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On inference rules in decision formal contexts

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Abstract

Rule acquisition is one of the main purposes in the analysis of decision formal contexts. In general, the number of implications in a decision formal context is an exponential increase to the scale of the database. So, it is important to introduce effective inference rules between implications for eliminating as many superfluous implications as possible. This study puts forward a criterion called 'strongness' to assess the effectiveness of inference rules in terms of eliminating superfluous implications. We define a new inference rule in decision formal contexts and prove that the proposed inference rule is stronger than the existing one. Furthermore, we figure out the exact number of the superfluous implications that we can additionally remove by using the proposed inference rule compared with the existing one.

Keywords: Formal concept analysis, concept lattice, formal context, decision formal context, inference rule, rule acquisition.

1. Introduction

The theory of *formal concept analysis* (FCA), originally proposed by Wille [1] in 1982, is oriented towards the discovery of *formal concepts* and their hierarchical structure at its early stage of development, but nowadays it has shown a trend of multidisciplinary intersection and fusion [2,3,4,5,6,7]. This theory is based on mathematical *order* theory, in particular the theory of *complete lattice*, and it has been considered to be an effective tool for conceptual data analysis and knowledge processing [8].

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In FCA, the basic notions are those of a *formal context* and a formal concept. A formal context is a triple (U,A,I) which consists of the object set U, the attribute set A and the incidence relation $I \subseteq U \times A$ indicating that each object of U has what attributes in A. A formal concept is an ordered pair (X,B) in which X, called the *extent*, is the set of objects having all the attributes in B and B, called the *intent*, is the set of attributes common to all the objects in X. The extent and intent of a formal concept uniquely determine each other. Wille [1] introduced a hierarchical order for formal concepts of a formal context and proved that the set of all formal concepts together with the hierarchical order forms a complete lattice, called the concept lattice of the formal context. In FCA, concept lattices are the basic tool used to analyze relational databases and have been applied to a variety of areas such as machine learning [9,10,11,12,13,14], knowledge discovery [15,16,17,18,19], software engineering [20] and other aspects [21,22,23].

In FCA, a useful way of characterizing dependencies between attributes of a formal context is via *implications* [24] or *association rules* [25]. These rules have drawn much attention in recent years. For example, generating all implications or association rules of a formal context was investigated in Refs. [25,26,27] and how to eliminate as many superfluous implications (association rules) as possible from all the implications (association rules) was discussed in Refs. [28,29,30,31]. Additionally, an approach of mining all fuzzy implications of a fuzzy formal context was developed in Ref. [32].

In the real world, a formal context often contains target attributes for the purpose of learning knowledge or making decision analysis [33,34]. Such a formal context (U,A,I) with target attributes $d_1,\cdots,d_k$ (none of them belongs to A) is called a *decision formal context* [34] or a *training context* [35] often denoted by (U,A,I,D,J), [i] where $D=\{d_1,\cdots,d_k\}$ and $J \subseteq U \times D$. Rule acquisition is one of the main purposes in the analysis of decision formal contexts. To the best of our knowledge, some studies (e.g., Refs. [34,36,37,38,39,40,41,42,43,44,45]) have been devoted to the rule acquisition in decision formal contexts. For instance, Zhang and Qiu [34] introduced the notion of a *decision rule* in decision formal contexts; furthermore, Li et al. [38] developed an algorithm to extract all non-redundant decision rules from a decision formal context using both the conditional concept lattice and decision concept lattice and this algorithm has been improved in Ref. [36] by using the conditional concept lattice only. Qu et al. [40] extended the notion of an implication defined in a formal context into the case of decision formal contexts and explored an approach to mine all non-redundant *decision implications*. Wu et al. [45] proposed the notion of a *granular rule* in decision formal contexts by means of the granular structures of concept lattices. Based on weakly closed label concept lattices, Li et al. [46] presented the notion of a *limitary decision implication* in decision formal contexts. It should be pointed out that decision rules, decision implications, granular rules and limitary decision implications are special kinds of implications. Totally speaking, the existing study on the rule acquisition in decision formal contexts mainly focuses on the issues of defining all kinds of special implications for meeting different requirements of analyzing a decision formal context and developing efficient approaches to extract the pre-defined non-redundant special implications. Note that the number of implications or even the non-redundant ones in a decision formal context is generally an exponential increase to the scale of the database. So, it is quite important to propose effective inference rules between implications of a decision formal context for eliminating as many superfluous implications as possible.

However, there has been little work concerning inference rules in decision formal contexts. Such a discussion is quite desirable because different inference rules may lead to different numbers of implications that can be removed from the whole implication set of a decision formal context. Motivated by this problem, in this paper we put forward a criterion called 'strongness' to assess the effectiveness of inference rules in terms of eliminating superfluous implications, and show that a stronger inference rule can eliminate more superfluous implications. Furthermore, we define a new inference rule in decision formal contexts which is stronger than the existing

[i] A decision formal context was sometimes rewritten as a formal decision context by some researchers.

one, and figure out the exact number of the superfluous implications that we can additionally remove by using the proposed inference rule.

The rest of the paper is organized as follows. Section 2 reviews some basic notions related to FCA. Section 3 proposes a criterion called 'strongness' to assess the effectiveness of inference rules in decision formal contexts, and proves that a stronger inference rule can eliminate more superfluous implications from the whole implication set of a decision formal context. Section 4 puts forward a new inference rule which is stronger than the existing one, and figures out the exact number of the superfluous implications that we can additionally remove by using the proposed inference rule. Section 5 gives an illustrative example. The paper is then concluded with a brief summary in Section 6.

