Extended 2-tuple linguistic hybrid aggregation operators and their application to multi-attribute group decision making

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Abstract

The aim of this paper is to develop some new 2-tuple linguistic hybrid aggregation operators, which are called the extended 2-tuple linguistic hybrid arithmetical weighted (ET-LHAW) operator, the extended 2-tuple linguistic hybrid geometric mean (ET-LHGM) operator, the induced ET-LHAW (IET-LHAW) operator and the induced ET-LHGM (IET-LHGM) operator. These operators do not only consider the importance of the elements but also reflect the importance of their ordered positions. Meantime, some desirable properties are studied, such as idempotency, boundary, etc. When the information about linguistic weight vectors is partly known, the models for the optimal linguistic weight vectors on an expert set, on an attribute set and on their ordered sets are established, respectively. Moreover, an approach to multi-attribute group decision making under linguistic environment is developed. Finally, a numerical example is offered to verify the developed method and to demonstrate its practicality and feasibility.

Keywords: multi-attribute group decision making; uncertain linguistic value; aggregation operator; TOPSIS method

1. Introduction

In the real world there are many situations in which problems are too complex or too ill-defined to be amenable for description in conventional quantitative expressions. The use of numerical based modelling to represent such problems is not always adequate. Often the experts that take part in this type of problems use linguistic descriptors to express their assessments regarding the uncertain knowledge they have about the problem [1-11]. A number of studies have recently focused on using the linguistic variables to model the problems and have produced successful results in different fields [12-21].

In the process of multi-attribute decision making, the linguistic decision information needs to be aggregated by means of some proper approaches so as to rank the given decision alternatives and then to select the most desirable one(s). Herrera and Martínez [5] developed the 2-tuple linguistic arithmetic mean (TLAM) operator, the 2-tuple linguistic weighted averaging (TLWA) operator and the 2-tuple linguistic ordered weighted averaging (TLOWA) operator. Jiang and Fan [22] defined the 2-tuple linguistic weighted geometric (TLWG) operator and the 2-tuple linguistic ordered weighted geometric (TLOWG) operator. Zhang and Fan [23] proposed the extended 2-tuple linguistic ordered weighted averaging (ET-LOWA) operator. Wei et al. [24] presented the 2-tuple linguistic hybrid weighted averaging (T-LHWA) operator. Xu and Huang [25] introduced the 2-tuple linguistic weighted geometric averaging (TLWGA) operator, the 2-tuple linguistic ordered weighted geometric averaging (TLOWGA) operator, and the 2-tuple linguistic hybrid geometric averaging (TLHGA) operator. Wei [26] defined the extended 2-tuple linguistic weighted geometric (ET-LWG) operator and the extended 2-tuple linguistic ordered weighted geometric (ET-LOWG) operator. Xu and Wang [27] gave the 2-tuple linguistic power average (2TLPA) operator and the 2-tuple linguistic power...
ordered weighted average (2TLPOWA) operator. Meanwhile, Wei [28] developed the generalized 2-tuple weighted average (G-2TWA) operator, the generalized 2-tuple linguistic ordered weighted average (G-2TLOWA) operator, and the induced generalized 2-tuple linguistic ordered weighted average (IG-2TLOWA) operator.

As Xu and Da [29] have noted, the ordered weighted operator only considers the importance of the ordered positions, while the weighted operator only gives the importance of elements. In order to both consider the importance of the elements and that of their ordered positions, Xu and Da [30, 31] defined the hybrid weighted averaging (HWA) operator and the hybrid weighted geometric mean (HWGM) operator. After the pioneer works of Xu and Da [30, 31], many hybrid aggregation operators are presented [24, 25, 32-34]. However, all above mentioned hybrid aggregation operators do not satisfy boundary or idempotency, which are desirable properties for aggregating a finite collection of arguments.

If the weight vector is exactly known, then we can use some aggregation operator to get the best choice(s). Sometimes, however, we are only able to provide uncertain information about the weights because of time pressure, lack of knowledge, or data, and their limited expertise related to the problem domain. In these cases, it would be necessary to develop a model to obtain such weight information.

In order to eliminate the issue in the existing hybrid operators, this study develops some new 2-tuple linguistic hybrid aggregation operators called the extended 2-tuple linguistic hybrid arithmetical weighted (ET-LHAW) operator, the induced extended 2-tuple linguistic hybrid arithmetical weighted (IET-LHAW) operator, the extended 2-tuple linguistic hybrid geometric mean (ET-LHGM) operator, and the induced extended 2-tuple linguistic hybrid geometric mean (IET-LHGM) operator. These operators do not only consider the importance of the elements but also reflect the importance of their ordered positions. Furthermore, based on distance deviation method, TOPSIS method and mean deviation method, the models for the optimal linguistic weight vectors are established, respectively. In Sect. 5, an approach to multi-attribute group decision making under linguistic environment is developed. In Sect. 6, an illustrative example is presented to verify the developed method. In Sect. 7, the conclusions are made.

2. Preliminaries

Let \( S = \{ s_i \mid i = 1, 2, \ldots, t \} \) be a linguistic term set with odd cardinality. Any label, \( s_i \), represents a possible value for a linguistic variable, and it should satisfy the following characteristics [4]:

1. The set is ordered: \( s_i > s_j \) if \( i > j \);
2. Max operator: \( \max(s_i, s_j) = s_i \) if \( s_i \geq s_j \);
3. Min operator: \( \min(s_i, s_j) = s_j \) if \( s_i \leq s_j \);
4. A negation operator: \( \neg(s_i) = s_j \) such that \( j = t-i \).

For example, \( S = \{ s_1; \text{extremely poor}, s_2; \text{very poor}, s_3; \text{poor}, s_4; \text{fair}, s_5; \text{good}, s_6; \text{very good}, s_7; \text{extremely good} \} \).

Herrera and Martínez [5] developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple \((s_i, \alpha_i)\), where \( s_i \) is a linguistic label from predefined linguistic term set \( S \) and \( \alpha_i \) is the value of symbolic translation with \( \alpha_i \in [-0.5, 0.5] \).

**Definition 2.1** [5] Let \( \beta \) be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set \( S \), i.e., the result of a symbolic aggregation operation, \( \beta \in [1, t] \), being \( t \) the cardinality of \( S \). Let \( i = \lfloor \text{round}(\beta) \rfloor \) and \( \alpha = \beta - i \) be two values, such that, \( i \in [1, t] \) and \( \alpha \in [-0.5, 0.5] \), then \( \alpha \) is called a symbolic translation.

**Definition 2.2** [5] Let \( S = \{ s_1, s_2, \ldots, s_t \} \) be a linguistic term set, and \( \beta \in [1, t] \) be a number value representing the aggregation result of linguistic symbolic, then the 2-tuple that expresses the equivalent information to \( \beta \) is obtained with the following function \( \Delta \):
\[ \Delta: [1, t] \rightarrow S = [-0.5, 0.5), \]
\[
\Delta(\beta) = (s_i, a_i), \text{ with } \begin{cases} 
    s_i = i - \text{round}(\beta), \\
    a_i = \beta - i, \quad a_i \in [-0.5, 0.5), 
\end{cases}
\]

where round(\( \beta \)) is the usual round operation, \( s_i \) has the closest index label to \( \beta \) and \( a_i \) is the value of the symbolic translation.

