A \cap h_B)(x) = \{0.1,0.3,0.4,0.5,0.6\}$.

Let $\alpha = 0.45$, then $(h_A)^+_{0.45}(x) = \{0.6\}$, $(h_A)^-_{0.45}(x) = \{0.1,0.3\}$, $(h_B)^+_{0.45}(x) = \{0.5,0.8\}$ and $(h_B)^-_{0.45}(x) = \{0.4\}$. We have $(h_A \cup^+_{0.45} h_B)(x) = \{0.6,0.8\}$, $(h_A \cap^+_{0.45} h_B)(x) = \{0.5,0.6\}$, $(h_A \cup^-_{0.45} h_B)(x) = \{0.4\}$ and $(h_A \cap^-_{0.45} h_B)(x) = \{0.1,0.3\}$.

Theorem 4. *For any h_1 and h_2 in $H(X)$, $x \in X$ and $\alpha \in [0,1]$,*

1. $(h_1 \cup^+_\alpha h_2)^c(x) = (((h_1)^+_\alpha)^c \cap ((h_2)^+_\alpha)^c)(x)$;
2. $(h_1 \cap^+_\alpha h_2)^c(x) = (((h_1)^+_\alpha)^c \cup ((h_2)^+_\alpha)^c)(x)$;
3. $(h_1 \cup^-_\alpha h_2)^c(x) = (((h_1)^-_\alpha)^c \cap ((h_2)^-_\alpha)^c)(x)$;
4. $(h_1 \cap^-_\alpha h_2)^c(x) = (((h_1)^-_\alpha)^c \cup ((h_2)^-_\alpha)^c)(x)$.

Proof. According to proposition 2, we have

$(h_1 \cup^+_\alpha h_2)^c(x) = ((h_1)^+_\alpha \cup (h_2)^+_\alpha)^c(x) = \{\gamma \in (h_1)^+_\alpha(x) \cup (h_2)^+_\alpha(x)| \; \gamma \geqslant max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}\}^c = \{1 - \gamma \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma \geqslant max\{((h_1)^+_\alpha)^-(x),((h_2)^+_\alpha)^-(x)\}\} = \{1 - \gamma \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; 1 - \gamma \leqslant 1 - max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}\} = \{\gamma'(= 1 - \gamma) \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma' \leqslant min\{1 - ((h_1)^+_\alpha)^-(x), 1 - ((h_2)^+_\alpha)^-(x)\}\} = \{\gamma' \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma' \leqslant min\{(((h_1)^+_\alpha)^c)^+(x),(((h_2)^+_\alpha)^c)^+(x)\}\} = (((h_1)^+_\alpha)^c \cap ((h_2)^+_\alpha)^c)(x)$.

$(h_1 \cap^+_\alpha h_2)^c(x) = ((h_1)^+_\alpha \cap (h_2)^+_\alpha)^c(x) = \{\gamma \in (h_1)^+_\alpha(x) \cup (h_2)^+_\alpha(x)| \; \gamma \leqslant min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}\}^c = \{1 - \gamma \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma \leqslant min\{((h_1)^+_\alpha)^+(x),((h_2)^+_\alpha)^+(x)\}\} = \{1 - \gamma \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; 1 - \gamma \geqslant 1 - min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}\} = \{\gamma'(= 1 - \gamma) \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma' \geqslant max\{1 - ((h_1)^+_\alpha)^+(x), 1 - ((h_2)^+_\alpha)^+(x)\}\} = \{\gamma' \in ((h_1)^+_\alpha)^c(x) \cup ((h_2)^+_\alpha)^c(x)| \; \gamma' \geqslant max\{(((h_1)^+_\alpha)^c)^-(x),(((h_2)^+_\alpha)^c)^-(x)\}\} = (((h_1)^+_\alpha)^c \cup ((h_2)^+_\alpha)^c)(x)$.

Similarly, we can prove that $(h_1 \cup^-_\alpha h_2)^c(x) = (((h_1)^-_\alpha)^c \cap ((h_2)^-_\alpha)^c)(x)$ and $(h_1 \cap^-_\alpha h_2)^c(x) = (((h_1)^-_\alpha)^c \cup ((h_2)^-_\alpha)^c)(x)$ are valid. □

For any h_1 and h_2 in $H(X)$, $x \in X$ and $\alpha \in [0,1]$, we denote

1. $(h_1 \oplus^+_\alpha h_2)(x) = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2|\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)\}$;
2. $(h_1 \otimes^+_\alpha h_2)(x) = \{\gamma_1\gamma_2|\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)\}$;
3. $(h_1 \oplus^-_\alpha h_2)(x) = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2|\gamma_1 \in (h_1)^-_\alpha(x), \gamma_2 \in (h_2)^-_\alpha(x)\}$;
4. $(h_1 \otimes^-_\alpha h_2)(x) = \{\gamma_1\gamma_2|\gamma_1 \in (h_1)^-_\alpha(x), \gamma_2 \in (h_2)^-_\alpha(x)\}$.

According to above mentioned analysis, the following corollary is obvious.

Corollary 5. *For any h_1 and h_2 in $H(X)$, $x \in X$ and $\alpha \in [0,1]$,*

1. *If $\alpha \leqslant min\{(h_1)^-(x),(h_2)^-(x)\}$, then $(h_1 \oplus^+_\alpha h_2)(x) = (h_1 \oplus h_2)(x)$ and $(h_1 \otimes^+_\alpha h_2)(x) = (h_1 \otimes h_2)(x)$;*

2. *If* $\alpha \geqslant max\{(h_1)^+(x),(h_2)^+(x)\}$*, then* $(h_1 \oplus^-_\alpha h_2)(x) = (h_1 \oplus h_2)(x)$ *and* $(h_1 \otimes^-_\alpha h_2)(x) = (h_1 \otimes h_2)(x)$.

Example 5. (Continues Example 4), $(h_A \oplus h_B)(x) = \{0.46,0.55,0.82,0.58,0.65, 0.86,0.76,0.8, 0.92\}$ and $(h_A \otimes h_B)(x) = \{0.04,0.05, 0.08,0.12,0.15,0.24,0.3,0.48\}$.

Let $\alpha = 0.1 = min\{(h_A)^-(x),(h_B)^-(x)\}$, then $(h_A \oplus^+_{0.1} h_B)(x) = (h_A \oplus h_B)(x)$ and $(h_A \otimes^+_{0.1} h_B)(x) = (h_A \otimes h_B)(x)$ are obvious. However, $(h_A \oplus^-_{0.1} h_B)(x) = \emptyset$ and $(h_A \otimes^-_{0.1} h_B)(x) = \emptyset$.

Let $\alpha = 0.8 = max\{(h_A)^+(x),(h_B)^+(x)\}$, then $(h_A \oplus^-_{0.8} h_B)(x) = (h_A \oplus h_B)(x)$ and $(h_A \otimes^-_{0.8} h_B)(x) = (h_A \otimes h_B)(x)$. However, $(h_A \oplus^+_{0.8} h_B)(x) = \emptyset$ and $(h_A \otimes^+_{0.8} h_B)(x) = \emptyset$.

Let $\alpha = 0.45$, then $(h_A \oplus^+_{0.45} h_B)(x) = \{0.8,0.92\}$, $(h_A \otimes^+_{0.45} h_B)(x) = \{0.3,0.48\}$, $(h_A \oplus^-_{0.45} h_B)(x) = \{0.46,0.58\}$ and $(h_A \otimes^-_{0.45} h_B)(x) = \{0.04,0.12\}$.

Theorem 6. *For any* h_1 *and* h_2 *in* $H(X)$*,* $x \in X$ *and* $\alpha \in [0,1]$*,*

1. $(h_1 \oplus^+_\alpha h_2)^c(x) = ((h_1)^c \otimes^-_{1-\alpha} (h_2)^c)(x)$;
2. $(h_1 \otimes^+_\alpha h_2)^c(x) = ((h_1)^c \oplus^-_{1-\alpha} (h_2)^c)(x)$;
3. $(h_1 \oplus^-_\alpha h_2)^c(x) = ((h_1)^c \otimes^+_{1-\alpha} (h_2)^c)(x)$;
4. $(h_1 \otimes^-_\alpha h_2)^c(x) = ((h_1)^c \oplus^+_{1-\alpha} (h_2)^c)(x)$.