2. Preliminaries

In this section, we briefly recall some basic notions related to FCA in order to make the paper self-contained.

Definition 1. [1] A formal context is a triple (U,A,I), where U is an object set called the universe of discourse, A is an attribute set, and $I \subseteq U \times A$ is a binary relation (also called an incidence relation) in which $(x,a) \in I$ represents that the object x has the attribute a and $(x,a) \notin I$ means the opposite.

Note that a formal context, a special information system with two-valued input data [47], can easily be represented by a two-dimensional table in which the rows are headed by the object names and the columns are headed by the attribute names.

Definition 2. [1] Let (U,A,I) be a formal context. For $X \subseteq U$ and $B \subseteq A$, two derivation operators are defined as follows:

$$\begin{aligned} X^{\uparrow} &= \{a \in A \mid \forall x \in X, (x,a) \in I\}, \\ B^{\downarrow} &= \{x \in U \mid \forall a \in B, (x,a) \in I\}. \end{aligned} \tag{1}$$

A pair (X,B) is called a formal concept (or simply a concept) of (U,A,I) if $X^{\uparrow} = B$ and $B^{\downarrow} = X$. In this case, X and B are called the extent and intent of (X,B), respectively.

By Definition 2, we know that a formal concept (X,B) has two constituent parts: the extent X containing exactly those objects shared by all the attributes in B, and the intent B containing exactly those attributes common to all the objects in X. The extent and intent of a formal concept uniquely determine each other, thereby allowing the latter to characterize the former completely.

For two formal concepts (X_1,B_1) and (X_2,B_2), if $X_1 \subseteq X_2$ or $B_2 \subseteq B_1$, then (X_1,B_1) is called a subconcept of (X_2,B_2) or equivalently (X_2,B_2) is called a superconcept of (X_1,B_1), denoted by $(X_1,B_1) \prec (X_2,B_2)$. The relation $\prec$ is called the hierarchical order of formal concepts. All formal concepts of a formal context (U,A,I) together with the hierarchical order $\prec$ form a complete lattice, called the concept lattice of (U,A,I), and we denote it by $\underline{\mathfrak{B}}(U,A,I)$. The *infimum* and *supremum* of two formal concepts (X_1,B_1) and (X_2,B_2) in $\underline{\mathfrak{B}}(U,A,I)$ are given by:

$$\begin{aligned} (X_1,B_1) \wedge (X_2,B_2) &= (X_1 \cap X_2, (B_1 \cup B_2)^{\downarrow\uparrow}), \\ (X_1,B_1) \vee (X_2,B_2) &= ((X_1 \cup X_2)^{\uparrow\downarrow}, B_1 \cap B_2). \end{aligned} \tag{2}$$

Proposition 1. [8] *Let (U,A,I) be a formal context. For $X, X_1, X_2 \subseteq U$ and $B, B_1, B_2 \subseteq A$, the following properties hold:*

(i) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}$;
(ii) $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$;
(iii) $X \subseteq X^{\uparrow\downarrow}$ and $B \subseteq B^{\downarrow\uparrow}$;
(iv) $X^{\uparrow} = X^{\uparrow\downarrow\uparrow}$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$;
(v) $(X^{\uparrow\downarrow}, X^{\uparrow}), (B^{\downarrow}, B^{\downarrow\uparrow}) \in \underline{\mathfrak{B}}(U,A,I)$.

It follows from (i), (ii), (iii) of Proposition 1 that the pair $(\uparrow,\downarrow)$ forms a *Galois connection* [8] between the partially ordered sets $(2^U,\subseteq)$ and $(2^A,\subseteq)$, where 2^U, 2^A are the power sets of U and A, respectively.

Definition 3. [25] Let (U,A,I) be a formal context. Then $\varphi : 2^A \to 2^A$ is called a Galois closure operator, where

$$\varphi(B) = B^{\downarrow\uparrow} \text{ for any } B \in 2^A. \tag{3}$$

If $\varphi(B) = B$, then B is called a closed set.

Obviously, for every formal concept (X,B) of a formal context (U,A,I), the intent B must be a closed set.

Definition 4. [48] Let (U,A,I) be a formal context, $(X,B) \in \underline{\mathfrak{B}}(U,A,I)$ and $E \subseteq B$. If $\varphi(E) = B$ and

$\varphi(F) \subset \varphi(E)$ for all $F \subset E$, then E is called a minimal generator of (X,B). We denote by $\mathrm{MG}(X,B)$ the set of all minimal generators of (X,B).

Note that a minimal generator of a formal concept (X,B) of a formal context (U,A,I) is the minimal combination of attributes to retrieve (X,B), while the intent B is the maximal combination of attributes to retrieve (X,B).

In FCA, dependencies between the attributes of a formal context (U,A,I) are described by means of implications. An implication between the attributes in A is a pair of subsets $B,C \subseteq A$, often denoted by $B \to C$ [26]. In this case, B and C are called the premise and conclusion of the implication $B \to C$, respectively.

Definition 5. [26] Let (U,A,I) be a formal context and $B,C \subseteq A$. $B \to C$ is said to be true (or valid) in (U,A,I) if each object $x \in U$ having all the attributes of B also has all the attributes of C. A true implication $B \to C$ is called an implication of the formal context (U,A,I).