**Definition 2.3** [5] Let \( S = \{ s_1, s_2, \ldots, s_l \} \) be a linguistic term set, and \( (s_i, a_i) \) be a 2-tuple. There is always a function \( \Delta^{-1} \):

\[ \Delta^{-1}: S = [-0.5, 0.5) \rightarrow [1, t], \]

\[ \Delta^{-1}(s_i, a_i) = i + a_i = \beta. \]

From Definitions 2.1 and 2.2, we can conclude that the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value 0 as symbolic translation:

\[ \Delta(s_i) = (s_i, 0). \]

Let \( (s_i, a_i) \) and \( (s_i, a_2) \) be any two 2-tuples, Herrera and Martínez [5] gave the following relationship between them.

1. If \( k < l \) then \( (s_i, a_i) \) is smaller than \( (s_i, a_2) \), denoted \( (s_i, a_i) < (s_i, a_2) \).
2. If \( k = l \), then
   a. If \( a_1 = a_2 \), then \( (s_i, a_i) \) and \( (s_i, a_2) \) represent the same information, denoted \( (s_i, a_1) = (s_i, a_2) \).
   b. If \( a_1 < a_2 \), then \( (s_i, a_1) \) is smaller than \( (s_i, a_2) \), denoted \( (s_i, a_1) < (s_i, a_2) \).
   c. If \( a_1 > a_2 \), then \( (s_i, a_1) \) is bigger than \( (s_i, a_2) \), denoted \( (s_i, a_1) > (s_i, a_2) \).

Xu and Da [29] found the ordered weighted operator only considers the importance of the ordered positions, but does not give the importance of their own. Based on the OWA operator [35] and the weighted arithmetic averaging (WAA) operator [36], Xu and Da [30] developed the hybrid weighted averaging (HWA) operator, which is defined as follows:

**Definition 2.4** [30] A HWA operator of dimension \( n \) is a mapping HWA: \( R^n \rightarrow R \), which has an associated weight vector \( w = (w_1, w_2, \ldots, w_n)^T \) on \( N = \{1, 2, \ldots, n\} \) such that \( w_j \in [0,1], \sum_{j=1}^{n} w_j = 1 \), denoted by

\[ \text{HWA}_{\omega} (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \]

where \( b_j \) is the \( j \)th largest value of the weighted argument \( n \omega, a_j \) \( (j = 1, 2, \ldots, n) \), \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( a_j \) \( (i=1,2,\ldots,n) \) with \( \omega_i > 0 \), \( \sum_{i=1}^{n} \omega_i = 1 \), and \( n \) is the balancing coefficient.

Based on the geometric mean (GM) operator and the OWG operator [37], Xu and Da [31] further developed the hybrid weighted geometric mean (HWGM) operator to aggregate the arguments in a similar way to the HWA operator as follows:

**Definition 2.5** [31] A HWGM operator of dimension \( n \) is a mapping HWGM: \( R^n \rightarrow R \), which has an associated weight vector \( w = (w_1, w_2, \ldots, w_n)^T \) such that \( w_j \in [0,1], \sum_{j=1}^{n} w_j = 1 \), denoted by

\[ \text{HWGM}_{\omega} (a_1, a_2, \ldots, a_n) = \Pi_{j=1}^{n} b_j^{w_j}, \]

where \( b_j \) is the \( j \)th largest value of the weighted arguments \( a_j^{w_j} \) \( (j = 1, 2, \ldots, n) \), \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( a_j \) \( (i=1,2,\ldots,n) \) with \( \omega_i > 0 \), \( \sum_{i=1}^{n} \omega_i = 1 \), and \( n \) is the balancing coefficient.

In the real world, there are many situations in which problems must deal with vague and imprecise information that usually involves uncertainty in their definition frameworks. It is hard to provide numerical precise information of the weights of elements and their ordered positions when the knowledge is vague. Often the experts that take part in this type of problems use linguistic descriptors to express their assessments regarding the uncertain knowledge they have about the problem.

Later, Xu [32] defined the linguistic hybrid geometric averaging (LHGA) operator; Xu and Huang [25] proposed the 2-tuple hybrid geometric averaging (THGA) operator; Wei et al. [24] presented the 2-tuple hybrid weighted averaging (T-HWA) operator; Xu [33] introduced the uncertain linguistic hybrid aggregation (ULHA) operator, whilst Wei [34] defined the uncertain linguistic hybrid geometric mean (ULHGM) operator. But all these operators can only be used in the setting of the weights of elements and their ordered positions both taking the form of real numbers. Further, all above mentioned hybrid aggregation operators do not satisfy boundary or idempotency, which are desirable properties for aggregating a finite collection of arguments. For example, let \( S = \{ s_i \mid i = 1,2,3,4,5 \} \) and \( X = \{ s_2, s_3, s_4, s_5 \} \). Assume that \( w = \{0.4, 0.3, 0.2, 0.1\} \) and \( \omega = \{0.1, 0.2, 0.3, 0.4\} \). By the HWA operator, it has \( \text{HWA}_{\omega} = (s_2, s_3, s_4, s_5) = s_{2,2} > s_5 \); by the HWGM operator, it

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has HWGM_{w,\alpha} = (s_2, s_3, s_4, s_5) = s_{5.6} > s_5. Furthermore, by the HWA operator, it has HWA_{w,\alpha} = (s_1, s_2, s_2, s_2) = s_{2,2} \neq s_{2,2}^\prime; by the HWGM operator, it has HWGM_{w,\alpha} = (s_2, s_2, s_2, s_2) = s_{2,2} \neq s_{2,2}.

3. New extended 2-tuple linguistic hybrid aggregation operators

In this section, we define four new extended 2-tuple linguistic hybrid aggregation operators for the group decision-making problems, in which the weights of experts, attributes, their ordered positions as well as the attribute preference values all take the form of linguistic information.

Based on the extended 2-tuple weighted average (ET-WA) operator [5] and the extended 2-tuple ordered weighted averaging (ET-OWA) operator [23], we define the following extended 2-tuple linguistic hybrid arithmetical weighted (ET-LHAW) operator:

**Definition 3.1** Let \( X = \{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\} \) be a set of 2-tuple linguistic variables, the ET-LHAW operator of dimension \( n \) is a mapping ET-LHAW: \( \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated 2-tuple linguistic weight vector \( hw = ((r_1, g_1), (r_2, g_2), \ldots, (r_n, g_n)) \), denoted by \((s^*, \alpha^*) = \text{ET-LHAW}_{w,\alpha}((s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n))\)

\[
=s^* = \sum_{j=1}^{n} \frac{\Delta^{-1}(r_j, g_j) \Delta^{-1}(t_j, \gamma_j)}{\sum_{j=1}^{n} \Delta^{-1}(r_j, g_j) \Delta^{-1}(\alpha_j, \epsilon_j)}
\]

where \( \Delta^{-1}(t_j, \gamma_j) \) is the \( j \)-th largest value of the weighted values \( \Delta^{-1}(s_j, \alpha_j) \Delta^{-1}(\alpha_j, \epsilon_j), (i=1,2,\ldots,n) \), \( lo\omega = ((\alpha_1, \epsilon_1), (\alpha_2, \epsilon_2), \ldots, (\alpha_n, \epsilon_n)) \) is the 2-tuple linguistic weight vector on \( X \).

Just like any hybrid aggregation operator, in Definition 3.1, \( hw \) is a 2-tuple linguistic weight vector defined on the ordered set, while \( lo\omega \) is a 2-tuple linguistic weight vector defined on the element set.