Proof. According to Proposition 2, we have

$(h_1 \oplus^+_\alpha h_2)^c(x) = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)} \{1 - (\gamma_1 + \gamma_2 - \gamma_1\gamma_2)\} = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)} \{(1 - \gamma_1)(1 - \gamma_2)\} = \cup_{\gamma'_1 \in ((h_1)^+_\alpha)^c(x), \gamma'_2 \in ((h_2)^+_\alpha)^c(x)} \{\gamma'_1\gamma'_2\} = \cup_{\gamma'_1 \in ((h_1)^c)^-_{1-\alpha}(x), \gamma'_2 \in ((h_2)^c)^-_{1-\alpha}(x)} \{\gamma'_1\gamma'_2\} = ((h_1)^c \otimes^-_{1-\alpha} (h_2)^c)(x)$.

$(h_1 \otimes^+_\alpha h_2)^c(x) = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)} \{1 - \gamma_1\gamma_2\} = \cup_{\gamma'_1 \in ((h_1)^+_\alpha)^c(x), \gamma'_2 \in ((h_2)^+_\alpha)^c(x)} \{1 - (1 - \gamma'_1)(1 - \gamma'_2)\} = \cup_{\gamma'_1 \in ((h_1)^c)^-_{1-\alpha}(x), \gamma'_2 \in ((h_2)^c)^-_{1-\alpha}(x)} \{\gamma'_1 + \gamma'_2 - \gamma'_1\gamma'_2\} = ((h_1)^c \oplus^-_{1-\alpha} (h_2)^c)(x)$.

In which, $\gamma'_1 = 1 - \gamma_1$ and $\gamma'_2 = 1 - \gamma_2$. Similarly, we can prove that $(h_1 \oplus^-_\alpha h_2)^c(x) = ((h_1)^c \otimes^+_{1-\alpha} (h_2)^c)(x)$ and $(h_1 \otimes^-_\alpha h_2)^c(x) = ((h_1)^c \oplus^+_{1-\alpha} (h_2)^c)(x)$ are valid. □

4. Algebraic structures of hesitant fuzzy sets with confidence level

Algebraic structures of fuzzy sets are widely discussed [2]. In this section, we discuss algebraic structures of hesitant fuzzy sets with confidence levels based on the above mentioned operations.

4.1. The semilattice of hesitant fuzzy sets with confidence levels

Here, we first analyze properties of operators $\cup^+_\alpha$, $\cup^-_\alpha$, $\cap^+_\alpha$ and $\cap^-_\alpha$ on hesitant fuzzy sets with confidence levels. Then we provide semilattices of hesitant fuzzy sets with confidence levels based on $\cup^+_\alpha$, $\cup^-_\alpha$, $\cap^+_\alpha$ and $\cap^-_\alpha$, respectively. According to Proposition 1 (1), (2) and (3), the following corollaries are obvious.

Corollary 7. *For any* h_1*,* h_2 *and* h_3 *in* $H(X)$*,* $x \in X$ *and* $\alpha \in [0,1]$*,*

1. $(h_1 \cup^+_\alpha h_1)(x) = (h_1)^+_\alpha(x)$;
2. $(h_1 \cup^+_\alpha h_2)(x) = (h_2 \cup^+_\alpha h_1)(x)$;
3. $((h_1 \cup^+_\alpha h_2) \cup^+_\alpha h_3)(x) = (h_1 \cup^+_\alpha (h_2 \cup^+_\alpha h_3))(x)$;
4. $(h_1 \cap^+_\alpha h_1)(x) = (h_1)^+_\alpha(x)$;
5. $(h_1 \cap^+_\alpha h_2)(x) = (h_2 \cap^+_\alpha h_1)(x)$;
6. $((h_1 \cap^+_\alpha h_2) \cap^+_\alpha h_3)(x) = (h_1 \cap^+_\alpha (h_2 \cap^+_\alpha h_3))(x)$;

Corollary 8. *For any* h_1*,* h_2 *and* h_3 *in* $H(X)$*,* $x \in X$ *and* $\alpha \in [0,1]$*,*

1. $(h_1 \cup^-_\alpha h_1)(x) = (h_1)^-_\alpha(x)$;
2. $(h_1 \cup^-_\alpha h_2)(x) = (h_2 \cup^-_\alpha h_1)(x)$;
3. $((h_1 \cup^-_\alpha h_2) \cup^-_\alpha h_3)(x) = (h_1 \cup^-_\alpha (h_2 \cup^-_\alpha h_3))(x)$;
4. $(h_1 \cap^-_\alpha h_1)(x) = (h_1)^-_\alpha(x)$;
5. $(h_1 \cap^-_\alpha h_2)(x) = (h_2 \cap^-_\alpha h_1)(x)$;
6. $((h_1 \cap^-_\alpha h_2) \cap^-_\alpha h_3)(x) = (h_1 \cap^-_\alpha (h_2 \cap^-_\alpha h_3))(x)$.

Accordingly, we can obtain the following theorem.

Theorem 9. *For any discourse set X and* $\alpha \in [0,1]$*,*

1. $(H(X), \cup_\alpha^+, h^1)$ *and* $(H(X), \cup_\alpha^-, h^1)$ *are* $\cup$-*semilattices, respectively;*

2. $(H(X), \cap_\alpha^+, h^0)$ *and* $(H(X), \cap_\alpha^-, h^0)$ *are* $\cap$-*semilattices, respectively.*

Example 6. Let $h_1(x) = \{0.7, 0.8\}$, $h_2(x) = \{0.4, 0.5, 0.6, 0.8\}$ and $h_3(x) = \{0.5, 0.6, 0.7\}$ be three hesitant fuzzy elements. Than $(h_1)^+_{0.6}(x) = \{0.7, 0.8\}, (h_2)^+_{0.6}(x) = \{0.6, 0.8\}, (h_3)^+_{0.6}(x) = \{0.6, 0.7\}$.

$(h_1 \cup^+_{0.6} h_1)(x) = \{r \in (h_1)^+_\alpha(x) | r \geqslant 0.7\} = \{0.7, 0.8\} = (h_1)^+_{0.6}(x)$; 2) $(h_1 \cup^+_{0.6} h_2)(x) = \{r \in \{0.7, 0.8\} \cup \{0.6, 0.8\} | r \geqslant max\{0.7, 0.6\}\} = \{0.7, 0.8\}$, $(h_2 \cup^+_{0.6} h_1)(x) = \{r \in \{0.6, 0.8\} \cup \{0.7, 0.8\} | r \geqslant max\{0.6, 0.7\}\} = \{0.7, 0.8\}$, *i.e.*, $(h_1 \cup^+_{0.6} h_2)(x) = (h_2 \cup^+_{0.6} h_1)(x)$; 3) $((h_1 \cup^+_{0.6} h_2) \cup^+_{0.6} h_3)(x) = \{r \in \{0.7, 0.8\} \cup \{0.6, 0.7\} | r \geqslant 0.7\} = \{0.7, 0.8\}$, $(h_1 \cup^+_{0.6} (h_2 \cup^+_{0.6} h_3))(x) = \{r \in \{0.7, 0.8\} \cup (\{0.6, 0.8\} \cup \{0.6, 0.7\}) | r \geqslant max\{0.7, max\{0.6, 0.6\}\}\} = \{r \in \{0.6, 0.7, 0.8\} | r \geqslant 0.7\} = \{0.7, 0.8\}$, *i.e.*, $((h_1 \cup^+_{0.6} h_2) \cup^+_{0.6} h_3)(x) = (h_1 \cup^+_{0.6} (h_2 \cup^+_{0.6} h_3))(x)$.