In general, the number of implications of a formal context (U,A,I) is an exponential increase to the scale of the formal context. So, it is important to introduce some effective inference rules between implications for eliminating more superfluous implications. The following inference rule is commonly used to discuss the redundancy of implications:

$$\frac{B_1 \subseteq B_2 \subseteq A, C_2 \subseteq C_1 \subseteq A, B_1 \to C_1}{B_2 \to C_2}, \tag{4}$$

which means that $B_2 \to C_2$ must be an implication of (U,A,I) if $B_1 \subseteq B_2 \subseteq A$, $C_2 \subseteq C_1 \subseteq A$ and $B_1 \to C_1$ is an implication of (U,A,I).

Remark 1. To prove whether

$$\frac{\text{'known conditions'}, B_1 \to C_1}{B_2 \to C_2}, \tag{5}$$

is an inference rule between implications of a formal context (U,A,I), it is sufficient to show that $B_2 \to C_2$ must be an implication of (U,A,I) if 'known conditions' are satisfied and $B_1 \to C_1$ is an implication of (U,A,I).

Except the inference rule introduced in Eq. (4), there have existed some other inference rules defined between implications of a formal context such as Armstrong rules [26]. More details can be found in Refs. [49,50]. Note that different inference rules may lead to different numbers of implications that can be removed from the implication set under consideration.

3. A criterion to assess the effectiveness of inference rules in decision formal contexts

Similar to the case in formal contexts, it is also important to propose effective inference rules between implications of a decision formal context for eliminating as many superfluous implications as possible. However, up to now, only the inference rule introduced in Eq. (4) has successfully been used in decision formal contexts. This hinders to some extent the in-depth study of the rule acquisition in decision formal contexts. In preparation for presenting a more effective inference rule in next section, we shall put forward a criterion called 'strongness' to assess the effectiveness of inference rules in terms of eliminating superfluous implications of a decision formal context. Before embarking on the strongness criterion, we formally introduce the notion of a decision formal context including several types of special implications.

Definition 6. [34,35] A decision formal context (or a training context) is a quintuple (U,A,I,D,J), where (U,A,I) and (U,D,J) with $A \cap D = \emptyset$ are two formal contexts, and A and D are called the conditional attribute set and the decision attribute set, respectively.

As usual, we assume in this paper that the decision formal contexts to be discussed are all finite. Moreover, in order to avoid confusion, the derivation operators $\uparrow$ and $\downarrow$ defined in Eq. (1) are expressed by different notations when they appear in different formal contexts of a decision formal context $\mathbb{K} = (U,A,I,D,J)$. Concretely, in the context (U,A,I), the notations $\uparrow$ and $\downarrow$ are reserved in their current forms, while in the context (U,D,J), they are changed as $\uparrow^{\sim}$ and $\downarrow^{\sim}$, respectively.

Example 1. Table 1 shows a decision formal context $\mathbb{K} = (U,A,I,D,J)$, where $U = \{1,2,3,4,5\}$, $A = \{a,b,c,d,e,f\}$ and $D = \{d_1,d_2\}$. In the table, each number 1 in the position (i,j) means that the i-th object has the j-th attribute, and each number 0 in the position (i,j) means the opposite.

Table 1. A decision formal context $\mathbb{K} = (U,A,I,D,J)$

U	a	b	c	d	e	f	d_1	d_2
1	1	0	0	0	0	0	1	0
2	0	1	0	1	0	0	1	0
3	1	0	1	0	0	1	1	0
4	0	1	0	0	1	0	1	0
5	1	1	1	0	1	0	0	1

Note that a decision formal context $\mathbb{K} = (U,A,I,D,J)$ can still be viewed as a formal context if one does not want to explicitly distinguish the conditional attributes from the decision attributes.

Definition 7. [40] Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context. A true implication $B \to C$ with $B \subseteq A$ and $C \subseteq D$ in $\mathbb{K}$ is called a decision implication (or simply an implication) of $\mathbb{K}$.

Thus, a decision implication is a special implication obtained by taking conditional attributes as the premise B and decision attributes as the conclusion C. It should be pointed out that except decision implications, other special implications were also introduced into decision formal contexts such as decision rules [34], granular rules [45] and limitary decision implications [46]. However, without loss of generality, in this paper we only take decision implications as a representative to elaborate the strongness criterion for assessing the effectiveness of inference rules in decision formal contexts since it is similar to other special implications.

By Definition 7, we have the following proposition.

Proposition 2. *Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context, $B \subseteq A$ and $C \subseteq D$. Then $B \to C$ is a decision implication of $\mathbb{K}$ if and only if $B^{\downarrow} \subseteq C^{\downarrow^{\sim}}$.*

Up to now, to the best of our knowledge, only the inference rule introduced in Eq. (4) has successfully been extended into decision formal contexts and it is of the following form:

$$\Pi^* : \quad \frac{B_1 \subseteq B_2 \subseteq A, C_2 \subseteq C_1 \subseteq D, B_1 \to C_1}{B_2 \to C_2}. \tag{6}$$

However, this inference rule is short of effectiveness in terms of eliminating superfluous decision implications (an illustrative example will be provided in Section 5). In preparation for presenting a more effective inference rule in next section, we give below the criterion called ‘strongness’ to assess the effectiveness of inference rules.

Let Π be an inference rule of a decision formal context $\mathbb{K} = (U,A,I,D,J)$. We denote it by $(B_1 \to C_1)\Pi(B_2 \to C_2)$ that the decision implication $B_2 \to C_2$ can be inferred by $B_1 \to C_1$ via the inference rule Π.