**Remark 3.1** If \((r_i, g_i) = (r_j, g_j)\) for all \( i, j = 1,2,\ldots,n \) with \( i \neq j \), then the ET-LHAW operator reduces to the ET-WA operator [5], and if \((\alpha_i, \epsilon_i) = (\alpha_j, \epsilon_j)\) for all \( i, j = 1,2,\ldots,n \) with \( i \neq j \), then the ET-LHAW operator degenerates to the ET-OWA operator [23].

By extending the extended 2-tuple weighted geometric (ET-WG) operator and the extended 2-tuple ordered weighted geometric (ET-OWG) operator [26], we introduce the following extended 2-tuple linguistic hybrid geometric mean (ET-LHGM) operator:

**Definition 3.2** Let \( X = \{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\} \) be a set of 2-tuple linguistic variables, the ET-LHGM operator of dimension \( n \) is a mapping ET-LHGM: \( \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated 2-tuple linguistic weight vector \( hw = ((r_1, g_1), (r_2, g_2), \ldots, (r_n, g_n)) \), denoted by \((s^*, \alpha^*) = \text{ET-LHGM}_{w,\alpha}((s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n))\)

\[
=s^* = \sum_{j=1}^{n} \frac{\Delta^{-1}(r_j, g_j) \Delta^{-1}(t_j, \gamma_j)}{\sum_{j=1}^{n} \Delta^{-1}(r_j, g_j) \Delta^{-1}(\alpha_j, \epsilon_j)}
\]

where \( \Delta^{-1}(t_j, \gamma_j) \) is the \( j \)-th largest value of the weighted values \( \Delta^{-1}(s_j, \alpha_j) \Delta^{-1}(\alpha_j, \epsilon_j), (i=1,2,\ldots,n) \), \( lo\omega = ((\alpha_1, \epsilon_1), (\alpha_2, \epsilon_2), \ldots, (\alpha_n, \epsilon_n)) \) is the 2-tuple linguistic weight vector on \( X \).

**Remark 3.2** If \((r_i, g_i) = (r_j, g_j)\) for all \( i, j = 1,2,\ldots,n \) with \( i \neq j \), then the ET-LHGM operator reduces to the ET-WG operator [9], and if \((\alpha_i, \epsilon_i) = (\alpha_j, \epsilon_j)\) for all \( i, j = 1,2,\ldots,n \) with \( i \neq j \), then the ET-LHGM operator degenerates to the ET-OWG operator [26].

From the induced uncertain linguistic ordered weighted geometric (IULOWG) operator [38], Definitions 3.1 and 3.2, we further introduce the induced extended 2-tuple linguistic hybrid arithmetical weighted (IET-LHAW) operator and the induced extended 2-tuple linguistic hybrid geometric mean (IET-LHGM) operator:

**Definition 3.3** Let \( X = \{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\} \) be a set of 2-tuple linguistic variables, the IET-LHAW operator of dimension \( n \) is a mapping IET-LHAW: \( \mathbb{R}^n \rightarrow \mathbb{R} \) defined on the set of second arguments of two tuples \(<u_1, (s_1, \alpha_1)> <u_2, (s_2, \alpha_2)> \ldots, <u_n, (s_n, \alpha_n)>\)

\[
\hat{\alpha} = \text{IET-LHAW}_{w,\alpha}((u_1, (s_1, \alpha_1)), (u_2, (s_2, \alpha_2)), \ldots, (u_n, (s_n, \alpha_n)))
\]

\[
\hat{s} = \Delta \left( \sum_{j=1}^{n} \Delta^{-1}(r_j, g_j) \Delta^{-1}(t_{j(i)}, \gamma_{j(i)}) \right)
\]

\[
\hat{s} \in S, \hat{\alpha} \in [-0.5, 0.5],
\]

(5)
where $u_{ij}$ is the $j$th largest value of $u_i$ ($i=1,2,\ldots,n$), \(\Delta^{-1}(t_j, r_j)\) is the weighted value \(\Delta^{-1}(s_j, \alpha_j)\Delta^{-1}(\omega_j, \varepsilon_j)\) \((i=1, 2, \ldots, n)\), \(hw=((r_1, g_1), (r_2, g_2), \ldots, (r_n, g_n))\) is the 2-tuple linguistic weight vector on ordered set \(N=\{1, 2, \ldots, n\}\), and \(lw=((\omega_1, \varepsilon_1), (\omega_2, \varepsilon_2), \ldots, (\omega_n, \varepsilon_n))\) is the 2-tuple linguistic weight vector on $X$.

Definition 3.4 Let $X=\{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\}$ be a set of 2-tuple linguistic variables, the IET-LHGM operator of dimension $n$ is a mapping \(\text{IET-LHGM}: \mathbb{R}^n \rightarrow \mathbb{R}\) defined on the set of second arguments of two tuples $<u_i, (s_i, \alpha_i)>, <u_j, (s_j, \alpha_j)>, \ldots, <u_n, (s_n, \alpha_n)>$, with a set of order-inducing variables $u_i(i=1, 2, \ldots, n)$, denoted by $\bigl(\hat{s}^*, \hat{\alpha}^*\bigr)=\text{IET-LHGM}_{\text{lkw}}\bigl(\langle u_i, (s_i, \alpha_i)\rangle, \langle u_j, (s_j, \alpha_j)\rangle, \ldots, \langle u_n, (s_n, \alpha_n)\rangle\bigr)$ with a set of order-inducing variables $u_j(i=1, 2, \ldots, n)$, \(\Delta^{-1}(t_j, r_j)\) is the weighted value \(\Delta^{-1}(s_j, \alpha_j)\Delta^{-1}(\omega_j, \varepsilon_j)\) \((i=1, 2, \ldots, n)\), \(hw=((r_1, g_1), (r_2, g_2), \ldots, (r_n, g_n))\) is the 2-tuple linguistic weight vector on ordered set \(N=\{1, 2, \ldots, n\}\), and \(lw=((\omega_1, \varepsilon_1), (\omega_2, \varepsilon_2), \ldots, (\omega_n, \varepsilon_n))\) is the 2-tuple linguistic weight vector on $X$.

However, if there is a tie between $<u_i, (s_i, \alpha_i)>$ and $<u_j, (s_j, \alpha_j)>$, we replace these by $k$ replicates of their average.

Remark 3.3: If $u_i=\Delta^{-1}(s_i, \alpha_i)\Delta^{-1}(\omega_i, \varepsilon_i)$, then the IET-LHAW operator degenerates to the ET-LHAW operator, and if $u_i=\Delta^{-1}(s_i, \alpha_i)\Delta^{-1}(\alpha_i, \varepsilon_i)$, the IET-LHGM operator reduces to the ET-LHGM operator. Thus, the ET-LHAW and ET-LHGM operators can be seen a special case of the IET-LHAW and IET-LHGM operators, respectively.

In the following, we study some desirable properties of these four operators.