$(h_2 \cup^+_{0.6} h_3)(x) = \{r \in \{0.6, 0.8\} \cup \{0.6, 0.7\} | r \geqslant max\{0.6, 0.6\}\} = \{0.6, 0.7, 0.8\}$, $((h_2 \cup^+_{0.6} h_3) \cap^+_{0.6} h_2)(x) = \{r \in \{0.6, 0.7, 0.8\} \cup \{0.6, 0.8\} | r \leqslant 0.8\} = \{0.6, 0.7, 0.8\} \neq (h_2)^+_{0.6}(x)$.

$(h_2 \cap^+_{0.6} h_3)(x) = \{r \in \{0.6, 0.8\} \cup \{0.6, 0.7\} | r \leqslant min\{0.8, 0.7\}\} = \{0.6, 0.7\}$, $((h_2 \cap^+_{0.6} h_3) \cup^+_{0.6} h_2)(x) = \{r \in \{0.6, 0.7\} \cup \{0.6, 0.8\} | r \geqslant max\{0.6, 0.6\}\} = \{0.6, 0.7, 0.8\} \neq (h_2)^+_{0.6}(x)$.

$(h_2)^-_{0.8}(x) = \{0.4, 0.5, 0.6, 0.8\}$ and $(h_3)^-_{0.8}(x) = \{0.5, 0.6, 0.7\}$. $(h_2 \cup^-_{0.8} h_3)(x) = \{r \in \{0.4, 0.5, 0.6, 0.8\} \cup \{0.5, 0.6, 0.7\} | r \geqslant max\{0.4, 0.5\}\} = \{0.5, 0.6, 0.7, 0.8\}$, $((h_2 \cup^-_{0.8} h_3) \cap^-_{0.8} h_2)(x) = \{r \in \{0.5, 0.6, 0.7, 0.8\} \cup \{0.4, 0.5, 0.6, 0.8\} | r \leqslant min \{0.8, 0.8\}\} = \{0.4, 0.5, 0.6, 0.7, 0.8\} \neq (h_2)^-_{0.8}(x)$.

$(h_2 \cap^-_{0.8} h_3)(x) = \{r \in \{0.4, 0.5, 0.6, 0.8\} \cup \{0.5, 0.6, 0.7\} | r \leqslant min\{0.8, 0.7\}\} = \{0.4, 0.5, 0.6, 0.7\}$, $((h_2 \cap^-_{0.8} h_3) \cup^-_{0.8} h_2)(x) = \{r \in \{0.4, 0.5, 0.6, 0.7\} \cup \{0.4, 0.5, 0.6, 0.8\} | r \geqslant max\{0.4, 0.4\}\} = \{0.4, 0.5, 0.6, 0.7, 0.8\} \neq (h_2)^-_{0.8}(x)$.

Example 6 shows that $\cup^+_\alpha$ and $\cap^+_\alpha$ ($\cup^+_\alpha$ and $\cap^+_\alpha$) are not absorptive. Generally, this means that $(H(X), \cup^+_\alpha, \cap^+_\alpha)$ ($(H(X), \cup^-_\alpha, \cap^-_\alpha)$) is not a lattice.

4.2. *Commutative monoid of hesitant fuzzy sets with confidence levels*

In this subsection, we discuss commutative monoid of hesitant fuzzy sets with confidence levels based on $\oplus^+_\alpha$, $\otimes^+_\alpha$, $\oplus^-_\alpha$ and $\otimes^-_\alpha$, respectively.

Theorem 10. *For any* h_1, h_2 *and* h_3 *in* $H(X)$, $x \in X$ *and* $\alpha \in [0, 1]$,

1. $(h_1 \oplus^+_\alpha h_2)(x) = (h_2 \oplus^+_\alpha h_1)(x)$;
2. $((h_1 \oplus^+_\alpha h_2) \oplus^+_\alpha h_3)(x) = (h_1 \oplus^+_\alpha (h_2 \oplus^+_\alpha h_3))(x)$;
3. $(h_1 \otimes^+_\alpha h_2)(x) = (h_2 \otimes^+_\alpha h_1)(x)$;
4. $((h_1 \otimes^+_\alpha h_2) \otimes^+_\alpha h_3)(x) = (h_1 \otimes^+_\alpha (h_2 \otimes^+_\alpha h_3))(x)$;

Proof. (1) and (3) is trivial. We only prove (2) and (4), in fact, $((h_1 \oplus^+_\alpha h_2) \oplus^+_\alpha h_3)(x) = ((\cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}) \oplus^+_\alpha h_3)(x) = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{(\gamma_1 + \gamma_2 - \gamma_1\gamma_2) + \gamma_3 - (\gamma_1 + \gamma_2 - \gamma_1\gamma_2)\gamma_3\} = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_1 + \gamma_2 + \gamma_3 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_3\} = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_1 + (\gamma_2 + \gamma_3 - \gamma_2\gamma_3) - \gamma_1(\gamma_2 + \gamma_3 - \gamma_2\gamma_3)\} = (h_1 \oplus^+_\alpha (\cup_{\gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_2 + \gamma_3 - \gamma_2\gamma_3\}))(x) = (h_1 \oplus^+_\alpha (h_2 \oplus^+_\alpha h_3))(x)$.

$((h_1 \otimes^+_\alpha h_2) \otimes^+_\alpha h_3)(x) = ((\cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x)} \{\gamma_1\gamma_2\}) \otimes^+_\alpha h_3)(x) = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_1\gamma_2\gamma_3\} = \cup_{\gamma_1 \in (h_1)^+_\alpha(x), \gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_1(\gamma_2\gamma_3)\} = (h_1 \otimes^+_\alpha (\cup_{\gamma_2 \in (h_2)^+_\alpha(x), \gamma_3 \in (h_3)^+_\alpha(x)} \{\gamma_2\gamma_3\}))(x) = (h_1 \otimes^+_\alpha (h_2 \otimes^+_\alpha h_3))(x)$. □

Similarly, we can prove the following corollary.

Corollary 11. *For any* h_1, h_2 *and* h_3 *in* $H(X)$, $x \in X$ *and* $\alpha \in [0, 1]$,

1. $(h_1 \oplus^-_\alpha h_2)(x) = (h_2 \oplus^-_\alpha h_1)(x)$;
2. $((h_1 \oplus^-_\alpha h_2) \oplus^-_\alpha h_3)(x) = (h_1 \oplus^-_\alpha (h_2 \oplus^-_\alpha h_3))(x)$;
3. $(h_1 \otimes^-_\alpha h_2)(x) = (h_2 \otimes^-_\alpha h_1)(x)$;
4. $((h_1 \otimes^-_\alpha h_2) \otimes^-_\alpha h_3)(x) = (h_1 \otimes^-_\alpha (h_2 \otimes^-_\alpha h_3))(x)$.

Example 7. Let three hesitant fuzzy sets for x be $h_1(x) = \{0.2, 0.3\}$, $h_2(x) = \{0.4, 0.6\}$ and $h_3(x) = \{0.5, 0.6\}$. $(h_1 \oplus h_2)(x) = \{0.52, 0.68, 0.58, 0.72\}$, $((h_1 \oplus h_2) \oplus h_3)(x) = \{0.76, 0.84, 0.79, 0.86, 0.808, 0.872, 0.832, 0.888\}$. $(h_2 \oplus h_3)(x) = \{0.7, 0.76, 0.8, 0.84\}$, $(h_1 \oplus (h_2 \oplus h_3))(x) = \{0.76, 0.808, 0.84, 0.872, 0.79, 0.832, 0.86, 0.888\}$, *i.e.*, $((h_1 \oplus h_2) \oplus h_3)(x) = (h_1 \oplus (h_2 \oplus h_3))(x)$. However, $(h_1 \oplus h_1)(x) = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} = \{0.36, 0.44, 0.51\} \neq h_1(x)$ and $(h_1 \otimes h_1)(x) = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \{\gamma_1\gamma_2\} = \{0.04, 0.06, 0.09\} \neq h_1(x)$.