Definition 8. Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and Π_1,Π_2 be two inference rules defined between decision implications of $\mathbb{K}$. If $(B_1 \to C_1)\Pi_1(B_2 \to C_2) \Rightarrow (B_1 \to C_1)\Pi_2(B_2 \to C_2)$ for any two decision implications $B_1 \to C_1, B_2 \to C_2$ of $\mathbb{K}$, then Π_2 is said to be stronger than Π_1 and we denote it by $\Pi_1 \leqslant \Pi_2$. If $\Pi_1 \leqslant \Pi_2$ and $\Pi_2 \leqslant \Pi_1$, then Π_1 is said to be equivalent to Π_2, denoted by $\Pi_1 \approx \Pi_2$. If $\Pi_1 \leqslant \Pi_2$ but $\Pi_1 \approx \Pi_2$ does not hold, then Π_2 is said to be strictly stronger than Π_1, denoted by $\Pi_1 < \Pi_2$.

Example 2. Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context. By Remark 1, it is easy to verify that

$$\Pi_0 : \quad \frac{B, B_0 \subseteq A, C \subseteq D, B \to C}{B \cup B_0 \to C} \tag{7}$$

is an inference rule defined between decision implications of $\mathbb{K}$. Furthermore, we can prove that Π^* specified in Eq. (6) is strictly stronger than Π_0. In fact, for any two decision implications $B_1 \to C_1, B_2 \to C_2$ of $\mathbb{K}$, if $(B_1 \to C_1)\Pi_0(B_2 \to C_2)$, it follows from Eq. (7) that $B_1 \subseteq B_2$ and $C_1 = C_2$. By Eq. (6), we have $(B_1 \to C_1)\Pi^*(B_2 \to C_2)$. According to Definition 8, we obtain $\Pi_0 \leqslant \Pi^*$. Furthermore, note that $(B \to C)\Pi^*(B \to C \backslash C_0)$ holds, where C_0 is a nonempty set, but $(B \to C)\Pi_0(B \to C \backslash C_0)$ cannot be obtained. Thus, $\Pi_0 < \Pi^*$ is at hand.

In what follows, we illustrate that a stronger inference rule can eliminate more superfluous decision implications from the whole set of decision implications of a decision formal context. To this end, we first propose the notions of Π-redundant, Π-complete and Π-base in decision formal contexts.

Definition 9. Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context, Σ be a set of decision implications and Π be an inference rule defined between decision implications of $\mathbb{K}$. If $B \to C \in \Sigma$ can be inferred by another decision implication $B_0 \to C_0 \in \Sigma$ via the inference rule Π, i.e., $(B_0 \to C_0)\Pi(B \to C)$, then $B \to C$ is said to be Π-redundant in Σ. Moreover, if every decision implication of $\mathbb{K}$ can be inferred by one in Σ via the inference rule Π, we say that Σ is Π-complete with respect to $\mathbb{K}$.

As usual in FCA, it is interesting to discuss a minimal set of decision implications which is Π-complete with respect to $\mathbb{K}$.

Definition 10. Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context, Σ be a set of decision implications and Π be an inference rule defined between decision implications of $\mathbb{K}$. If each $B \to C \in \Sigma$ is not Π-redundant in Σ and Σ is Π-complete with respect to $\mathbb{K}$, then Σ is called a Π-base of $\mathbb{K}$.

By Definition 10, we can obtain a Π-base of $\mathbb{K}$ from a Π-complete set Σ of decision implications by removing Π-redundant decision implications from Σ one by one until none of the remainder is Π-redundant in Σ.

With a Π-base Σ of $\mathbb{K}$, we can eliminate superfluous decision implications from the whole set of decision implications of a decision formal context. That is, Π-redundant decision implications can be removed without any effect on decision-implications-based data analysis.

Proposition 3. *Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and Π be an inference rule defined between decision implications of $\mathbb{K}$. If Σ_1, Σ_2 are Π-bases of $\mathbb{K}$, then the cardinalities of Σ_1 and Σ_2 are the same, i.e., $|\Sigma_1| = |\Sigma_2|$.*

Proof. To prove $|\Sigma_1| = |\Sigma_2|$, it is sufficient to show that there exists a bijection from Σ_1 to Σ_2 since both of them are finite.

Note that Σ_2 is a Π-base of $\mathbb{K}$. Then for any $B_1 \to C_1 \in \Sigma_1$, there exists $B_2 \to C_2 \in \Sigma_2$ such that $(B_2 \to C_2)\Pi(B_1 \to C_1)$. Meanwhile, there also exists $B_3 \to C_3 \in \Sigma_1$ such that $(B_3 \to C_3)\Pi(B_2 \to C_2)$ because Σ_1 is a Π-base of $\mathbb{K}$. Combining $(B_3 \to C_3)\Pi(B_2 \to C_2)$ with $(B_2 \to C_2)\Pi(B_1 \to C_1)$, we obtain $(B_3 \to C_3)\Pi(B_1 \to C_1)$. Thus, $(B_3 \to C_3)$ and $(B_1 \to C_1)$ are the same due to the minimality of Σ_1. As a result, $\Pi : \Sigma_1 \longrightarrow \Sigma_2$ is a bijection in terms of inference relationship. □

Proposition 3 says that we can choose any Π-base of a decision formal context to eliminate superfluous decision implications since the number of the decision implications that can be removed is the same (no matter which Π-base is selected) once the inference rule is determined.

However, it should be pointed out that different inference rules may lead to different numbers of decision implications that can be removed by their respective bases from the whole set of decision implications. Of course, it is better to find such a base that is of less elements under the pre-defined inference rule, since in this case more superfluous decision implications can be eliminated without any effect on data analysis.