Proposition 3.1 (Commutativity) Let $X=\{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\}$ be a set of 2-tuple linguistic variables, and $(s'_i, \alpha'_i) (i=1, 2, \ldots, n)$ be a permutation of $(s_i, \alpha_i)$. Then

\[\begin{align*}
\text{IET-LHAW}_{\text{lkw}}\bigl(\langle u_i, (s_i, \alpha_i)\rangle, \langle u_j, (s_j, \alpha_j)\rangle, \ldots, \langle u_n, (s_n, \alpha_n)\rangle\bigr) &= \text{IET-LHAW}_{\text{lkw}}\bigl(\langle u_i, (s'_i, \alpha'_i)\rangle, \langle u_j, (s'_j, \alpha'_j)\rangle, \ldots, \langle u_n, (s'_n, \alpha'_n)\rangle\bigr) \\
\text{IET-LHGM}_{\text{lkw}}\bigl(\langle u_i, (s_i, \alpha_i)\rangle, \langle u_j, (s_j, \alpha_j)\rangle, \ldots, \langle u_n, (s_n, \alpha_n)\rangle\bigr) &= \text{IET-LHGM}_{\text{lkw}}\bigl(\langle u_i, (s'_i, \alpha'_i)\rangle, \langle u_j, (s'_j, \alpha'_j)\rangle, \ldots, \langle u_n, (s'_n, \alpha'_n)\rangle\bigr).
\end{align*}\]

Corollary 3.1 (Idempotency) Let $X=\{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\}$ and \(X'\) be sets of 2-tuple linguistic variables, and $(s'_i, \alpha'_i) (i=1, 2, \ldots, n)$ be a permutation of $(s_i, \alpha_i)$. Then \(X=\{(s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)\}\) be two sets of 2-tuple linguistic variables. If $(s_i, \alpha_i) \leq (s'_i, \alpha'_i)$ for each $i=1, 2, \ldots, n$, then
\[
\begin{align*}
\text{IET-LHAW}_{\text{lkw}}\bigl(\langle u_i, (s_i, \alpha_i)\rangle, \langle u_j, (s_j, \alpha_j)\rangle, \ldots, \langle u_n, (s_n, \alpha_n)\rangle\bigr) &\leq \text{IET-LHAW}_{\text{lkw}}\bigl(\langle u_i, (s'_i, \alpha'_i)\rangle, \langle u_j, (s'_j, \alpha'_j)\rangle, \ldots, \langle u_n, (s'_n, \alpha'_n)\rangle\bigr) \\
\text{IET-LHGM}_{\text{lkw}}\bigl(\langle u_i, (s_i, \alpha_i)\rangle, \langle u_j, (s_j, \alpha_j)\rangle, \ldots, \langle u_n, (s_n, \alpha_n)\rangle\bigr) &\leq \text{IET-LHGM}_{\text{lkw}}\bigl(\langle u_i, (s'_i, \alpha'_i)\rangle, \langle u_j, (s'_j, \alpha'_j)\rangle, \ldots, \langle u_n, (s'_n, \alpha'_n)\rangle\bigr).
\end{align*}\]
The interval linguistic weight vector is

\[ \text{IET-LHGM}_{i/o} < u_i(s_i, \alpha_i), \ldots, u_s(s_e, \alpha_e) > \]

**Corollary 3.3** Let \( X=\{s_1, \alpha_1\}, \ldots, \{s_n, \alpha_n\} \) be a set of 2-tuple linguistic variables. If \( \{s_i, \alpha_i\} = (\sigma, \alpha) \) for each \( i=1, 2, \ldots, n \), then

ET-LHAW_{i/o} ((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) = (\sigma, \alpha)

and

ET-LHGM_{i/o} ((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) = (\sigma, \alpha).

**Proposition 3.4** (Boundary) Let \( X=\{s_1, \alpha_1\}, \ldots, \{s_n, \alpha_n\} \) be a set of 2-tuple linguistic variables, then

\[
\min((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) \leq \text{IET-LHAW}_{i/o} ((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) \leq \max((s_1, \alpha_1), \ldots, (s_n, \alpha_n))
\]

and

\[
\min((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) \leq \text{IET-LHGM}_{i/o} ((s_1, \alpha_1), \ldots, (s_n, \alpha_n)) \leq \max((s_1, \alpha_1), \ldots, (s_n, \alpha_n)).
\]

**Corollary 3.4** Let \( X=\{s_1, \alpha_1\}, \ldots, \{s_m, \alpha_m\} \) be a set of 2-tuple linguistic variables, then

\[
\min((s_1, \alpha_1), \ldots, (s_m, \alpha_m)) \leq \text{IET-LHAW}_{i/o} ((s_1, \alpha_1), \ldots, (s_m, \alpha_m)) \leq \max((s_1, \alpha_1), \ldots, (s_m, \alpha_m))
\]

and

\[
\min((s_1, \alpha_1), \ldots, (s_m, \alpha_m)) \leq \text{IET-LHGM}_{i/o} ((s_1, \alpha_1), \ldots, (s_m, \alpha_m)) \leq \max((s_1, \alpha_1), \ldots, (s_m, \alpha_m)).
\]

**4. The models for the optimal linguistic weight vectors**

Based on distance deviation method, TOPSIS method and mean deviation method, we research the models for the optimal linguistic weight vectors.

Consider a multi-attribute group decision-making problem, in which both the weights and the attribute preference values take the form of linguistic variables. Let \( A=\{a_1, a_2, \ldots, a_m\} \) be the set of alternatives, \( C=\{c_1, c_2, \ldots, c_n\} \) be the set of attributes, and \( E=\{e_1, e_2, \ldots, e_g\} \) be the set of experts. Assume that \( \tilde{Q}^h = (q^h)_{mn} \) is the decision matrix, where \( q^h \in S \) is a preference value, which takes the form of linguistic variables, given by the expert \( e_h \), for the alternative \( a_i \in A \) w.r.t. the attribute \( c_j \in C \).

The linguistic weight vectors be evaluated using the linguistic term set \( Z \). For example, \( Z=\{z_1, z_2, z_3, z_4, z_5, z_6, z_7\} \) is the linguistic term set.

**4.1. The model for the weight vector on the expert set**

For each attribute \( c_j (j=1, 2, \ldots, n) \), calculate the distance \( d_{ij}^h \) between the expert \( e_h \) and each expert in \( E\{e_k\} \) as follows:

\[
d_{ij}^h = \sum_{l=1}^{q} \sum_{i=1}^{m} |\Delta^{-1}(q^h_{ij}, 0) - \Delta^{-1}(q^h_{ij}, 0)|.
\]

Based on distance deviation method, we build the following model for the optimal linguistic weight vector on the expert set \( E \) for the attribute \( c_j (j=1, 2, \ldots, n) \).

\[
\min \sum_{i=1}^{m} d_{ij}^h \cdot \Delta^{-1}(z_{ij}^h, \alpha_{ij}^h)
\]

s.t. \( \Delta^{-1}(z_{ij}^h, \alpha_{ij}^h) \in [\Delta^{-1}(z_{ij}^h, 0), \Delta^{-1}(z_{ij}^h, 0)], e_i \in E \). (7)

It is equivalent to

\[
\min \lambda \left\{ \sum_{i=1}^{m} d_{ij}^h \cdot \Delta^{-1}(z_{ij}^h, \alpha_{ij}^h) - \lambda \sum_{i=1}^{m} \Delta^{-1}(z_{ij}^h, \alpha_{ij}^h) = 0 \right\}
\]

(8)

s.t. \( \Delta^{-1}(z_{ij}^h, \alpha_{ij}^h) \in [\Delta^{-1}(z_{ij}^h, 0), \Delta^{-1}(z_{ij}^h, 0)], e_i \in E \)

The optimal linguistic weights obtained by this
method have the following desirable characteristics: the closer an expert’s evaluation values are to other experts’, the larger the weight measure is. This can avoid unduly high or low evaluation values induced by experts’ limited knowledge or expertise.