Theorem 10 and Corollary 11 show that the operations $\oplus^+_\alpha$, $\otimes^+_\alpha$, $\oplus^-_\alpha$ and $\otimes^-_\alpha$ are commutative and associative. However, Example 7 shows that the operations $\oplus^+_\alpha$, $\otimes^+_\alpha$, $\oplus^-_\alpha$ and $\otimes^-_\alpha$ are not idempotent.

According to proposition 3 (3) and (4), Theorem 10 and Corollary 11, the following theorem is obvious.

Theorem 12. *For any discourse set X and $\alpha \in [0,1]$,*

1. $(H(X), \oplus^+_\alpha, h^0)$ *and* $(H(X), \oplus^-_\alpha, h^0)$ *are commutative monoid and isotone in both arguments, respectively;*

2. $(H(X), \otimes^+_\alpha, h^1)$ *and* $(H(X), \otimes^-_\alpha, h^1)$ *are commutative monoid and isotone in both arguments, respectively.*

4.3. *The lattice of hesitant fuzzy sets with confidence levels*

To obtain the lattice of hesitant fuzzy sets, we consider a special case of $h \in H(X)$. Formally, for any $h \in H(X)$, h corresponds to the special hesitant fuzzy set $\overline{h}(x) = [h^-(x), h^+(x)]$ for any $x \in X$, *i.e.*, $h(x)$ is a closed interval of $[0,1]$ for any $x \in X$. In [19,20], the special case of $h \in H(X)$ is also used to explain relationship between IFSs and HFSs, *i.e.*, an IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle\}$ corresponds to a HFS $\overline{h}(x) = [\mu_A(x), 1 - \nu_A(x)]$. Conversely, a HFS $h \in H(X)$ defines an IFS $A_{env}(f) = \{\langle x, \mu_A(x), \nu_A(x)\rangle\}$ (called as the envelope of h), where $\mu_A(x) = h^-(x)$ and $\nu_A(x) = 1 - h^+(x)$. For any $h \in H(X)$ and $\alpha \in [0,1]$, we have

$$\overline{h}^+_\alpha(x) = \begin{cases} \overline{h}(x), & 0 \leqslant \alpha \leqslant h^-(x), \\ [\alpha, h^+(x)], & h^-(x) < \alpha \leqslant h^+(x), \\ \emptyset, & h^+(x) < \alpha \leqslant 1. \end{cases}$$

$$\overline{h}^-_\alpha(x) = \begin{cases} \emptyset, & 0 \leqslant \alpha < h^-(x), \\ [h^-(x), \alpha], & h^-(x) \leqslant \alpha < h^+(x), \\ \overline{h}(x), & h^+(x) \leqslant \alpha \leqslant 1. \end{cases}$$

Definition 2. For any h_1 and h_2 in $H(X)$, $h_1 \sim h_2$ if and only if $(h_1)^-(x) = (h_2)^-(x)$ and $(h_1)^+(x) = (h_2)^+(x)$ for any $x \in X$.

It can be easily proved that the binary relation $\sim$ on $H(X)$ is reflexive, symmetric and transitive, *i.e.*, the binary relation $\sim$ is an equivalence relation on $H(X)$ and $H(X)/\sim = \{[h] | h \in H(X), \forall h_1 \in H(X), h \sim h_1 \Longrightarrow h_1 \in [h]\}$. Due to $\overline{h}^+(x) = h^+(x)$ and $\overline{h}^-(x) = h^-(x)$, we have $\overline{h} \in [h]$ for any $h \in H(X)$.

We consider all closed interval hesitant fuzzy sets on X, denoted by $CIH(X)$, *i.e.*,

$$\begin{aligned} CIH(X) &= \{\overline{h} | h \in H(X)\} \\ &= \{\langle x, [h^-(x), h^+(x)]\rangle | \forall h \in H(X), x \in X\}, \end{aligned}$$

then there exists a bijective mapping I of $H(X)/\sim$ to $CIH(X)$, *i.e.*,

$$\begin{aligned} I \;:\; & H(X)/\sim \longrightarrow CIH(X), \\ & [h] \longmapsto \overline{h}. \end{aligned}$$

Theorem 13. *For any $\overline{h}_1$ and $\overline{h}_2$ in $CIH(X)$, $x \in X$ and $\alpha \in [0,1]$,*

1. $((\overline{h}_1 \cup^+_\alpha \overline{h}_2) \cap^+_\alpha \overline{h}_1)(x) = ((\overline{h}_1 \cap^+_\alpha \overline{h}_2) \cup^+_\alpha \overline{h}_1)(x) = (\overline{h}_1)^+_\alpha(x)$*;*

2. $((\overline{h}_1 \cup^-_\alpha \overline{h}_2) \cap^-_\alpha \overline{h}_1)(x) = ((\overline{h}_1 \cap^-_\alpha \overline{h}_2) \cup^-_\alpha \overline{h}_1)(x) = (\overline{h}_1)^+_\alpha(x)$*.*

Proof. If $(\overline{h}_1)^+_\alpha(x) \cap (\overline{h}_2)^+_\alpha(x) \neq \emptyset$, then $(\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x) = \{\gamma \in (\overline{h}_1)^+_\alpha(x) \cup (\overline{h}_2)^+_\alpha(x) | \gamma \geqslant max\{((\overline{h}_1)^+_\alpha)^-(x), ((\overline{h}_2)^+_\alpha)^-(x)\}\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] | \gamma \geqslant max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}\} = [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]$.

If $(\overline{h}_1)^+_\alpha(x) \cap (\overline{h}_2)^+_\alpha(x) = \emptyset$, no loss generalization, let $((\overline{h}_1)^+_\alpha)^+(x) < ((\overline{h}_2)^+_\alpha)^-(x)$, then $(\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x) = \{\gamma \in (\overline{h}_1)^+_\alpha(x) \cup$

$(\bar{h}_2)^+_\alpha(x)|\gamma \geqslant max\{((\bar{h}_1)^+_\alpha)^-(x), ((\bar{h}_2)^+_\alpha)^-(x)\}\} = [((h_2)^+_\alpha)^-(x)\}, ((h_2)^+_\alpha)^+(x)] = [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]$.

Hence, $((\bar{h}_1 \cup^+_\alpha \bar{h}_2) \cap^+_\alpha \bar{h}_1)(x) = \{\gamma \in [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup (\bar{h}_1)^+_\alpha(x)|\gamma \leqslant min\{max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_1)^+_\alpha)^+(x)\}\} = \{\gamma \in [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)]|\gamma \leqslant ((h_1)^+_\alpha)^+(x)\} = \{\gamma \in [((h_1)^+_\alpha)^-(x), max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]| \gamma \leqslant ((h_1)^+_\alpha)^+(x)\} = [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)] = (\bar{h}_1)^+_\alpha(x)$.

If $(\bar{h}_1)^+_\alpha(x) \cap (\bar{h}_2)^+_\alpha(x) \neq \emptyset$, then $(\bar{h}_1 \cap^+_\alpha \bar{h}_2)(x) = \{\gamma \in (\bar{h}_1)^+_\alpha(x) \cup (\bar{h}_2)^+_\alpha(x)|\gamma \leqslant min\{((\bar{h}_1)^+_\alpha)^+(x), ((\bar{h}_2)^+_\alpha)^+(x)\}\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]|\gamma \leqslant min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}\} = [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]$.

If $(\bar{h}_1)^+_\alpha(x) \cap (\bar{h}_2)^+_\alpha(x) = \emptyset$, no loss generalization, let $((\bar{h}_1)^+_\alpha)^+(x) < ((\bar{h}_2)^+_\alpha)^-(x)$, then $(\bar{h}_1 \cap^+_\alpha \bar{h}_2)(x) = \{\gamma \in (\bar{h}_1)^+_\alpha(x) \cup (\bar{h}_2)^+_\alpha(x)|\gamma \leqslant min\{((\bar{h}_1)^+_\alpha)^+(x), ((\bar{h}_2)^+_\alpha)^+(x)\}\} = [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)] = [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}]$. Hence, $((\bar{h}_1 \cap^+_\alpha \bar{h}_2) \cup^+_\alpha \bar{h}_1)(x) = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup (\bar{h}_1)^+_\alpha(x)|\gamma \geqslant max\{min\{((h_1)^+_\alpha)^-(x), ((h_2)^-_\alpha)^+(x)\}, ((h_1)^+_\alpha)^-(x)\}\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)]|\gamma \geqslant ((h_1)^+_\alpha)^-(x)\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_1)^+_\alpha)^+(x)]|\gamma \geqslant ((h_1)^+_\alpha)^-(x)\} = [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)] = (\bar{h}_1)^+_\alpha(x)$.

