Some properties of implications via aggregation functions and overlap functions

H. Zapata[1], H. Bustince[2], L. De Miguel[2], C. Guerra[2].

[1] Universidad Central de Venezuela,
Venezuela

E-mail: hugozapata1@hotmail.com

[2] Departamento de Automatica y Computacion, Universidad Publica de Navarra,
Campus Arrosadia, 31006, Pamplona, Navarra, Spain.
E-mail: bustince@unavarra.es, carlos.guerra@unavarra.es, laura.demiguel@unavarra.es

Abstract

In this work, using the identification between implication operators and aggregation functions, we study the implication operators that are recovered from overlap functions. In particular, we focus in which properties of implication operators are preserved. We also study how negations can be defined in terms of overlap functions.

Keywords: Implication operator, aggregation function, negation, overlap function.

1. Introduction

The concept of implication operator was introduced in 1982 by Dombi[15]. It arises from the minimal set of properties that the operators usually used in aggregation or data fusion procedures must fulfill.

From the works of Bandler and Kohout[3] in 1980 and of Trillas and Valverde[24] in 1981, Klir and Folger in 1988[19] settle that the logic operation of implication is as essential for approximate reasoning as it is for reasoning within classical two-valued logic. In general, a fuzzy implication I is a function $I : [0,1]^2 \to [0,1]$ such that for every two possible truth values a and b of propositions p and q respectively, $I(a,b)$ represents the truth value of the conditional proposition if p, then q.

Implication operators have been widely studied and used in different fields[1,2,20]. For instance, in image processing they have been used to build measures for image comparison[6,8] when we are working with algorithms which allow us to extract objects from the considered image[7]. In order to use such algorithms in an image composed of a single object and the background, the first thing to do is to represent both the object and the background by means of a fuzzy set.

Moreover, it is well-known that implication operators can be recovered from aggregation functions and vice-versa[14]. In particular, certain properties of implication operators can be directly translated into the aggregation function setting, and properties of aggregation functions admit a straight interpretation in terms of implication operators.

It has been proved that in multiclass problems the difficulty in providing an optimal solution is directly proportional to the overlapping that exists between the considered classes. This fact, together with the idea of consistency index for fuzzy rules given by

Zadeh[26], led to the introduction of the concept of overlap function[10]. Overlap functions are mappings $[0,1]^2 \to [0,1]$ that measure the degree of overlapping between the membership degree of an element to two fuzzy sets. Moreover, duality allows to define also the concept of grouping function[13] in such a way that some of the most relevant properties demanded to triplets in terms of t-norms and t-conorms are preserved. However, neither overlap nor grouping functions need to be associative, which is an advantage in some fields of application where associativity is not a natural property to be demanded.

Since overlap (and grouping) functions are particular instances of aggregation functions, in this work we aim at the following objectives:

1. To study how negations can be recovered from overlap functions.

2. To analyze the properties of implication operators obtained from overlap functions.

This paper is organized as follows. After recalling some preliminaries notions in Section 2, we study in Section 3 those negations constructed from overlap functions. In Section 4, the relationship between implication operators and overlap functions is investigated. We finish in Section 5 with some conclusions and acknowledgments.

2. Preliminaries

In this section we recall some well-known concepts in order to fix the notations used further on. A reference for these concepts is[17].

Definition 1. An automorphism on the unit interval is an increasing, continuous bijection $\varphi : [0,1] \to [0,1]$.

Definition 2. A negation is a decreasing function $N : [0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. A negation is called:

- *frontier* if $N(x) \in \{0,1\}$ if and only if $x \in \{0,1\}$.
- *strict* if N is continuous and strictly decreasing.
- *strong* if N is involutive, that is, $N(N(x)) = x$ for all $x \in [0,1]$.

Example 1. The least negation N_* is defined by

$$N_*(x) = \begin{cases} 1 & \text{if } x = 0 \text{ ;} \\ 0 & \text{otherwise.} \end{cases}$$

The greatest negation N^* is given by

$$N^*(x) = \begin{cases} 0 & \text{if } x = 1 \text{ ;} \\ 1 & \text{otherwise.} \end{cases}$$

The most used negation is the standard (or Zadeh's) negation $N_Z(x) = 1 - x$. Notice that N_Z is strong. The following characterization of strong negations was proven by Trillas.