Proposition 4. *Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and Π_1, Π_2 be two inference rules defined between decision implications of $\mathbb{K}$. If $\Pi_1 \leqslant \Pi_2$, then every Π_1-base Σ is Π_2-complete with respect to $\mathbb{K}$.*

Proof. Since Σ is a Π_1-base of $\mathbb{K}$, then by Definition 10 Σ is Π_1-complete with respect to $\mathbb{K}$. Thus, it follows from Definition 9 that every decision implication $B \to C$ of $\mathbb{K}$ can be inferred by a decision implication $B_0 \to C_0 \in \Sigma$ via the inference rule Π_1. In other words, $(B_0 \to C_0)\Pi_1(B \to C)$ holds. Moreover, by Definition 8, we obtain $(B_0 \to C_0)\Pi_2(B \to C)$ due to $\Pi_1 \leqslant \Pi_2$. Therefore, every decision implication of $\mathbb{K}$ can be inferred by one in Σ via the inference rule Π_2. Consequently, Σ is Π_2-complete with respect to $\mathbb{K}$. □

By Proposition 4, if Π_2 is stronger than Π_1, a Π_2-base can be generated from the Π_1-base Σ by further removing Π_2-redundant decision implications (if any) from Σ. In this case, the cardinality of the Π_2-base is not more than that of the Π_1-base. This means that with a stronger inference rule, more superfluous decision implications can be removed from the whole set of decision implications of a decision formal context. So, it is necessary to propose stronger inference rules for effectively eliminating superfluous decision implications.

4. A new inference rule in decision formal contexts

In the previous section, we have shown that it is quite important to propose effective inference rules for eliminating more superfluous decision implications of a decision formal context. To achieve this task, it is sufficient to propose stronger inference rules. In what follows, we put forward a new inference rule in decision formal contexts which is strictly stronger than the existing one in Eq. (6). Also, we figure out the exact number of the superfluous decision implications that we can additionally remove by using the proposed inference rule compared with the existing one.

Definition 11. Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context. We denote

$$\Pi^{\#}: \quad \frac{\varphi(B_1) \subseteq \varphi(B_2) \subseteq A, C_2 \subseteq C_1 \subseteq D, B_1 \to C_1}{B_2 \to C_2}, \tag{8}$$

where the Galois closure operator φ is specified in Eq. (3).

Proposition 5. *Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and $\Pi^{\#}$ be specified in Eq. (8). Then $\Pi^{\#}$ is an inference rule defined between decision implications of $\mathbb{K}$.*

Proof. According to Remark 1, to complete the proof, it is sufficient to show that $B_2 \to C_2$ must be a decision implication of $\mathbb{K}$ if $\varphi(B_1) \subseteq \varphi(B_2) \subseteq A$, $C_2 \subseteq C_1 \subseteq D$ and $B_1 \to C_1$ is a decision implication of $\mathbb{K}$.

Since $\varphi(B_1) \subseteq \varphi(B_2)$ and $C_2 \subseteq C_1$, we have $B_2^{\downarrow} \subseteq B_1^{\downarrow}$ and $C_1^{\downarrow^{\sim}} \subseteq C_2^{\downarrow^{\sim}}$ according to (ii), (iv) of Proposition 1. Moreover, we obtain $B_1^{\downarrow} \subseteq C_1^{\downarrow^{\sim}}$ because $B_1 \to C_1$ is a decision implication of $\mathbb{K}$. To sum up, it follows $B_2^{\downarrow} \subseteq B_1^{\downarrow} \subseteq C_1^{\downarrow^{\sim}} \subseteq C_2^{\downarrow^{\sim}}$ yielding $B_2^{\downarrow} \subseteq C_2^{\downarrow^{\sim}}$. Based on Proposition 2, we conclude that $B_2 \to C_2$ is a decision implication of $\mathbb{K}$. □

Then, a new inference rule $\Pi^{\#}$ is defined between decision implications of a decision formal context. In what follows, we discuss some useful properties of the proposed inference rule.

Proposition 6. *Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and $\Pi^*, \Pi^{\#}$ be specified in Eqs. (6) and (8), respectively. Then $\Pi^{\#}$ is stronger than Π^*, i.e., $\Pi^* \leqslant \Pi^{\#}$.*

Proof. For any two decision implications $B_1 \to C_1, B_2 \to C_2$ of $\mathbb{K}$, if $(B_1 \to C_1)\Pi^*(B_2 \to C_2)$, it follows from Eq. (6) that $B_1 \subseteq B_2$ and $C_2 \subseteq C_1$. Note that $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow} \Rightarrow \varphi(B_1) \subseteq \varphi(B_2)$. Then according to Eq. (8), we obtain $(B_1 \to C_1)\Pi^{\#}(B_2 \to C_2)$. By Definition 8, $\Pi^* \leqslant \Pi^{\#}$ is at hand. □

Furthermore, we use the following example to illustrate that $\Pi^* \approx \Pi^{\#}$ does not hold.