Accordingly, $((\bar{h}_1 \cup^+_\alpha \bar{h}_2) \cap^+_\alpha \bar{h}_1)(x) = ((\bar{h}_1 \cap^+_\alpha \bar{h}_2) \cup^+_\alpha \bar{h}_1)(x) = (\bar{h}_1)^+_\alpha(x)$ is valid. Similarly, we can prove $((\bar{h}_1 \cup^-_\alpha \bar{h}_2) \cap^-_\alpha \bar{h}_1)(x) = ((\bar{h}_1 \cap^-_\alpha \bar{h}_2) \cup^-_\alpha \bar{h}_1)(x) = (\bar{h}_1)^+_\alpha(x)$. □

Theorem 14. *For any $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$ in $CIH(X)$, $x \in X$ and $\alpha \in [0,1]$,*

1. $((\bar{h}_1 \cup^+_\alpha \bar{h}_2) \cap^+_\alpha \bar{h}_3)(x) = ((\bar{h}_1 \cap^+_\alpha \bar{h}_3) \cup^+_\alpha (\bar{h}_2 \cap^+_\alpha \bar{h}_3))(x)$;
2. $((\bar{h}_1 \cap^+_\alpha \bar{h}_2) \cup^+_\alpha \bar{h}_3)(x) = ((\bar{h}_1 \cup^+_\alpha \bar{h}_3) \cap^+_\alpha (\bar{h}_2 \cup^+_\alpha \bar{h}_3))(x)$;
3. $((\bar{h}_1 \cup^-_\alpha \bar{h}_2) \cap^-_\alpha \bar{h}_3)(x) = ((\bar{h}_1 \cap^-_\alpha \bar{h}_3) \cup^-_\alpha (\bar{h}_2 \cap^-_\alpha \bar{h}_3))(x)$;
4. $((\bar{h}_1 \cap^-_\alpha \bar{h}_2) \cup^-_\alpha \bar{h}_3)(x) = ((\bar{h}_1 \cup^-_\alpha \bar{h}_3) \cap^-_\alpha (\bar{h}_2 \cup^-_\alpha \bar{h}_3))(x)$.

Proof. According to Theorem 13, we have

$((\bar{h}_1 \cup^+_\alpha \bar{h}_2) \cap^+_\alpha \bar{h}_3)(x) = \{\gamma \in [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup (\bar{h}_3)^+_\alpha(x)|\gamma \leqslant min\{max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}\} = \{\gamma \in [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup [((h_3)^+_\alpha)^-(x), ((h_3)^+_\alpha)^+(x)]|\gamma \leqslant max\{min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}\} = \{\gamma \in [min\{max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}, max\{max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}]|\gamma \leqslant max\{min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}\} = [min\{max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}]$.

$((\bar{h}_1 \cap^+_\alpha \bar{h}_3) \cup^+_\alpha (\bar{h}_2 \cap^+_\alpha \bar{h}_3))(x) = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}] \cup [min\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}]|\gamma \geqslant max\{min\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, min\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), (h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}]|\gamma \geqslant min\{max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}\} = [min\{max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, min\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}]$.

Accordingly, $((\bar{h}_1 \cup^+_\alpha \bar{h}_2) \cap^+_\alpha \bar{h}_3)(x) = ((\bar{h}_1 \cap^+_\alpha \bar{h}_3) \cup^+_\alpha (\bar{h}_2 \cap^+_\alpha \bar{h}_3))(x)$ is valid.

$((\bar{h}_1 \cap^+_\alpha \bar{h}_2) \cup^+_\alpha \bar{h}_3)(x) = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup (\bar{h}_3)^+_\alpha(x)|\gamma \geqslant max\{min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}\} = \{\gamma \in [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] \cup [((h_3)^+_\alpha)^-(x), ((h_3)^+_\alpha)^+(x)]|\gamma \geqslant min\{max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}\} = \{\gamma \in [min\{min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, ((h_3)^+_\alpha)^-(x)\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}]|\gamma \geqslant min\{max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}\} = [min\{max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}]$.

$((\bar{h}_1 \cup^+_\alpha \bar{h}_3) \cap^+_\alpha (\bar{h}_2 \cup^+_\alpha \bar{h}_3))(x) = \{\gamma \in [max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}] \cup [max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}]|\gamma \leqslant min\{max\{((h_1)^+_\alpha)^+(x),$

$((h_3)^+_\alpha)^+(x)\}, max\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}\} = \{\gamma \in [min\{max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}, max\{max\{((h_1)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}, max\{((h_2)^+_\alpha)^+(x), ((h_3)^+_\alpha)^+(x)\}\}] | \gamma \leqslant max\{min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}\} = [min\{max\{((h_1)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}, max\{((h_2)^+_\alpha)^-(x), ((h_3)^+_\alpha)^-(x)\}\}, max\{min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}, ((h_3)^+_\alpha)^+(x)\}]$.

Accordingly, $((\overline{h}_1 \cap^+_\alpha \overline{h}_2) \cup^+_\alpha \overline{h}_3)(x) = ((\overline{h}_1 \cup^+_\alpha \overline{h}_3) \cap^+_\alpha (\overline{h}_2 \cup^+_\alpha \overline{h}_3))(x)$ is valid. Similarly, we can prove $((h_1 \cup^-_\alpha h_2) \cap^-_\alpha h_3)(x) = ((h_1 \cap^-_\alpha h_3) \cup^-_\alpha (h_2 \cap^-_\alpha h_3))(x)$ and $((h_1 \cap^-_\alpha h_2) \cup^-_\alpha h_3)(x) = ((h_1 \cup^-_\alpha h_3) \cap^-_\alpha (h_2 \cup^-_\alpha h_3))(x)$. □

Theorem 13 and 14 show that operators $\cup^+_\alpha$ and $\cap^+_\alpha$ ($\cup^-_\alpha$ and $\cap^-_\alpha$)) in $CIH(X)$ are absorptive and distributive, respectively. Combining Theorem 7, Corollary 8, Theorem 13 and 14, the following theorem is valid.

Theorem 15. *For any discourse set X and* $\alpha \in [0,1]$,

1. $(CIH(X), \cup^+_\alpha, \cap^+_\alpha, h^0, h^1)$ *is a distributive lattice with* h^0 *and* h^1 *are its least and the greatest elements, respectively;*

2. $(CIH(X), \cup^-_\alpha, \cap^-_\alpha, h^0, h^1)$ *is a distributive lattice with* h^0 *and* h^1 *are its least and the greatest elements, respectively.*

According to the bijective mapping $I : H(X)/\sim \longrightarrow CIH(X)$, for any $[h_1]$ and $[h_2]$ in $H(X)/\sim$, $x \in X$, we define

1. $([h_1] \cup^+_\alpha [h_2])(x) = (\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x)$;
2. $([h_1] \cap^+_\alpha [h_2])(x) = (\overline{h}_1 \cap^+_\alpha \overline{h}_2)(x)$;
3. $([h_1] \cup^-_\alpha [h_2])(x) = (\overline{h}_1 \cup^-_\alpha \overline{h}_2)(x)$;
4. $([h_1] \cap^-_\alpha [h_2])(x) = (\overline{h}_1 \cap^-_\alpha \overline{h}_2)(x)$.

Then the following corollary is obvious.