Theorem 1. [23] *A function* $N : [0,1] \to [0,1]$ *is a strong negation if and only if there exists an automorphism* $\varphi : [0,1] \to [0,1]$ *such that*

$$N(x) = \varphi^{-1}(1 - \varphi(x))$$

for all $x \in [0,1]$.

That is, a strong negation can be seen as a perturbation, via an automorphism φ, of Zadeh's negation.

Definition 3. An (*n*-ary) aggregation function is an increasing mapping $A : [0,1]^n \to [0,1]$ such that $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.

In the present work, we mainly deal with binary ($n = 2$) aggregation functions.

Example 2.

- The greatest aggregation function A^* is defined by

$$A^*(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \text{ ;} \\ 1 & \text{otherwise.} \end{cases}$$

- The least aggregation function A_* is defined by

$$A_*(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 1 \text{ ;} \\ 0 & \text{otherwise.} \end{cases}$$

Given a mapping $\varphi : [0,1] \to [0,1]$ and an aggregation function $A : [0,1]^2 \to [0,1]$, we denote by A_φ the mapping defined by

$$A_\varphi(x,y) = A(\varphi(x), \varphi(y)) \,.$$

If φ is increasing, $\varphi(0) = 0$ and $\varphi(1) = 1$, then A_φ is also an aggregation function.

Definition 4. An aggregation function A is called associative if the identity

$$A(A(x,y),z) = A(x,A(y,z))$$

holds for all $x,y,z \in [0,1]$.

Definition 5. Let A be a binary aggregation function.

(i) An element $e_A \in [0,1]$ is said to be a neutral element for A if $A(x,e_A) = A(e_A,x) = x$ for all $x \in [0,1]$.

(ii) An element $x_0 \in]0,1]$ is called a zero-divisor for A if there exists $x \in]0,1]$ such that $A(x,x_0) = A(x_0,x) = 0$.

(iii) An element $a \in [0,1]$ is said to be an annihilator for A if $A(a,x) = A(x,a) = a$ for all $x \in [0,1]$.

Several studies have focused on specific classes of aggregation operators, from both a theoretical and a practical point of view. Two of the classes receiving more attention are those of t-norms and overlap functions.

Definition 6. A t-norm is an associative symmetric aggregation function $T : [0,1]^2 \to [0,1]$ such that $T(x,1) = (1,x) = x$ for all $x \in [0,1]$ (that is, 1 is a neutral element for T).

Definition 7. An overlap function is a function $G_O : [0,1]^2 \to [0,1]$ such that:

(O1) G_O is symmetric;
(O2) $G_O(x,y) = 0$ if and only if $xy = 0$;
(O3) $G_O(x,y) = 1$ if and only if $xy = 1$;
(O4) G_O is increasing;
(O5) G_O is continuous.

A deep study on overlap functions can be found in[10].

Example 3.

1. Any continuous t-norm without divisors of zero is an example of overlap function. In particular, both $G_O(x,y) = \min(x,y)$ and $G_O(x,y) = xy$ are t-norms and consequently overlap functions.

2. The function $G_O(x,y) = (xy)^p$ for $p > 0$ is an overlap function. Notice that for $p \neq 1$ G_O is not associative.

3. The function $G_O(x,y) = \min(x^p, y^p)^q$ for $p,q > 0$ is an overlap function. Notice that for $p,q < 1$, $G_O(x,y) \geqslant \min(x,y)$.

Another key concept for our work is that of implication operator[3,24]. We start recalling its definition (in the sense of Fodor[16]) and the main properties that can be demanded to such an operator.

Definition 8. An implication operator is a function $I : [0,1]^2 \to [0,1]$ such that

(FA) If $x \leqslant z$ then $I(x,y) \geqslant I(z,y)$ for all $y \in [0,1]$ (decreasing in the first variable).

(SI) If $y \leqslant z$ then $I(x,y) \leqslant I(x,z)$ for all $x \in [0,1]$ (increasing in the second variable).