4.2. The model for the weight vector on the ordered set \( Q \)

Consider the linguistic weights on the ordered set \( Q = \{1, 2, \ldots, q\} \) for each pair \((i, j)\) \((i=1, 2, \ldots, m; j=1, 2, \ldots, n)\). Calculate the arithmetic averaging value \( d_y \) for each pair \((i, j)\) \((i=1, 2, \ldots, m; j=1, 2, \ldots, n)\) as follows:

\[
d_y = \frac{1}{q} \sum_{i=1}^{q} \Delta^{-1}(q_y^i, 0),
\]

and compute the distance \( d_y^k \) between \( \Delta^{-1}(q_y^k, 0) \) and \( d_y \) as follows:

\[
d_y^k = |\Delta^{-1}(q_y^k, 0) - d_y| \quad \forall k \in Q.
\]

Then, reorder \( d_y^k \) \((k \in Q)\) in decreasing order \( d_y^{(1)} \geq d_y^{(2)} \geq \ldots \geq d_y^{(q)} \). Based on mean deviation method, we establish the following model for the optimal linguistic weight vector on the ordered set \( Q \) for each pair \((i, j)\) \((i=1, 2, \ldots, m; j=1, 2, \ldots, n)\).

\[
\min \sum_{i=1}^{q} d_y^k \Delta^{-1}(z, \alpha) \sum_{i=1}^{n} \Delta^{-1}(z, \alpha)
\]

s.t. \( \Delta^{-1}(z, \alpha) \in \Delta^{-1}(z, \alpha_i), k \in Q \) \((9)\). It is equivalent to

\[
\min \lambda
\]

s.t. \[
\sum_{i=1}^{q} d_y^k \Delta^{-1}(z, \alpha) - \lambda \sum_{i=1}^{n} \Delta^{-1}(z, \alpha) = 0 \] \((10)\).

If there is more than one \( d_y^k \) equal, then we reorder their positions according to their subscripts in increasing order.

4.3. The model for the weight vector on the attribute set

Based on TOPSIS method \([40]\), we introduce the following model for the optimal linguistic weight vector on attribute set \( C \).

Let \( \hat{Q} = (q_y, \alpha)_{mn} \) be the comprehensive 2-tuple linguistic decision matrix. Transform the interval linguistic weight vector

\[
l\hat{w}_y = \{(z_{i1}, z_{i2}), (z_{i1}, z_{i2}), \ldots, (z_{in}, z_{in})\}
\]
on attribute set \( C \) and the interval linguistic weight vector

\[
l\hat{w}_y = \{(z_{i1}, z_{i2}), (z_{i1}, z_{i2}), \ldots, (z_{in}, z_{in})\}
\]
on ordered set \( N \) for each \( i=1, 2, \ldots, m \) into the 2-tuple interval linguistic weight vectors

\[
l\hat{w}_y = \{(z_{i1}, 0), (z_{i1}, 0), \ldots, (z_{in}, 0), (z_{in}, 0)\}
\]

and

\[
l\hat{w}_y = \{(z_{i1}, 0), (z_{i1}, 0), \ldots, (z_{in}, 0), (z_{in}, 0)\}
\]

with \( z_{i1}, \ldots, z_{in}, z_{i1}, \ldots, z_{in} \in N \) for each \( j=1, 2, \ldots, n \).

Let \( \hat{R}_i = (\hat{r}_i^+, \ldots, \hat{r}_i^-) \) and \( \hat{R}_j = (\hat{r}_j^+, \ldots, \hat{r}_j^-) \) be the positive and negative ideal vectors, respectively, where \( \hat{r}_i^+ = \max(q_y, \alpha)_i \) and \( \hat{r}_j^- = \min(q_y, \alpha)_i \) for all \( j=1, 2, \ldots, n \).

For each \( i=1, 2, \ldots, m \), calculate the distances \( d_i^+ \) and \( d_i^- \) as follows:

\[
d_i^+ = \Delta^{-1}(q_y, \alpha) - \Delta^{-1}(\hat{r}_i^-),
\]

\[
d_i^- = \Delta^{-1}(q_y, \alpha) - \Delta^{-1} \hat{r}_i^+.
\]

Since all alternatives are non inferior, we build the following model for the optimal linguistic weight vector on the attribute set \( C \).

\[
\min \sum_{i=1}^{m} \sum_{j=1}^{n} d_i^+ \Delta^{-1}(z, \alpha) + d_i^- \sum_{i=1}^{n} \Delta^{-1}(z, \alpha)
\]

s.t. \( \Delta^{-1}(z, \alpha) \in \Delta^{-1}(z, \alpha_i), c_j \in C \). \((11)\).

It is equivalent to

\[
\min \lambda
\]

s.t. \[
\sum_{i=1}^{m} \sum_{j=1}^{n} d_i^+ \Delta^{-1}(z, \alpha) - \lambda \sum_{i=1}^{n} \Delta^{-1}(z, \alpha) = 0 \]

\[
\Delta^{-1}(z, \alpha) \in \Delta^{-1}(z, \alpha_i), c_j \in C \]. \((12)\)

4.4. The model for the weight vector on the ordered set \( N \)

Similar to the model for the optimal linguistic weight vector on the ordered set \( Q \), we introduce the following model for the optimal linguistic weight vector on the ordered set \( N \) for each \( i=1, 2, \ldots, m \).

For each \( i=1, 2, \ldots, m \), calculate the arithmetic average

\[
d_i = \frac{1}{n} \sum_{i=1}^{n} \Delta^{-1}(q_y, \alpha)
\]

and compute the distance \( d_i^+ \) between \( \Delta^{-1}(q_y, \alpha) \)
and \( d_j \) as follows:

\[
 d'_j = \Delta^{-1}(q_j, \alpha_j) - d_j
\]

Then, reorder \( d'_j \) in decreasing order \( d'_{(j)} \geq d'_{(j+1)} \geq \ldots \geq d'_{(n)} \). If there is more than one \( d'_j \) equal, then we reorder their positions according to their subscripts in increasing order.

Based on mean deviation method, we established the following model for the optimal linguistic weight vector on the ordered set \( N \) for each \( i = 1, 2, \ldots, m \).

\[
\min \sum_{j=1}^{n} d'_{(j)} \frac{\Delta^{-1}(z'_j, \alpha'_j)}{\sum_{j=1}^{n} \Delta^{-1}(z'_j, \alpha'_j)}
\]

s.t. \( \Delta^{-1}(z'_j, \alpha'_j) \in \{ \Delta^{-1}(z_j, 0), \Delta^{-1}(z_j, 0) \} \), \( j \in N \). (13)

It is equivalent to

\[
\min \lambda \left\{ \sum_{j=1}^{n} d'_{(j)} \Delta^{-1}(z'_j, \alpha'_j) - \lambda \sum_{j=1}^{n} \Delta^{-1}(z'_j, \alpha'_j) = 0 \right\}
\]

\[
\Delta^{-1}(z'_j, \alpha'_j) \in \{ \Delta^{-1}(z_j, 0), \Delta^{-1}(z_j, 0) \}, \ j \in N
\]

5. A new approach to multi-attribute group decision making under linguistic environment

This section develops an approach to multi-attribute group decision making under linguistic environment. The main decision procedure can be described as follows:

**Step 1:** Transform the linguistic decision matrix \( \tilde{Q} = (q_{ij})_{mn} \) into 2-tuple linguistic decision matrix \( \tilde{Q}' = (q'_{ij})_{mn} \), and convert the interval linguistic weight vector

\[
\tilde{\omega}_E = \left[ (z_{b_1}, z_{c_1}), (z_{b_2}, z_{c_2}), \ldots, (z_{b_n}, z_{c_n}) \right]
\]

on the expert set \( E \) for each attribute \( c_j \) \( (j=1, 2, \ldots, n) \) and the interval linguistic weight vector

\[
\tilde{\omega}_O = \left[ (z_{b_1}, z_{c_1}), (z_{b_1}, z_{c_1}), \ldots, (z_{b_1}, z_{c_1}) \right]
\]

on the ordered set \( O = \{1, 2, \ldots, q\} \) into the 2-tuple interval linguistic weight vectors

\[
\tilde{\omega}'_E = \left[ (z'_{b_1}, 0), (z'_{c_1}, 0), (z'_{b_1}, 0), (z'_{c_1}, 0), \ldots, (z'_{b_1}, 0), (z'_{c_1}, 0) \right]
\]

and

\[
\tilde{\omega}'_O = \left[ (z_{b_1}, 0), (z_{c_1}, 0), (z_{b_1}, 0), (z_{c_1}, 0), \ldots, (z_{b_1}, 0), (z_{c_1}, 0) \right],
\]

with \( z'_{b_k}, z'_{c_k}, z_{b_k}, z_{c_k} \in Z \) for each \( k \in Q \).

**Step 2:** Utilize the model (8) to calculate the optimal linguistic weight vector on the experts set \( E \) for the attribute \( c_j \) \( (j=1, 2, \ldots, n) \).

**Step 3:** Utilize the model (10) to calculate the optimal linguistic weight vector on the ordered set \( O \) for each pair \((i, j)(i=1, 2, \ldots, m; j=1, 2, \ldots, n)\).

**Step 4:** Use the IET-LHAW operator or the IET-LHGM operator to calculate the comprehensive 2-tuple linguistic decision matrix \( \tilde{Q} = (q_{ij})_{mn} \), where \((z'_{b_j}, z'_{c_j})\) \( (k \in Q) \) is the optimal 2-tuple linguistic weight of the expert \( c_j \) \( (j=1, 2, \ldots, n) \), and \((z'_{b_j}, z'_{c_j})\) is the optimal 2-tuple linguistic weight of the \( k \)th position of the pair \((i, j)(i=1, 2, \ldots, m; j=1, 2, \ldots, n)\).

**Step 5:** Transform the interval linguistic weight vector

\[
\tilde{\omega}_E = \left[ (z_{b_1}, z_{c_1}), (z_{b_2}, z_{c_2}), \ldots, (z_{b_n}, z_{c_n}) \right]
\]

on the attribute set \( C \) and the interval linguistic weight vector

\[
\tilde{\omega}_O = \left[ (z_{b_1}, z_{c_1}), (z_{b_1}, z_{c_1}), \ldots, (z_{b_1}, z_{c_1}) \right]
\]

on the ordered set \( N \) for each \( i = 1, 2, \ldots, m \) into the 2-tuple interval linguistic weight vectors

\[
\tilde{\omega}_E = \left[ (z_{b_1}, 0), (z_{c_1}, 0), (z_{b_1}, 0), (z_{c_1}, 0), \ldots, (z_{b_1}, 0), (z_{c_1}, 0) \right]
\]

and

\[
\tilde{\omega}_O = \left[ (z_{b_1}, 0), (z_{c_1}, 0), (z_{b_1}, 0), (z_{c_1}, 0), \ldots, (z_{b_1}, 0), (z_{c_1}, 0) \right],
\]

with \( z_{b_1}, z_{c_1}, z_{b_1}, z_{c_1} \in Z \) for each \( j \in N \).

**Step 6:** Utilize the model (12) to calculate the optimal linguistic weight vector on the attribute set \( C \).

**Step 7:** Utilize the model (14) to calculate the optimal linguistic weight vector on the ordered set \( Q \) for each \( i = 1, 2, \ldots, m \).

**Step 8:** Again use the IET-LHAW operator or the IET-LHGM operator to compute the comprehensive 2-tuple linguistic values \((s_{ij}, \alpha_{ij})\) \( (i=1, 2, \ldots, m) \), where \((z'_{b_j}, z'_{c_j})\) \( (j \in N) \) is the optimal 2-tuple linguistic weight of the attribute \( c_{ij} \) and \((z'_{b_j}, z'_{c_j})\) is the optimal 2-tuple linguistic weight of the \( j \)th position.
6. A practical example

Let us suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from Ref. [41]). There is a panel with five possible alternatives \( A = \{a_1, a_2, a_3, a_4, a_5\} \) to invest the money: \( a_1 \) is a car company; \( a_2 \) is a food company; \( a_3 \) is a computer company; \( a_4 \) is an arms company; \( a_5 \) is a TV company. The investment company must take a decision according to the following four attributes \( C = \{c_1, c_2, c_3, c_4\} \): \( c_1 \) is the risk analysis; \( c_2 \) is the growth analysis; \( c_3 \) is the social-political impact analysis; \( c_4 \) is the environmental impact analysis. The five possible alternatives \( a_i (i = 1,2,3,4,5) \) are to be evaluated using the linguistic term set \( S = \{s_i\} \): extremely poor, \( s_2 \): very poor, \( s_3 \): poor, \( s_4 \): fair, \( s_5 \): very good, \( s_6 \): extremely good. By three experts \( E = \{e_1, e_2, e_3\} \) under the above four attributes, and construct the decision matrices \( \hat{Q}^k = (q_{ij}^k)_{4 \times 4} \) \((k = 1,2,3)\) as follows:

\[
\hat{Q}^1 =
\begin{pmatrix}
  c_1 & c_2 & c_3 & c_4 \\
  s_4 & s_5 & s_1 & s_3 & a_1 \\
  s_3 & s_2 & s_4 & s_3 & a_2 \\
  s_2 & s_3 & s_5 & s_3 & a_3 \\
  s_1 & s_4 & s_5 & s_3 & a_4 \\
  \end{pmatrix}
\]

\[
\hat{Q}^2 =
\begin{pmatrix}
  c_1 & c_2 & c_3 & c_4 \\
  s_3 & s_4 & s_2 & s_2 & a_1 \\
  s_2 & s_3 & s_5 & s_5 & a_2 \\
  s_1 & s_4 & s_5 & s_5 & a_3 \\
  s_2 & s_3 & s_5 & s_5 & a_4 \\
  \end{pmatrix}
\]

\[
\hat{Q}^3 =
\begin{pmatrix}
  c_1 & c_2 & c_3 & c_4 \\
  s_3 & s_3 & s_2 & s_8 & a_1 \\
  s_2 & s_3 & s_5 & s_2 & a_2 \\
  s_1 & s_3 & s_5 & s_2 & a_3 \\
  s_2 & s_3 & s_5 & s_2 & a_4 \\
  \end{pmatrix}
\]