Corollary 16. *For any discourse set X and* $\alpha \in [0,1]$,

1. $(H(X)/\sim, \cup^+_\alpha, \cap^+_\alpha, [h^0], [h^1])$ *is a distributive lattice with* $[h^0]$ *and* $[h^1]$ *are its least and the greatest elements, respectively;*

2. $(H(X)/\sim, \cup^-_\alpha, \cap^-_\alpha, [h^0], [h^1])$ *is a distributive lattice with* $[h^0]$ *and* $[h^1]$ *are its least and the greatest elements, respectively.*

4.4. *The residuated lattice of hesitant fuzzy sets with confidence levels*

According to Theorem 15, $\forall \overline{h}_1, \overline{h}_2 \in CIH(X)$, $\overline{h}_1 \leqslant^+_\alpha \overline{h}_2$ is equivalent to the conditions $(\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x) = (\overline{h}_2)^+_\alpha(x)$ or $(\overline{h}_1 \cap^+_\alpha \overline{h}_2)(x) = (\overline{h}_1)^+_\alpha(x)$ for any $x \in X$. $\overline{h}_1 \leqslant^-_\alpha \overline{h}_2$ is equivalent to the conditions $(\overline{h}_1 \cup^-_\alpha \overline{h}_2)(x) = (\overline{h}_2)^-_\alpha(x)$ or $(\overline{h}_1 \cap^-_\alpha \overline{h}_2)(x) = (\overline{h}_1)^-_\alpha(x)$ for any $x \in X$. For simplicity, $\overline{h}_1 \leqslant^+_\alpha \overline{h}_2$ and $\overline{h}_1 \leqslant^-_\alpha \overline{h}_2$ are also denoted by $\overline{h}_1(x) \leqslant^+_\alpha \overline{h}_2(x)$ and $\overline{h}_1(x) \leqslant^-_\alpha \overline{h}_2(x)$ for any $x \in X$, respectively.

Corollary 17. *For any* $\overline{h}_1$, $\overline{h}_2$ *in* $CIH(X)$ *and* $x \in X$

1. $\overline{h}_1(x) \leqslant^+_\alpha \overline{h}_2(x)$ *if and only if* $((\overline{h}_1)^+_\alpha)^-(x) \leqslant ((\overline{h}_2)^+_\alpha)^-(x)$ *and* $((\overline{h}_1)^+_\alpha)^+(x) \leqslant ((\overline{h}_2)^+_\alpha)^+(x)$;

2. $\overline{h}_1(x) \leqslant^-_\alpha \overline{h}_2(x)$ *if and only if* $((\overline{h}_1)^-_\alpha)^-(x) \leqslant ((\overline{h}_2)^-_\alpha)^-(x)$ *and* $((\overline{h}_1)^-_\alpha)^+(x) \leqslant ((\overline{h}_2)^-_\alpha)^+(x)$.

Proof. If $((\overline{h}_1)^+_\alpha)^-(x) \leqslant ((\overline{h}_2)^+_\alpha)^-(x)$ and $((\overline{h}_1)^+_\alpha)^+(x) \leqslant ((\overline{h}_2)^+_\alpha)^+(x)$ for any $x \in X$, then $(\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x) = [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] = [((h_2)^+_\alpha)^-(x), ((h_2)^+_\alpha)^+(x)] = (\overline{h}_2)^+_\alpha(x)$, $(\overline{h}_1 \cap^+_\alpha \overline{h}_2)(x) = [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] = [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)] = (\overline{h}_1)^+_\alpha(x)$. Hence, $\overline{h}_1(x) \leqslant^+_\alpha \overline{h}_2(x)$.

If $\overline{h}_1(x) \leqslant^+_\alpha \overline{h}_2(x)$, then for any $x \in X$, $(\overline{h}_1 \cup^+_\alpha \overline{h}_2)(x) = [max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] = (\overline{h}_2)^+_\alpha(x) = [((h_2)^+_\alpha)^-(x), ((h_2)^+_\alpha)^+(x)]$, *i.e.*, $max\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\} = ((h_2)^+_\alpha)^-(x)$ and $max\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\} = ((h_2)^+_\alpha)^+(x)$, we have $((\overline{h}_1)^+_\alpha)^-(x) \leqslant ((\overline{h}_2)^+_\alpha)^-(x)$ and $((\overline{h}_1)^+_\alpha)^+(x) \leqslant ((\overline{h}_2)^+_\alpha)^+(x)$.

$(\overline{h}_1 \cap^+_\alpha \overline{h}_2)(x) = [min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\}, min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\}] = (\overline{h}_1)^+_\alpha(x) = [((h_1)^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)]$, i.e., $min\{((h_1)^+_\alpha)^-(x), ((h_2)^+_\alpha)^-(x)\} = ((h_1)^+_\alpha)^-(x)$ and $min\{((h_1)^+_\alpha)^+(x), ((h_2)^+_\alpha)^+(x)\} = ((h_1)^+_\alpha)^+(x)$, we have $((\overline{h}_1)^+_\alpha)^-(x) \leqslant ((\overline{h}_2)^+_\alpha)^-(x)$ and $((\overline{h}_1)^+_\alpha)^+(x) \leqslant ((\overline{h}_2)^+_\alpha)^+(x)$.

Accordingly, 1) is valid. Similarly, we can prove 2). □

Based on operations $\cap^+_\alpha$ ($\cap^-_\alpha$) and $\otimes^+_\alpha$ ($\otimes^-_\alpha$) on $CIH(X)$, we can define the following residual implications on $CIH(X)$: $\forall \overline{h}_1, \overline{h}_2 \in CIH(X)$ and $\forall x \in X$,

$(\overline{h}_1 \to^+_{\alpha,\cap} \overline{h}_2)(x) = \cup^+_\alpha\{\overline{h}(x)|(\overline{h}_1 \cap^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)\}$,

$(\overline{h}_1 \to^-_{\alpha,\cap} \overline{h}_2)(x) = \cup^-_\alpha\{\overline{h}(x)|(\overline{h}_1 \cap^-_\alpha \overline{h})(x) \leqslant^-_\alpha \overline{h}_2(x)\}$,

$(\overline{h}_1 \to^+_{\alpha,\otimes} \overline{h}_2)(x) = \cup^+_\alpha\{\overline{h}(x)|(\overline{h}_1 \otimes^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)\}$,

$(\overline{h}_1 \to^-_{\alpha,\otimes} \overline{h}_2)(x) = \cup^-_\alpha\{\overline{h}(x)|(\overline{h}_1 \otimes^-_\alpha \overline{h})(x) \leqslant^-_\alpha \overline{h}_2(x)\}$.

Due to $(\overline{h}_1 \cap^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)$ if and only if $min\{((h_1)^+_\alpha)^-(x), (h^+_\alpha)^-(x)\} \leqslant ((h_2)^+_\alpha)^-(x)$ and $min\{((h_1)^+_\alpha)^+(x), (h^+_\alpha)^+(x)\} \leqslant ((h_2)^+_\alpha)^+(x)$, we have

1) if $((h_1)^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)$, then $max\{(h^+_\alpha)^-(x)|min\{((h_1)^+_\alpha)^-(x), (h^+_\alpha)^-(x)\} \leqslant ((h_2)^+_\alpha)^-(x)\} = 1$;

2) if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)$, then $max\{(h^+_\alpha)^-(x)|min\{((h_1)^+_\alpha)^-(x), (h^+_\alpha)^-(x)\} \leqslant ((h_2)^+_\alpha)^-(x)\} = ((h_2)^+_\alpha)^-(x)$ and $max\{(h^+_\alpha)^+(x)|\min\{((h_1)^+_\alpha)^+(x), (h^+_\alpha)^+(x)\} \leqslant ((h_2)^+_\alpha)^+(x)\} = 1$;

3) if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) > ((h_2)^+_\alpha)^+(x)$, then $max\{(h^+_\alpha)^-(x)|min\{((h_1)^+_\alpha)^-(x), (h^+_\alpha)^-(x)\} \leqslant ((h_2)^+_\alpha)^-(x)\} = ((h_2)^+_\alpha)^-(x)$ and $max\{(h^+_\alpha)^+(x)|min\{((h_1)^+_\alpha)^+(x), (h^+_\alpha)^+(x)\} \leqslant ((h_2)^+_\alpha)^+(x)\} = ((h_2)^+_\alpha)^+(x)$.