(DF) $I(0,x) = 1$ for all $x \in [0,1]$ (dominance of falsity).

(DT) $I(x,1) = 1$ for all $x \in [0,1]$ (dominance of truth of consequent).

(BC) $I(1,0) = 0$ (the boundary condition).

Definition 8 recalls the minimal properties of implication operators. Nevertheless, their study has led to several other properties which further characterize their behaviour.

Definition 9. [1,4] The following properties can be used to outline the behaviour of an implication operator I.

(NP) $I(1,x) = x$ for all $x \in [0,1]$ (neutrality of truth).

(EP) $I(x,I(y,z)) = I(y,I(x,z))$ for all $x,y,z \in [0,1]$ (exchange principle).

(OP) $I(x,y) = 1$ if and only if $x \leqslant y$ (ordering property).

(SN) $N(x) = I(x,0)$ is a strong negation.

(IP) $I(x,x) = 1$ for all $x \in [0,1]$ (identity principle).

(CP) $I(x,y) = I(N(y),N(x))$ for some strong negation N and for all $x,y \in [0,1]$ (contrapositive property).

(IA) $I(x,y) \geqslant y$ for all $x,y \in [0,1]$.

(IB) I is continuous.

(*IC*) $I(x, N(x)) = N(x)$ for some strong negation N and for all $x \in [0, 1]$.

Apart from the properties in the previous definition, two other conditions are of great relevance for specific applications[12].

(*P1*) $I(x, y) = 1$ if and only if $x = 0$ or $y = 1$.
(*P2*) $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$.

3. Overlap functions and negations

In this section we stablish the relationship of the concepts of overlap functions and negations.

Proposition 2. *Let $F : [0, 1] \to [0, 1]$ be a function. Then the following statements are equivalent.*

(*a.*) *F is a negation.*

(*b.*) *For every overlap function $G_O : [0, 1]^2 \to [0, 1]$ the function*

$$N_{G_O}(x) = G_O(1, F(x))$$

is a negation.

Proof. If F is a negation and $G_O : [0, 1]^2 \to [0, 1]$ is an overlap function, then N_{G_O} is decreasing. Moreover

$$N_{G_O}(0) = G_O(1, F(0)) = G_O(1, 1) = 1$$

and

$$N_{G_O}(1) = G_O(1, F(1)) = G_O(1, 0) = 0,$$

so N_{G_O} is a negation.
Let us assume statement (b.) holds for every overlap function G_O. If G_O is an overlap function with neutral element 1, then

$$N_{G_O}(x) = G_O(1, F(x)) = F(x),$$

and since N_{G_O} is a negation, the result follows. $\square$

Corollary 3. *Let $F : [0, 1] \to [0, 1]$ be a function. Then the following statements are equivalent.*

(*a.*) *F is a frontier negation.*

(*b.*) *For every overlap function G_O, N_{G_O} defined as in Proposition 2 is a frontier negation.*

Proof. It is straight from the boundary conditions in the definition of overlap function. $\square$

Example 4. Let be $p > 0$ and $G_O(x, y) = (xy)^p$.

1. If $F = N_*$

$$N_{G_O}(x) = G_O(1, N_*(x)) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and we recover N_*.

2. If $F = N^*$

$$N_{G_O}(x) = G_O(1, N^*(x)) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise,} \end{cases}$$

and we recover N^*.

3. However, if we consider $F(x) = 1 - x^q$ for some $q > 0$, we see that

$$N(x) = G_O(1, F(x)) = (1 - x^q)^p$$

which is different from F as long as $p \neq 1$.

From the previous results we characterize negations through aggregation functions that satisfy suitable boundary conditions. We can also recover any negation from these aggregation functions using one fixed strong negation.

Lemma 4. *Let $\varphi : [0, 1] \to [0, 1]$ be an increasing continuous function such that $\varphi(x) = 0$ if and only if $x = 0$ and $\varphi(x) = 1$ if and only if $x = 1$. The function G, given by*

$$G(x, y) = \varphi(x)\varphi(y)$$

is an overlap function.
Proof. Straight. $\square$

Remark 1. Note that, in fact, the same result holds if we define

$$G(x, y) = G_O(\varphi(x), \varphi(y))$$

for some overlap function G_O.
Then we can state the main result.