The weight vectors on the expert set \( E \), on the ordered set \( Q = \{1,2,3\} \), on the attribute set \( C \) and on the ordered set \( N = \{1,2,3,4\} \) are all evaluated using the linguistic term set \( Z = \{z_i\} \): extremely unimportant, \( z_2 \): very unimportant, \( z_3 \): unimportant; \( z_4 \): fair, \( z_5 \): very important, \( z_6 \): extremely important. Assume that the experts’ weight vectors are given by

\[
l\vec{a}_i = ([z_2,z_3],[z_2,z_3],[z_2,z_3],[z_5,z_6]),
\]

\[
l\vec{b}_i = ([z_3,z_6],[z_3,z_6],[z_5,z_6],[z_5,z_6]),
\]

\[
l\vec{c}_i = ([z_1,z_4],[z_4,z_1],[z_2,z_3],[z_3,z_5]),
\]

and the weight vector on the ordered set \( Q \) is defined by

\[
l\vec{\omega}_0 = ([z_2,z_3],[z_2,z_3],[z_2,z_3],[z_2,z_3]).
\]

According to the judgment of the three experts, the weights of attributes are known as

\[
l\vec{\omega}_k = ([z_1,z_4],[z_4,z_1],[z_2,z_3],[z_3,z_5]),
\]

and the weights of the ordered positions are known as

\[
l\vec{\omega}_k = ([z_2,z_3],[z_2,z_3],[z_2,z_3],[z_2,z_3]) \text{ for each } k = 1,2,3,4,5.
\]

In the following, we can utilize the proposed procedure to get the most desirable investment company(s).

**Step 1:** Transform the linguistic decision matrices \( \hat{Q}^k = (q_{ij}^k)_{4 \times 4} \) into the 2-tuple linguistic decision matrices

\[
\hat{Q}^1 =
\begin{pmatrix}
  (s_4,0) & (s_5,0) & (s_1,0) & (s_3,0) \\
  (s_3,0) & (s_2,0) & (s_4,0) & (s_3,0) \\
  (s_2,0) & (s_4,0) & (s_1,0) & (s_3,0) \\
  (s_1,0) & (s_4,0) & (s_5,0) & (s_3,0) \\
  \end{pmatrix}
\]

\[
\hat{Q}^2 =
\begin{pmatrix}
  (s_3,0) & (s_4,0) & (s_2,0) & (s_2,0) \\
  (s_2,0) & (s_3,0) & (s_5,0) & (s_3,0) \\
  (s_1,0) & (s_4,0) & (s_5,0) & (s_3,0) \\
  (s_2,0) & (s_3,0) & (s_5,0) & (s_3,0) \\
  \end{pmatrix}
\]

\[
\hat{Q}^3 =
\begin{pmatrix}
  (s_3,0) & (s_3,0) & (s_2,0) & (s_3,0) \\
  (s_3,0) & (s_3,0) & (s_2,0) & (s_3,0) \\
  (s_3,0) & (s_3,0) & (s_2,0) & (s_3,0) \\
  (s_3,0) & (s_3,0) & (s_2,0) & (s_3,0) \\
  \end{pmatrix}
\]

Transform the interval linguistic weight vectors on the expert set \( E \) for each attribute \( c_j \) \((j = 1,2,3,4)\) into the 2-tuple linguistic weight vectors

\[
l\vec{\omega}_k = ([z_2,0),(z_3,0)],(z_2,0),(z_3,0)],(z_2,0),(z_4,0))
\]
Step 3: Form the model (10), we get the following optimal linguistic weight vector on the ordered set $Q$ into the 2-tuple linguistic weight vector 

$$lW_Q = \left(\left(\frac{(z_0,0),(z_0,0)}{((z_0,0),(z_0,0))}\right)\right)_{i=1}^{s_{4}}$$

and transform the interval linguistic weight vector on the ordered set $Q$ into the 2-tuple linguistic weight vector 

$$lW_Q = \left(\left(\frac{(z_0,0),(z_0,0)}{((z_0,0),(z_0,0))}\right)\right)_{i=1}^{s_{4}}$$

Step 4: Let $u_i = d_i^{(k=1,2,3)}$ for each pair $(i, j)$ $(i=1, 2, 3, 4; j=1,2,3,4)$. Use the IET-LHAW operator, we get the comprehensive 2-tuple linguistic value $(q_i, \alpha_i)$, e.g., $i=j=1$, 

$$(q_{i1}, \alpha_{i1}) = \text{IET-LHAW}_{(q_{i1}, \alpha_{i1})} \left(\frac{(z_0,0),(z_0,0)}{((z_0,0),(z_0,0))}\right)$$

Step 5: Transform the interval linguistic weight vector on the attribute set $C$ into the 2-tuple linguistic weight vector 

$$lW_C = \left(\left(\frac{(z_0,0),(z_0,0)}{((z_0,0),(z_0,0))}\right)\right)_{i=1}^{s_{4}}$$

and the interval linguistic weight vector on the ordered set $N$ for each $i (i=1,2,3,4,5; j=1,2,3,4)$. From $\hat{Q}$ and $\hat{R}$, we get the distance matrices $D' = (d_i')_{s_{4}}$ and $D^* = (d_i^*)_{s_{4}}$ as follows:
Solve the above model, it has

\[ \text{min } \lambda \]

\[ 2.25 \Delta^{-1}(z_{i,1}, \alpha_i) + 2.11 \Delta^{-1}(z_{i,2}, \alpha_i) + 2.31 \Delta^{-1}(z_{i,3}, \alpha_i) \]

\[ + 2.48 \Delta^{-1}(z_{i,4}, \alpha_i) - \lambda \sum_{j=1}^{4} \Delta^{-1}(z_{i,j}, \alpha_i) = 0 \]

\[ \Delta^{-1}(z_{i,1}, \alpha_i) \in [5,6], \Delta^{-1}(z_{i,2}, \alpha_i) \in [4,6], \]

\[ \Delta^{-1}(z_{i,3}, \alpha_i) \in [3,4], \Delta^{-1}(z_{i,4}, \alpha_i) \in [5,7] \]

Solve the above model, it has

\[ \Delta^{-1}(z_{i,1}, \alpha_i) = 5, \Delta^{-1}(z_{i,2}, \alpha_i) = 4, \]

\[ \Delta^{-1}(z_{i,3}, \alpha_i) = 3, \Delta^{-1}(z_{i,4}, \alpha_i) = 5. \]

Step 7: From the model (14), we get the following model for the optimal linguistic weight vector on the ordered set \( N \) for \( i = 1 \):

\[ \text{min } \lambda \]

\[ 0.93 \Delta^{-1}(z_{i,1}', \alpha_i') + 0.74 \Delta^{-1}(z_{i,2}', \alpha_i') + 0.49 \Delta^{-1}(z_{i,3}', \alpha_i') \]

\[ + 0.3 \Delta^{-1}(z_{i,4}', \alpha_i') - \lambda \sum_{j=1}^{4} \Delta^{-1}(z_{i,j}', \alpha_i') = 0 \]

\[ \Delta^{-1}(z_{i,1}', \alpha_i') \in [1,2], \Delta^{-1}(z_{i,2}', \alpha_i') \in [2,3], \]

\[ \Delta^{-1}(z_{i,3}', \alpha_i') \in [3,4], \Delta^{-1}(z_{i,4}', \alpha_i') \in [5,6] \]

Solve the above model, it has

\[ \Delta^{-1}(z_{i,1}', \alpha_i') = 1, \Delta^{-1}(z_{i,2}', \alpha_i') = 2, \]

\[ \Delta^{-1}(z_{i,3}', \alpha_i') = 3, \Delta^{-1}(z_{i,4}', \alpha_i') = 6. \]