Accordingly, the residual implication $\to^+_{\alpha,\cap}$ on $CIH(X)$ is rewritten by

$$(\overline{h}_1 \to^+_{\alpha,\cap} \overline{h}_2)(x) = \cup^+_\alpha\{\overline{h}(x)|(\overline{h}_1 \cap^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)\},$$

in which, $(\overline{h}_1 \to^+_{\alpha,\cap} \overline{h}_2)(x) = h^1$ if $((h_1)^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)$. $(\overline{h}_1 \to^+_{\alpha,\cap} \overline{h}_2)(x) = [((h_2)^+_\alpha)^-(x), 1]$ if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)$.

$(\overline{h}_1 \to^+_{\alpha,\cap} \overline{h}_2)(x) = [((h_2)^+_\alpha)^-(x), ((h_2)^+_\alpha)^+(x)]$ if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) > ((h_2)^+_\alpha)^+(x)$.

Similarly, the residual implication $\to^-_{\alpha,\cap}$ on $CIH(X)$ is rewritten by

$$(\overline{h}_1 \to^-_{\alpha,\cap} \overline{h}_2)(x) = \cup^-_\alpha\{\overline{h}(x)|(\overline{h}_1 \cap^-_\alpha \overline{h})(x) \leqslant^-_\alpha \overline{h}_2(x)\},$$

in which, $(\overline{h}_1 \to^-_{\alpha,\cap} \overline{h}_2)(x) = h^1$ if $((h_1)^-_\alpha)^-(x) \leqslant ((h_2)^-_\alpha)^-(x)$. $(\overline{h}_1 \to^-_{\alpha,\cap} \overline{h}_2)(x) = [((h_2)^-_\alpha)^-(x), 1]$ if $((h_1)^-_\alpha)^-(x) > ((h_2)^-_\alpha)^-(x)$ and $((h_1)^-_\alpha)^+(x) \leqslant ((h_2)^-_\alpha)^+(x)$.

$(\overline{h}_1 \to^-_{\alpha,\cap} \overline{h}_2)(x) = [((h_2)^-_\alpha)^-(x), ((h_2)^-_\alpha)^+(x)]$ if $((h_1)^-_\alpha)^-(x) > (h_2)^-_\alpha)^-(x)$ and $((h_1)^-_\alpha)^+(x) > ((h_2)^-_\alpha)^+(x)$

By $(\overline{h}_1 \otimes^+_\alpha \overline{h})(x) = \cup_{\gamma_1 \in (\overline{h}_1)^+_\alpha(x), \gamma_2 \in \overline{h}^+_\alpha(x)}\{\gamma_1\gamma_2\} = [((h_1)^+_\alpha)^-(x)(h^+_\alpha)^-(x), ((h_1)^+_\alpha)^+(x)(h^+_\alpha)^+(x)]$, $(\overline{h}_1 \otimes^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)$ if and only if $((h_1)^+_\alpha)^-(x)(h^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x)(h^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)$, we have

1) if $((h_1)^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)$, then $max\{(h^+_\alpha)^-(x)|((h_1)^+_\alpha)^-(x)(h^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)\} = 1$;

2) if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)$, then $max\{(h^+_\alpha)^-(x)|((h_1)^+_\alpha)^-(x)(h^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)\} = \frac{((h_2)^+_\alpha)^-(x)}{((h_1)^+_\alpha)^-(x)}$, $max\{(h^+_\alpha)^+(x)|((h_1)^+_\alpha)^+(x)(h^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)\} = 1$;

3) if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) > ((h_2)^+_\alpha)^+(x)$, then $max\{(h^+_\alpha)^-(x)|((h_1)^+_\alpha)^-(x)(h^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)\} = \frac{((h_2)^+_\alpha)^-(x)}{((h_1)^+_\alpha)^-(x)}$, $max\{(h^+_\alpha)^+(x)|((h_1)^+_\alpha)^+(x)(h^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)\} = \frac{((h_2)^+_\alpha)^+(x)}{((h_1)^+_\alpha)^+(x)}$.

Accordingly, the residual implication $\to^+_{\alpha,\otimes}$ on $CIH(X)$ is rewritten by

$$(\overline{h}_1 \to^+_{\alpha,\otimes} \overline{h}_2)(x) = \cup^+_\alpha\{\overline{h}(x)|(\overline{h}_1 \otimes^+_\alpha \overline{h})(x) \leqslant^+_\alpha \overline{h}_2(x)\},$$

in which, $(\overline{h}_1 \to^+_{\alpha,\otimes} \overline{h}_2)(x) = h^1$ if $((h_1)^+_\alpha)^-(x) \leqslant ((h_2)^+_\alpha)^-(x)$. $(\overline{h}_1 \to^+_{\alpha,\otimes} \overline{h}_2)(x) = [\frac{((h_2)^+_\alpha)^-(x)}{((h_1)^+_\alpha)^-(x)}, 1]$ if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) \leqslant ((h_2)^+_\alpha)^+(x)$. $(\overline{h}_1 \to^+_{\alpha,\otimes} \overline{h}_2)(x) = [a, b]$ if $((h_1)^+_\alpha)^-(x) > ((h_2)^+_\alpha)^-(x)$ and $((h_1)^+_\alpha)^+(x) > ((h_2)^+_\alpha)^+(x)$, where $a = min\{\frac{((\overline{h}_2)^+_\alpha)^-(x)}{((\overline{h}_1)^+_\alpha)^-(x)}, \frac{((\overline{h}_2)^+_\alpha)^+(x)}{((\overline{h}_1)^+_\alpha)^+(x)}\}$ and $b = max\{\frac{((\overline{h}_2)^+_\alpha)^-(x)}{((\overline{h}_1)^+_\alpha)^-(x)}, \frac{((\overline{h}_2)^+_\alpha)^+(x)}{((\overline{h}_1)^+_\alpha)^+(x)}\}$.

Similarly, the residual implication $\to^-_{\alpha,\otimes}$ on $CIH(X)$ is rewritten by

$$(\overline{h}_1 \to^-_{\alpha,\otimes} \overline{h}_2)(x) = \cup^-_\alpha\{\overline{h}(x)|(\overline{h}_1 \otimes^-_\alpha \overline{h})(x) \leqslant^-_\alpha \overline{h}_2(x)\},$$

in which, $(\overline{h}_1 \to^-_{\alpha,\otimes} \overline{h}_2)(x) = h^1$ if $((h_1)^-_\alpha)^-(x) \leqslant ((h_2)^-_\alpha)^-(x)$. $(\overline{h}_1 \to^-_{\alpha,\otimes} \overline{h}_2)(x) = [\frac{((\overline{h}_2)^-_\alpha)^-(x)}{((\overline{h}_1)^-_\alpha)^-(x)}, 1]$ if $((h_1)^-_\alpha)^-(x) > ((h_2)^-_\alpha)^-(x)$ and $((h_1)^-_\alpha)^+(x) \leqslant ((h_2)^-_\alpha)^+(x)$. $(\overline{h}_1 \to^-_{\alpha,\otimes} \overline{h}_2)(x) = [c, d]$ if $((h_1)^-_\alpha)^-(x) > ((h_2)^-_\alpha)^-(x)$ and $((h_1)^-_\alpha)^+(x) >$

$((h_2)^-_\alpha)^+(x)$, where $c = min\{\frac{((\overline{h}_2)^-_\alpha)^-(x)}{((\overline{h}_1)^-_\alpha)^-(x)}, \frac{((\overline{h}_2)^-_\alpha)^+(x)}{((\overline{h}_1)^-_\alpha)^+(x)}\}$ and $d = max\{\frac{((\overline{h}_2)^-_\alpha)^-(x)}{((\overline{h}_1)^-_\alpha)^-(x)}, \frac{((\overline{h}_2)^-_\alpha)^+(x)}{((\overline{h}_1)^-_\alpha)^+(x)}\}$.