Theorem 5. *Let N_0 be a strict negation and $F : [0, 1] \to [0, 1]$ a mapping. Then the following statements are equivalent.*

(a.) F is a continuous frontier negation.

(b.) There exists an overlap function $G_O : [0,1]^2 \to [0,1]$ such that

$$F(x) = G_O(1, N_0(x))$$

for all $x \in [0,1]$. Moreover, this overlap function is given by

$$G_O(x,y) = F(N_0^{-1}(x))F(N_0^{-1}(y)) .$$

Proof. The fact that statement *(b.)* implies statement *(a.)* follows from Corollary 3 because G_O is continuous and every strict negation is a frontier negation.

If *(a.)* holds, and F is a continuous frontier negation, we define

$$\varphi(x) = F(N_0^{-1}(x)),$$

which satisfies the properties in the statement of Lemma 4. If

$$G_O(x,y) = \varphi(x)\varphi(y)$$

then we have that

$$N_{G_O}(x) = G_O(1, N_0(x)) = F(N_0^{-1}(N_0(x))) = F(x)$$

and the result follows. $\square$

In particular, if we fix N_0 as the standard negation, namely, $N_0 = N_Z$, we arrive at the following result.

Corollary 6. *A function $F : [0,1] \to [0,1]$ is a continuous frontier negation if and only if there exists an overlap function $G_O : [0,1]^2 \to [0,1]$ such that $F(x) = G_O(1, 1-x)$ for all $x \in [0,1]$. This overlap function is given by $G_O(x,y) = F(1-x)F(1-y)$.*
Proof. Straight. $\square$

Notice that, in general, the overlap function G_O needs not be unique. For instance, observe that the standard negation $N(x) = 1 - x$ can be recovered by any overlap function which has neutral element 1.

4. Implication operators and overlap functions

For the development of this section the following result is crucial.

Proposition 7. [14] *Let N be a negation and A a binary aggregation function such that $A(0,1) = A(1,0) = 0$. Then, the function $I_A : [0,1]^2 \to [0,1]$ given by*

$$I_A(x,y) = N(A(x,N(y)))$$

is an implication operator.
Proof. Straight. $\square$

Recall that an overlap function G_O is a particular instance of aggregation function such that $G_O(0,1) = G_O(1,0) = 0$. So we could start just writing a corollary of Proposition 7. However, we can provide a slightly more general result.

Proposition 8. *If $G_O : [0,1]^2 \to [0,1]$ is an overlap function and N_1 and N_2 are strong negations, then $I(x,y) = N_1(G_O(x, N_2(y)))$ is a continuous implication operator.*
Proof. Straight. $\square$

Remark 2. The restriction of N_1 and N_2 being strong negations can be relaxed, and the same result holds if we demand that both are only strict (and hence continuous), and even if N_1 and N_2 are just continuous negations.

Even if they are negations, without any other additional requirement, we recover an implication operator, although not necessarily continuous.

In any case, involution has an impact in the subsequent examples.

Taking into account the already known methods for constructing overlap functions in terms of automorphisms, as well as Trillas' characterization theorem of strong negations, we can state the following.

Example 5. If $\varphi, \varphi_1, \varphi_2 : [0,1] \to [0,1]$ are automorphisms, and $G_O(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$ then

$$I(x,y) = \varphi_1^{-1}(1 - \varphi_1(\varphi^{-1}(\varphi(x)\varphi(\varphi_2^{-1}(1-\varphi_2(y))))))$$

is an implication operator.

Example 6.

- Let $\varphi_1(x) = \varphi_2(x) = \varphi(x) = x$ and $G_O(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$. Then $N_1(x) = N_2(x) = 1-x$ and $G_O(x,y) = \varphi^{-1}(\varphi(x)\varphi(y)) = xy$. The expression of I is $I(x,y) = N_1(G_O(x, N_2(y))) = N_1(G_O(x, 1-y)) = N_1(x(1-y)) = 1 - x(1-y) = 1 - x + xy$.