Example 3. Let $\mathbb{K} = (U,A,I,D,J)$ be the decision formal context shown in Table 1. It is easy to verify that $\{a,b\} \to \{d_2\}$ is a decision implication of $\mathbb{K}$. Note that $\varphi(\{a,b\}) = \{a,b\}^{\downarrow\uparrow} = \{a,b,c,e\}$ and $\varphi(\{a,e\}) = \{a,e\}^{\downarrow\uparrow} = \{a,b,c,e\}$. Then it follows from Eq. (8) that $\{a,e\} \to \{d_2\}$ can be inferred by $\{a,b\} \to \{d_2\}$ via the inference rule $\Pi^{\#}$, i.e., $(\{a,b\} \to \{d_2\})\Pi^{\#}(\{a,e\} \to \{d_2\})$. However, by Eq. (6), $\{a,e\} \to \{d_2\}$ cannot be inferred by $\{a,b\} \to \{d_2\}$ via the inference rule Π^*. So, based on Definition 8, $\Pi^{\#}$ is not equivalent to Π^*. That is, $\Pi^* \approx \Pi^{\#}$ does not hold.

Combining Proposition 6 with Example 3, $\Pi^{\#}$ is strictly stronger than Π^*, which means that more superfluous decision implications can be removed from the whole set of decision implications of a decision formal context by using the proposed inference rule $\Pi^{\#}$. In other words, a more effective inference rule $\Pi^{\#}$ is obtained in terms of eliminating superfluous decision implications.

Obviously, the difference set between a Π^*-base Σ and a $\Pi^{\#}$-base Σ_0 [ii] is composed of the superfluous decision implications that we can additionally remove by using the proposed inference rule $\Pi^{\#}$

[ii] Without loss of generality, it is always assumed that the $\Pi^{\#}$-base is obtained from the Π^*-base by further removing $\Pi^{\#}$-redundant decision implications.

compared with the existing one Π^*.

In what follows, we figure out the exact number of the difference set between a Π^*-base and a $\Pi^\#$-base for illustrating to what extent the effectiveness is improved in terms of eliminating superfluous decision implications of a decision formal context. Before embarking on this issue, we define some notations in decision formal contexts.

Let $\mathbb{K} = (U,A,I,D,J)$ be a decision formal context and $\mathfrak{B}(U,A,I)$, $\mathfrak{B}(U,D,J)$ be the concept lattices of the formal contexts (U,A,I) and (U,D,J), respectively. Define

$$\begin{aligned} \mathfrak{U}(U,A,I) &= \{X \mid (X,B) \in \mathfrak{B}(U,A,I)\}, \\ \mathfrak{U}(U,D,J) &= \{Y \mid (Y,C) \in \mathfrak{B}(U,D,J)\}. \end{aligned} \tag{9}$$

That is, $\mathfrak{U}(U,A,I)$ is the set of all extents of (U,A,I) and $\mathfrak{U}(U,D,J)$ is the set of all extents of (U,D,J).

By Definition 4, we know that $\mathrm{MG}(X,B)$ denotes the set of all minimal generators of the formal concept (X,B). Here, we further denote

$$\mathrm{MG}(U,A,I) = \bigcup\{\mathrm{MG}(X,B) \mid (X,B) \in \mathfrak{B}(U,A,I)\}. \tag{10}$$

That is, $\mathrm{MG}(U,A,I)$ is the set of minimal generators of all formal concepts of (U,A,I).

In addition, for $E \in \mathrm{MG}(U,A,I)$, $X \in \mathfrak{U}(U,A,I)$, $Y \in \mathfrak{U}(U,D,J)$, we define mappings $\alpha,\ \beta : \mathfrak{U}(U,A,I) \times \mathfrak{U}(U,D,J) \longrightarrow \{0,1\}$ and $\gamma : \mathrm{MG}(U,A,I) \times \mathfrak{U}(U,D,J) \longrightarrow \{0,1\}$ as follows:

$$\alpha(X,Y) = \begin{cases} 1, & \text{if } X \subseteq Y, \text{ and } X \subset X_0 \Rightarrow X_0 \not\subseteq Y \\ & \text{for any } X_0 \in \mathfrak{U}(U,A,I), \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

$$\beta(X,Y) = \begin{cases} 1, & \text{if } X \subseteq Y, \text{ and } Y_0 \subset Y \Rightarrow X \not\subseteq Y_0 \\ & \text{for any } Y_0 \in \mathfrak{U}(U,D,J), \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

and

$$\gamma(E,Y) = $$

$$\begin{cases} 1, & \text{if } E^{\downarrow} \subseteq Y, X_0 \in \mathfrak{U}(U,A,I) \text{ with } E^{\downarrow} \subset X_0 \subseteq Y \\ & \text{implies } E_0 \not\subset E \text{ for any } E_0 \in \mathrm{MG}(X_0,B_0), \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Then how to find a Π^*-base of a decision formal context $\mathbb{K} = (U,A,I,D,J)$ can be described as follows: (see Ref. [40] for details)

Algorithm 1. Find a Π^*-base of a decision formal context $\mathbb{K}$.

Input: A decision formal context $\mathbb{K} = (U,A,I,D,J)$.
Output: A Π^*-base of $\mathbb{K}$.

(1) Generate all minimal generators from (U,A,I) by TITANIC algorithm [48] and assign them to MG.

(2) Use MG to generate the concept lattice of (U,A,I) and assign it to $\mathscr{L}_A$; construct the concept lattice of (U,D,J) and assign it to $\mathscr{L}_D$.

(3) Initialize $\Sigma = \emptyset$.

(4) For every $E \in \mathrm{MG}$ and $(Y,C) \in \mathscr{L}_D$ with $\beta(E^{\downarrow},Y) = 1$,

  (a) if $\alpha(E^{\downarrow},Y) = 1$, then $\Sigma \leftarrow \Sigma \cup \{E \rightarrow C\}$ and skip step (b).