Similar to the calculating of the optimal linguistic weight vector on the ordered set \( N \) for \( i = 1 \), we get the following optimal linguistic weight vectors on the ordered set \( N \) for \( i = 2,3,4,5 \).

\[ \Delta^{-1}(z_{i,1}', \alpha_i') = 1, \Delta^{-1}(z_{i,2}', \alpha_i') = 2, \Delta^{-1}(z_{i,3}', \alpha_i') = 3, \Delta^{-1}(z_{i,4}', \alpha_i') = 5, \]

\[ \Delta^{-1}(z_{i,1}', \alpha_i') = 1, \Delta^{-1}(z_{i,2}', \alpha_i') = 2, \Delta^{-1}(z_{i,3}', \alpha_i') = 3, \Delta^{-1}(z_{i,4}', \alpha_i') = 6, \]

\[ \Delta^{-1}(z_{i,1}', \alpha_i') = 1, \Delta^{-1}(z_{i,2}', \alpha_i') = 2, \Delta^{-1}(z_{i,3}', \alpha_i') = 3, \Delta^{-1}(z_{i,4}', \alpha_i') = 6, \]

Step 8: Let \( u_{ij} = d_{ij} \) (\( i = 1,2,3,4,5 \)). By the IET-LHAW operator, we obtain the comprehensive 2-tuple linguistic value \((q_i, \alpha_i)\) of alternative \( a_i \), e.g. \( i = 1 \),

\[ (q_1, \alpha_1) = \text{IET-LHAW}_{\alpha_{1,i}}, \]

\[ < u_{11}, (q_{11}, \alpha_{11}) >, < u_{12}, (q_{12}, \alpha_{12}) >, \]

\[ < u_{13}, (q_{13}, \alpha_{13}) >, < u_{14}, (q_{14}, \alpha_{14}) > \]

\[ = (s_{1}, 0.45) \]

Similar to the calculation of \((q_i, \alpha_i)\), we get the comprehensive 2-tuple linguistic values \((q_i, \alpha_i)\) of alternatives \( a_i \) (\( i = 2,3,4,5 \)) as follows:

\[ (q_2, \alpha_2) = (s_2, 0.23), (q_3, \alpha_3) = (s_3, 0.04), \]

\[ (q_4, \alpha_4) = (s_4, 0.11), (q_5, \alpha_5) = (s_5, 0.8) \]

Step 9: According to the comprehensive 2-tuple linguistic values \((q_i, \alpha_i)\) (\( i = 1,2,3,4,5 \)) and the relationship between 2-tuple linguistic arguments [5], it has

\[ (q_i, \alpha_i) < (q_i, \alpha_i) < (q_i, \alpha_i) < (q_i, \alpha_i) < (q_i, \alpha_i) \]

Namely, \( a_5 \) (arms company) is the best choice.

In the following, we use the IET-LHGM operator to get the best choice\( (s) \) in the above example.

Step1’-Step3’: See Step1-Step3.

Step 4’: Let \( u_{ik} = d_{ik} \) (\( k = 1,2,3,4 \)) for each pair \((i, j) \) (\( i = 1,2,3,4,5, \) \( j = 1,2,3,4 \)). Use the IET-LHGM operator, we obtain the comprehensive 2-tuple linguistic value \((q_{ij}', \alpha_{ij}')\), e.g. \( i = 1 \),

\[ (q_{ij}', \alpha_{ij}') = \text{IET-LHGM}_{\alpha_{ij}}, \]

\[ < u_{ij}, (q_{ij}', \alpha_{ij}') > \]

\[ = (s_{ij}', 0.16) \]

Similar to the calculation of \((q_{ij}', \alpha_{ij}')\), we get the following comprehensive 2-tuple linguistic matrix

\[ \tilde{Q}' = \begin{pmatrix} (s_{1i}, 0.16) & (s_{2i}, 0.06) & (s_{3i}, 0.45) & (s_{4i}, 0.04) \\ (s_{12}, 0.27) & (s_{22}, 0.27) & (s_{32}, 0.06) & (s_{42}, 0.39) \\ (s_{13}, -0.15) & (s_{23}, -0.45) & (s_{33}, -0.41) & (s_{43}, 0.19) \\ (s_{14}, 0.02) & (s_{24}, 0.33) & (s_{34}, 0.15) & (s_{44}, -0.01) \\ (s_{15}, -0.21) & (s_{25}, -0.46) & (s_{35}, -0.44) & (s_{45}, 0.24) \end{pmatrix} \]

Step 5’: See Step 5.

Step 6’: By the model (12), we get the following model for the optimal linguistic weight vector on the attribute set \( C \).

\[ \text{min } \lambda \]
Solve the above model, it has comprehensive 2-tuple linguistic values $(\ell_1, \ell_2)$ and $(\ell_3, \ell_4)$ ordered set following optimal linguistic weight vectors on the alternative.

Similar to the calculation of the optimal linguistic weight vector on the ordered set $\{1,2,3,4,5\}$, from the model (14), we get the following optimal linguistic weight vector on the ordered set $S = \{1,2,3,4,5\}$ for each $i = 1,2,3,4,5$.

Step 7: From the model (14), we get the following optimal linguistic weight vector on the ordered set $S = \{1,2,3,4,5\}$ for each $i = 1,2,3,4,5$.

Similar to the calculation of the optimal linguistic weight vector on the ordered set $S = \{1,2,3,4,5\}$ for each $i = 1,2,3,4,5$.

Step 8: Let $u_i = d_{i,j}^u$ ($j = 1,2,3,4$) for each $i = 1,2,3,4,5$. By the IET-LHGM operator, we obtain the following optimal linguistic weight vector on the ordered set $S = \{1,2,3,4,5\}$ for each $i = 1,2,3,4,5$.

Step 9: According to the comprehensive 2-tuple linguistic values $(\ell_1, \ell_2)$, $(\ell_3, \ell_4)$ and the relationship between 2-tuple linguistic arguments [5], it gets $(\ell_1, \ell_2) < (\ell_3, \ell_4) < (\ell_1, \ell_2) < (\ell_3, \ell_4) < (\ell_1, \ell_2)$. Namely, $a_4$ (arms company) is the best choice.

In this example, by the IET-LHAW and IET-LHGM operators we obtain the slightly different ranking results, but the best choice is the alternative $a_4$ in both the cases.

7. Conclusions

From the given example in Section 2, we know that if the HWA or HWGM operator is directly applied to aggregate a set of linguistic variables for the predefined linguistic term set $S$, then it cannot guarantee the aggregation value fall in the scope of $S$. As we know, the same number’s arithmetic or geometric average equals to itself. However, for the HWA or HWGM operator this conclusion does not hold, which contradicts people’s intuition.

We have defined four aggregation operators for 2-tuple linguistic variables. These operators do not only consider the importance of the elements but also reflect the importance of their ordered positions. It is worth pointing out that most of the existing linguistic aggregation operators are special cases of our operators. When the information about linguistic weight vectors is partly known, the models for the optimal linguistic weight vectors on the expert set, on the attribute set and on their ordered sets are built, respectively. An approach to multi-attribute group decision making under linguistic environment is developed, where the weights of experts, attributes and their ordered positions all take the form of linguistic arguments. Finally, a financial decision-making problem under linguistic environment has been given to illustrate our approach.

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