- Consider now the automorphisms $\varphi_1(x) = x^2$, $\varphi(x) = \varphi_2(x) = x$, and the overlap function $G_O(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$. Then $N_1(x) = (1 - x^2)^{1/2}$, $N_2(x) = 1 - x$ and $G_O(x,y) = xy$. The expression of I is $I(x,y) = N_1(G_O(x,N_2(y))) = N_1(G_O(x,1-y)) = N_1(x(1-y)) = (1 - x^2(1 - y)^2)^{1/2}$.

Proposition 9. *If N_1, N_2 are strong negations and for $i = 1, \ldots, n$, $G_{O,i}(x,y)$ are overlap functions and $w_i \in [0,1]$ are such that $\sum_{i=1}^n w_i = 1$, then*

$$I(x,y) = N_1\left(\sum_{i=1}^n (w_i G_{O,i}(x,N_2(y)))\right)$$

is an implication operator.

Proof. In[18] it was proven that the convex combination of overlap function is also an overlap function so the result is straight by Proposition 8.

Remark 3. Notice that the previous theorem does not hold for every class of aggregation functions, an example being the t-norms, which convex combination does not necessarily yield another t-norm.

From the construction method of overlap functions by means of rational functions[10] we can also state the following example.

Example 7. If N_1 and N_2 are strong negations and

$$G_O(x,y) = \frac{f(x,y)}{f(x,y) + h(x,y)}$$

for some $f, h : [0,1]^2 \to [0,1]$ such that

1. f and h are symmetric;

2. f is non decreasing and h is non increasing;

3. $f(x,y) = 0$ if and only if $xy = 0$;

4. $h(x,y) = 0$ if and only if $xy = 1$;

5. f and h are continuous;

then

$$I(x,y) = N_1\left(\frac{f(x,N_2(y))}{f(x,N_2(y)) + h(x,N_2(y))}\right) \quad (1)$$

is an implication operator.

Remark 4. As in Remark 2 strong negations are not required, since the result holds for strict negations.

Example 8. If $f(x,y) = (xy)^\alpha$, $h(x,y) = 1 - (xy)^\alpha$ and $N(x) = N_1(x) = N_2(x) = (1 - x^w)^{1/w}$ for $\alpha, w > 0$, then eq. (1) yields the following implication operator

$$I(x,y) = \left(1 - x^{\alpha w}(1 - y^w)^\alpha\right)^{1/w}.$$

In this situation I does not necessarily verify the properties of Definition 9.

In the remainder of this section we study the restrictions to be imposed to overlaps in order Proposition 8 to yield implications fulfilling specific properties.

Corollary 10. *Let N_1 and N_2 be strong negations and let G_O be an overlap function. Then, if $I(x,y) = N_1(G_O(x,N_2(y))$, the following statements hold:*

1 *I satisfies (SN) with N_1 if and only if $G_O(x,1) = x$ for all $x \in [0,1]$.*

2 *$I(x,N_2(x)) = N_1(x)$ if and only if $G_O(x,x) = x$ for all $x \in [0,1]$.*

Proof. Straight. $\square$

Proposition 11. *Let N_1 and N_2 be strong negations and let G_O be an overlap function. Then, $I(x,y) = N_1(G_O(x,N_2(y))$ does not satisfy neither (OP) nor (IP) with respect to N_2.*

Proof. As $x \leqslant N_2(x)$ or $x > N_2(x)$ for all $x \in [0,1]$, if I verifies the property (OP) then:

$$I(x,N_2(x)) = 1 = N_1(G_O(x,x))$$
$$\text{if and only if } G_O(x,x) = 0$$

or

$$I(N_2(x),x) = N_1(G_O(N_2(x),N_2(x))) = 1$$

if and only if $G_O(N_2(x),N_2(x)) = 0$.

But this is a contradiction with the fact that $G_O(x,y) = 0$ if and only if $xy = 0$.

If I verifies the property (IP) then $1 = I(x,x) = N_1(G_O(x,N_2(x)))$, hence $G_O(x,N_2(x)) = 0$ for all $x \in [0,1]$. But this is also a contradiction with $G_O(x,y) = 0$ if and only if $xy = 0$. $\square$

The following result[10] characterizes associative overlap functions.