  (b) if $\gamma(E,Y) = 1$, then $\Sigma \leftarrow \Sigma \cup \{E \rightarrow C\}$ and label $E \rightarrow C$ with 'N'. [iii]

(5) Output Σ and end the algorithm.

According to Proposition 4, with the Π^*-base Σ output by Algorithm 1, we can further obtain a $\Pi^\#$-base Σ_0 by removing $\Pi^\#$-redundant decision implications.

Algorithm 2. Find a $\Pi^\#$-base Σ_0 from the Π^*-base Σ.

Input: The Π^*-base Σ output by Algorithm 1.
Output: A $\Pi^\#$-base Σ_0.

(1) Remove such decision implications from Σ that are labeled with 'N'.

(2) Initialize $\Sigma_0 = \emptyset$.

(3) Choose any decision implication $E \rightarrow C$ from Σ.

[iii] Such an action of labeling is to facilitate our subsequent discussion.

(4) If there does not exist $E_0 \to C_0 \in \Sigma_0$ such that $\varphi(E_0) = \varphi(E)$ and $C_0 = C$, then $\Sigma_0 \leftarrow \Sigma_0 \cup \{E \to C\}$ and $\Sigma \leftarrow \Sigma \backslash \{E \to C\}$; otherwise, $\Sigma \leftarrow \Sigma \backslash \{E \to C\}$.

(5) If $\Sigma \neq \emptyset$, go back to step (3).

(6) Output Σ_0 and end the algorithm.

Proposition 7. *Σ_0 output by Algorithm 2 is a $\Pi^{\#}$-base of the input decision formal context $\mathbb{K}$ with certainty.*

Proof. Suppose Σ is a Π^*-base of $\mathbb{K}$ output by Algorithm 1. Then, it follows from Proposition 4 that Σ is $\Pi^{\#}$-complete with respect to $\mathbb{K}$. By Definition 9, it is easy to verify that all decision implications in Σ being labeled with 'N' are $\Pi^{\#}$-redundant in Σ. Thus, removing them in step (1) of Algorithm 2 does not affect the $\Pi^{\#}$-completeness of Σ. By Eq. (8), if $\varphi(E_0) = \varphi(E)$, $C_0 = C$ and $E_0 \to C_0$ is a decision implication of $\mathbb{K}$, we have $(E_0 \to C_0)\Pi^{\#}(E \to C)$, which means that removing these decision implications in step (4) of Algorithm 2 still preserves the $\Pi^{\#}$-completeness of Σ. Moreover, it can be known from Definition 9 and Eq. (8) that the remainder (i.e., Σ_0) does not contain any $\Pi^{\#}$-redundant decision implication. Consequently, by Definition 10, Σ_0 is a $\Pi^{\#}$-base of $\mathbb{K}$. $\square$

Now we are ready to figure out the exact number of the difference set between a Π^*-base and a $\Pi^{\#}$-base.

Let $\mathbb{K} = (U, A, I, D, J)$ be a decision formal context. Denote

$$\overline{\mathfrak{U}}(U,A,I) = \{X \in \mathfrak{U}(U,A,I) \mid \text{there exists } Y \in \mathfrak{U}(U,D,J) \text{ such that } \alpha(X,Y) = \beta(X,Y) = 1\}$$

and

$$\overline{\mathrm{MG}}(U,A,I) = \{E \in \mathrm{MG}(U,A,I) \mid \text{there exists } Y \in \mathfrak{U}(U,D,J) \text{ such that } \gamma(E,Y) = \beta(E^{\downarrow},Y) = 1\},$$

where $\mathfrak{U}(U,A,I)$ and $\mathfrak{U}(U,D,J)$, $\mathrm{MG}(U,A,I)$, $\alpha(X,Y)$, $\beta(X,Y)$, and $\gamma(E,Y)$ are specified in Eqs. (9), (10), (11), (12), (13), respectively.

Theorem 8. *Let $\mathbb{K} = (U, A, I, D, J)$ be a decision formal context, Σ be a Π^*-base, and Σ_0 be a $\Pi^{\#}$-base of $\mathbb{K}$ which is obtained by removing $\Pi^{\#}$-redundant decision implications from Σ. Then the number of the difference set between Σ and Σ_0 is*

$$\mu = \sum_{X \in \overline{\mathfrak{U}}(U,A,I)} (|MG(X,B)| - 1) + \left| \overline{MG}(U,A,I) \Big\backslash \bigcup_{X \in \overline{\mathfrak{U}}(U,A,I)} MG(X,B) \right|.$$

Proof. It can be known from Algorithms 1 and 2 that

$$\left| \overline{\mathrm{MG}}(U,A,I) \Big\backslash \bigcup_{X \in \overline{\mathfrak{U}}(U,A,I)} \mathrm{MG}(X,B) \right|$$

is the number of the decision implications labeled with 'N', and

$$\sum_{X \in \overline{\mathfrak{U}}(U,A,I)} (|\mathrm{MG}(X,B)| - 1)$$

is the number of the decision implications that are removed in step (4) of Algorithm 2. So, μ is the number of the difference set between Σ and Σ_0. $\square$

Note that in general minimal generators of all formal concepts of a big formal context (U,A,I) have an extremely large scale [40]. Under such a circumstance, even the cardinality of $\bigcup_{X \in \overline{\mathfrak{U}}(U,A,I)} \mathrm{MG}(X,B)$ is large, and hence $\sum_{X \in \overline{\mathfrak{U}}(U,A,I)} (|\mathrm{MG}(X,B)| - 1)$ is big, let alone μ. This illustrates in theory that the proposed inference rule $\Pi^{\#}$ has a significant advantage in eliminating superfluous decision implications compared with the existing one.