If $\alpha \leqslant min\{(\overline{h}_1)^-(x), (\overline{h}_2)^-(x)\}$ ($\alpha \geqslant max\{(\overline{h}_1)^+(x), (\overline{h}_2)^+(x)\}$), then $(\overline{h}_1)^+_\alpha(x)$ and $(\overline{h}_2)^+_\alpha(x)$ ($(\overline{h}_1)^-_\alpha(x)$ and $(\overline{h}_2)^-_\alpha(x)$) are reduced to $\overline{h}_1(x)$ and $\overline{h}_2(x)$. Hence, $\rightarrow^+_{\alpha,\cap}$, $\rightarrow^-_{\alpha,\cap}$, $\rightarrow^+_{\alpha,\otimes}$, $\rightarrow^-_{\alpha,\otimes}$, $\leqslant^+_\alpha$ and $\leqslant^-_\alpha$ are reduced to $\rightarrow_\cap$, $\rightarrow_\otimes$ and $\leqslant$.

Example 8. Let three hesitant fuzzy sets for x be $\overline{h}_1(x) = [0.2, 0.5]$, $\overline{h}_2(x) = [0.4, 0.6]$ and $\overline{h}_3(x) = [0.3, 0.7]$. Then, $(\overline{h}_1 \rightarrow_\cap \overline{h}_2)(x) = h^1 = \{\langle x, \{1\}\rangle | x \in X\}$, $(\overline{h}_2 \rightarrow_\cap \overline{h}_3)(x) = [0.3, 1]$, $(\overline{h}_2 \rightarrow_\cap \overline{h}_1)(x) = \overline{h}_1(x) = [0.2, 0.5]$, $(\overline{h}_2 \rightarrow_\otimes \overline{h}_3)(x) = [\frac{3}{4}, 1]$ and $(\overline{h}_2 \rightarrow_\otimes \overline{h}_1)(x) = [min\{\frac{0.2}{0.4}, \frac{0.5}{0.6}\}, max\{\frac{0.2}{0.4}, \frac{0.5}{0.6}\}] = [0.5, \frac{5}{6}]$.

Based on Theorem 12, Theorem 15 and the above mentioned analysis about residual implications on $CIH(X)$, we have the following residuated lattice of $CIH(X)$.

Theorem 18. *For any discourse set X and $\alpha \in [0,1]$,*

1. $(CIH(X), \cup^+_\alpha, \cap^+_\alpha, \rightarrow^+_{\alpha,\cap}, h^0, h^1)$ *is a residuated lattice;*
2. $(CIH(X), \cup^-_\alpha, \cap^-_\alpha, \rightarrow^-_{\alpha,\cap}, h^0, h^1)$ *is a residuated lattice;*
3. $(CIH(X), \cup^+_\alpha, \cap^+_\alpha, \otimes^+_\alpha, \rightarrow^+_{\alpha,\otimes}, h^0, h^1)$ *is a residuated lattice;*
4. $(CIH(X), \cup^-_\alpha, \cap^-_\alpha, \otimes^-_\alpha, \rightarrow^-_{\alpha,\otimes}, h^0, h^1)$ *is a residuated lattice.*

According to the bijective mapping $I : H(X)/\sim \longrightarrow CIH(X)$, for any $[h_1]$ and $[h_2]$ in $H(X)/\sim$, $x \in X$, we define

1. $([h_1] \oplus^+_\alpha [h_2])(x) = \cup_{\gamma_1 \in (\overline{h}_1)^+_\alpha(x), \gamma_2 \in (\overline{h}_2)^+_\alpha(x)}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
2. $([h_1] \oplus^-_\alpha [h_2])(x) = \cup_{\gamma_1 \in (\overline{h}_1)^-_\alpha(x), \gamma_2 \in (\overline{h}_2)^-_\alpha(x)}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
3. $([h_1] \otimes^+_\alpha [h_2])(x) = \cup_{\gamma_1 \in (\overline{h}_1)^+_\alpha(x), \gamma_2 \in (\overline{h}_2)^+_\alpha(x)}\{\gamma_1\gamma_2\}$;
4. $([h_1] \otimes^-_\alpha [h_2])(x) = \cup_{\gamma_1 \in (\overline{h}_1)^-_\alpha(x), \gamma_2 \in (\overline{h}_2)^-_\alpha(x)}\{\gamma_1\gamma_2\}$;
5. $([h_1] \rightarrow^+_{\alpha,\cap} [h_2])(x) = (\overline{h}_1 \rightarrow^+_{\alpha,\cap} \overline{h}_2)(x)$;
6. $([h_1] \rightarrow^-_{\alpha,\cap} [h_2])(x) = (\overline{h}_1 \rightarrow^-_{\alpha,\cap} \overline{h}_2)(x)$;
7. $([h_1] \rightarrow^+_{\alpha,\otimes} [h_2])(x) = (\overline{h}_1 \rightarrow^+_{\alpha,\otimes} \overline{h}_2)(x)$;
8. $([h_1] \rightarrow^-_{\alpha,\otimes} [h_2])(x) = (\overline{h}_1 \rightarrow^-_{\alpha,\otimes} \overline{h}_2)(x)$.

Then the following corollaries can be easily proved.

Corollary 19. *For any discourse set X and $\alpha \in [0,1]$,*

1. $(H(X)/\sim, \oplus^+_\alpha, h^0)$ *and* $(H(X)/\sim, \oplus^-_\alpha, [h^0])$ *are commutative monoid and isotone in both arguments, respectively;*
2. $(H(X)/\sim, \otimes^+_\alpha, h^1)$ *and* $(H(X)/\sim, \otimes^-_\alpha, [h^1])$ *are commutative monoid and isotone in both arguments, respectively.*

Corollary 20. *For any discourse set X and $\alpha \in [0,1]$,*

1. $(H(X)/\sim, \cup^+_\alpha, \cap^+_\alpha, \rightarrow^+_{\alpha,\cap}, [h^0], [h^1])$ *is a residuated lattice;*
2. $(H(X)/\sim, \cup^-_\alpha, \cap^-_\alpha, \rightarrow^-_{\alpha,\cap}, [h^0], [h^1])$ *is a residuated lattice;*
3. $(H(X)/\sim, \cup^+_\alpha, \cap^+_\alpha, \otimes^+_\alpha, \rightarrow^+_{\alpha,\otimes}, [h^0], [h^1])$ *is a residuated lattice;*
4. $(H(X)/\sim, \cup^-_\alpha, \cap^-_\alpha, \otimes^-_\alpha, \rightarrow^-_{\alpha,\otimes}, [h^0], [h^1])$ *is a residuated lattice.*

5. Conclusions

In this paper, properties of operators and algebraic structure of hesitant fuzzy sets with confidence levels are investigated, four kinds of semilattices on hesitant fuzzy sets are provided by operators $\cup^+_\alpha$, $\cup^-_\alpha$, $\cap^+_\alpha$ and $\cap^-_\alpha$, respectively. Four kinds of commutative monoid on hesitant fuzzy sets are provided by operators $\oplus^+_\alpha$, $\otimes^+_\alpha$, $\oplus^-_\alpha$ and $\otimes^-_\alpha$, respectively. Based on closed interval hesitant fuzzy sets, an equivalence relation on hesitant fuzzy sets is defined, then lattices and distributive lattices on hesitant fuzzy sets are constructed. Based on distributive lattices on hesitant fuzzy sets, residuated lattices of hesitant fuzzy sets are constructed by residual implications $\rightarrow^+_{\alpha,\cap}$, $\rightarrow^-_{\alpha,\cap}$, $\rightarrow^+_{\alpha,\otimes}$ and $\rightarrow^-_{\alpha,\otimes}$, which is induced by intersection or $\otimes$ with $\alpha-$confidence level, respectively. Results of the paper may be useful for hesitant information processing, such as approximate reasoning and decision making in hesitant situation.

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