Proposition 12. [10] *If G_O is an overlap function satisfying*

$$G_O(x, G_O(y,z)) = G_O(y, G_O(x,z))$$

for all $x,y,z \in [0,1]$, then the following items hold.

1. *G_O is associative;*

2. *If G_O has neutral element 1, then*

3. *If G_O has neutral element 1, then G_O is a continuous t-norm.*

Theorem 13. *If N is a strong negation, $G_O(x,y)$ is an overlap function and*

$$I(x,y) = N(G_O(x,N(y))), \qquad (2)$$

then:

1. *$I(x,y)$ satisfies (CP).*

2. *$I(x,y)$ is a continuous function; namely, (IB) is satisfied.*

Proof. Straight. □

In the following example we provide counterexamples for the properties of Definition 9 to implication operators which are built from overlap functions.

Example 9. Consider the family of overlap functions $G_O(x,y) = (xy)^p$ for $p > 0$. From Eq. (2), let I_p be the implication operator constructed through G_O with exponent p using $N(x) = 1 - x$.

1. If $p = 2$, then $I_2(1,x) = 1 - 1^2(1-x)^2 = 2x - x^2 \neq x$, so I_2 does not satisfy property (NP).

2. If $p = 2$, then

$$\begin{aligned}
I_2(1/2, I(1/3,1/4)) &= I_2(1/2, 15/16) \\
&= 1023/1024 \neq I_2(1/3, I_2(1/2,1/4)) \\
&= I_2(1/3, 55/64) = 4087/4096,
\end{aligned}$$

so I_2 does not satisfy (EP).

3. If $p = 1$, then $I_1(x,x) = 1 - x(1-x) = 1 - x + x^2 \neq 1$ for all $x \in]0,1[$, so I_1 does not satisfy (IP).

4. If $p = 1$ $I_1(1/3, 1/2) = 1 - 1/3 \cdot 1/2 = 1 - 1/6 = 5/6 \neq 1$, so I_1 does not satisfy (OP).

5. If $p = 2$, then $N_I(x) = I_2(x,0) = 1 - x^2$. Hence N_I is a negation, but it is not a strong negation because $N_I(N_I(x)) = N_I(1 - x^2) = 1 - (1 - x^2)^2 \neq x$. So I_2 does not satisfy (SN).

6. Let $p = \frac{1}{3}$, $I_{\frac{1}{3}}(1, 7/8) = 1 - (1 \cdot 1/8)^{1/3} = 1 - 1/2 = 1/2 < 7/8$, so $I_{\frac{1}{3}}$ does not satisfy (IA).

Example 10. Let be I an implication operator constructed as eq. (2). Then:

1. If $G_O(x,y) = (xy)^p$, for $p > 0$ and

$$N(x) = \frac{1-x}{1+\lambda x},$$

with $\lambda > -1$ then

$$I(x,y) = \frac{1 - \left(\frac{x - xy}{1 + \lambda y}\right)^p}{1 + \lambda \left(\frac{x - xy}{1 + \lambda y}\right)^p}.$$

This implication function does not necessarily verify the properties of Definition 9.

2. If $G(x,y) = \frac{2xy}{x+y}$, then

$$I(x,y) = N\left(\frac{2xN(y)}{x + N(y)}\right).$$

verifies the property (IC) because

$$I(x, N(x)) = N(G(x,x)) = N(x).$$

However, it does not necessarily verify any other property of Definition 9.

5. Conclusions

In this work we have studied the construction of negations and implication operators through a particular class of aggregation functions, namely, overlap functions. The analysis that we have carried out can be extended to the dual of overlap functions, grouping functions, and most of the results will just need a rephrasing.

However, this is only the first step of a more ambitious program. Since overlap and grouping functions can provide, in some situations (where associativity is not required), a replacement of t-norms and t-conorms, it seems natural to develop an extensive study on both residual implications and (S,N)-implications defined in terms of overlap and grouping functions, with an eye kept on possible applications, specially in the field of image processing.

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