5. An illustrative example

In the previous section, we have illustrated in theory that compared with the existing inference rule Π^*, the proposed inference rule $\Pi^{\#}$ has a significant advantage in eliminating superfluous decision implications of a decision formal context. In this section, we use an example to show this advantage vividly.

Example 4. Let $\mathbb{K} = (U,A,I,D,J)$ be the decision formal context shown in Table 1. Then the minimal generators of all formal concepts of (U,A,I) are listed in Table 2. It is easy to verify that

$$\overline{\mathfrak{U}}(U,A,I) = \{U,\{2\},\{3\},\{5\},\emptyset\}$$

and $\overline{\mathrm{MG}}(U,A,I) = \{\emptyset,\{d\},\{f\},\{a,b\},\{a,e\},\{b,c\},\{c,e\},\{a,d\},\{b,f\},\{c,d\},\{d,e\},\{d,f\},\{e,f\}\}$.

Table 2. Minimal generators of all formal concepts of (U,A,I)

No.	Formal concepts	Minimal generators
1	$(U,\emptyset)$	$\emptyset$
2	$(\{1,3,5\},\{a\})$	$\{a\}$
3	$(\{2,4,5\},\{b\})$	$\{b\}$
4	$(\{3,5\},\{a,c\})$	$\{c\}$
5	$(\{4,5\},\{b,e\})$	$\{e\}$
6	$(\{2\},\{b,d\})$	$\{d\}$
7	$(\{3\},\{a,c,f\})$	$\{f\}$
8	$(\{5\},\{a,b,c,e\})$	$\{a,b\},\{a,e\},\{b,c\},\{c,e\}$
9	$(\emptyset,A)$	$\{a,d\},\{b,f\},\{c,d\},$ $\{d,e\},\{d,f\},\{e,f\}$

Moreover, we compute

$$\bigcup_{X\in\overline{\mathfrak{U}}(U,A,I)} \mathrm{MG}(X,B) = \overline{\mathrm{MG}}(U,A,I)$$

and

$$\begin{aligned}\mu &= \sum_{X\in\overline{\mathfrak{U}}(U,A,I)} (|\mathrm{MG}(X,B)|-1) + \\ &\quad \left|\overline{\mathrm{MG}}(U,A,I) \Big\backslash \bigcup_{X\in\overline{\mathfrak{U}}(U,A,I)} \mathrm{MG}(X,B)\right| \\ &= \sum_{X\in\overline{\mathfrak{U}}(U,A,I)} (|\mathrm{MG}(X,B)|-1) \\ &= 8.\end{aligned}$$

That is, the number of the superfluous decision implications that can additionally be removed by the proposed inference rule $\Pi^{\#}$ compared with the existing one Π^{*} is eight. Furthermore, Table 3 shows a Π^{*}-base of $\mathbb{K}$ and Table 4 shows a $\Pi^{\#}$-base of $\mathbb{K}$ which is obtained by removing $\Pi^{\#}$-redundant decision implications from the Π^{*}-base. From these tables, we can see that the number of the difference set between the Π^{*}-base and the $\Pi^{\#}$-base is indeed eight.

Table 3. A Π^{*}-base of $\mathbb{K}$

No.	Decision implications
1	$\emptyset \to \emptyset$
2	$\{d\} \to \{d_1\}$
3	$\{f\} \to \{d_1\}$
4	$\{a,b\} \to \{d_2\}$
5	$\{a,e\} \to \{d_2\}$
6	$\{b,c\} \to \{d_2\}$
7	$\{c,e\} \to \{d_2\}$
8	$\{a,d\} \to \{d_1,d_2\}$
9	$\{b,f\} \to \{d_1,d_2\}$
10	$\{c,d\} \to \{d_1,d_2\}$
11	$\{d,e\} \to \{d_1,d_2\}$
12	$\{d,f\} \to \{d_1,d_2\}$
13	$\{e,f\} \to \{d_1,d_2\}$

Table 4. A $\Pi^{\#}$-base of $\mathbb{K}$

No.	Decision implications
1	$\emptyset \to \emptyset$
2	$\{d\} \to \{d_1\}$
3	$\{f\} \to \{d_1\}$
4	$\{a,b\} \to \{d_2\}$
5	$\{a,d\} \to \{d_1,d_2\}$

6. Conclusion

Since the number of decision implications of a decision formal context is generally an exponential increase to the scale of the database, it is important to propose effective inference rules between decision implications for eliminating as many superfluous decision implications as possible. Motivated by this problem, in this paper we have put forward the strongness criterion to assess the effectiveness of inference rules in decision formal contexts and a stronger inference rule which is more effective than the existing one. Moreover, we have figured out the exact number of the superfluous decision implications that we can additionally remove by using the proposed inference rule compared with the existing one and an illustrative example has been used to show this advantage vividly.

From the point of view of real applications, the results obtained in this paper need to be further extended to the case of fuzzy decision formal contexts [51], incomplete decision formal contexts [52] or even real decision formal contexts [53,54] since in practice the relationship between some objects and attributes of a decision formal context may be fuzzy-valued, interval-valued or real-valued.

Besides, in order to make an in-depth study of the rule acquisition in decision formal contexts, more effective inference rules with respect to the strongness criterion and other practical criteria for evaluating inference rules are needed and deserve to be studied in future work.

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