

Cooperative Fuzzy Games with a Coalition Structure and Interval Payoffs

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Abstract

Based on the extension Hukuhara difference between interval numbers, a generalized form of cooperative fuzzy games with a coalition structure and interval payoffs is proposed, which can be seen as an extension of crisp case. The interval Owen value for this kind of fuzzy games is studied, and its explicit form is given. When the fuzzy games are convex, the proposed interval Owen value is an interval population monotonic allocation function (IPMAF), and belongs to the associated core. Furthermore, we discuss a special kind of fuzzy games with a coalition structure and interval payoffs, and study the interval Owen value and the core of it. Some properties are also examined, which are coincided with the crisp case.

Keywords: Cooperative fuzzy game, Coalition structure, Interval Owen value, Extension Hukuhara difference

1. Introduction

The Shapley value (see [1]) is a well-known solution concept in cooperative game theory. Some researchers (see [2-6]) who have investigated the Shapley value have focused their attention on fuzzy coalition and real-valued characteristic function.

In some situations, players can only know imprecise information regarding the real outcome of cooperation. For this kind of games, Mares [7] and Vlach [8] researched games with crisp coalitions and fuzzy payoffs. Borkotokey [9] and Yu and Zhang [10] studied games with fuzzy coalitions and fuzzy payoffs. Since interval numbers can conveniently describe the lower and upper bounds of the coalition values. Alparslan Gök et al. [11, 12] and Branzei et al. [13] researched games

with crisp coalitions and interval payoffs. Due to the limitation of fuzzy set operations, the discussions in [10-12] are all based on the Hukuhara difference (see [14]) for fuzzy sets directly or indirectly.

In 1977, Owen [15] introduced games with a priori unions which are known as the Owen coalition structures where the probability of cooperation among coalitions is considered. By extending the Shapley value for traditional games, Owen [15] proposed the Owen value for games with a coalition structure. The axiomatic systems of the Owen value are considered in [16-19]. Later, Owen [20] gave another payoff index for games with a coalition structure, which is called the Banzhaf-Owen value, and discussed its axiomatic system. More researches about axiomatic characterizations of the Banzhaf-Owen value can be

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seen in [21, 22]. Alonso-Meijide and Fiestras-Janeiro [23] pointed the Banzhaf-Owen value dissatisfies symmetry in quotient games, and gave another solution concept for games with a coalition structure, which is known as the symmetric Banzhaf value. But all above researches are only introduced for crisp case.

In this paper, based on the extension Hukuhara difference between interval numbers, we will research cooperative fuzzy games with a coalition structure and interval payoffs. As we will see, the extension Hukuhara difference between interval numbers can be used to more situations than the Hukuhara difference. Inspired by Sprumont [24], we show the given interval Owen value is an IPMAF when the fuzzy games are convex.

This paper is organized as follows: In the next section, we recall some notations and basic definitions, which will be used in the following. In section 3, an axiomatic definition of the interval Owen value for fuzzy games with a coalition structure and interval payoffs is proposed, and the explicit form of the interval Owen value is given. Meantime, some properties are researched. In section 4, we pay an attention to discuss the interval Owen value for a special kind of fuzzy games with a coalition structure and interval payoffs, which can be seen as an extension of fuzzy games introduced by Tsurumi et al. [4].

2. Preliminaries

2.1. The Extension Hukuhara Difference between Interval Numbers

Let \mathbb{R} and \mathbb{R}_+ be the sets of real numbers and positive real numbers, respectively.

Definition 2.1 Let $\bar{a} = [a^-, a^+]$, if $a^- \leq a^+$ and $a^-, a^+ \in \mathbb{R}$, then \bar{a} is said to be an interval number; if $a^- \leq a^+$ and $a^-, a^+ \in \mathbb{R}_+$, then \bar{a} is said to be a positive interval number.

Let $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ be two interval numbers. In order to ensure that $\bar{a} - \bar{b}$ is an interval number, the difference between \bar{a} and \bar{b} is defined as $\bar{a} - \bar{b} = [a^- - b^+, a^+ - b^-]$.

In generally, we can not have $\bar{a} - \bar{b} + \bar{b} = \bar{a}$. The Hukuhara difference for fuzzy sets, proposed by Banks and Jacobs [14], can well deal with this problem. When it is applied to interval numbers, the Hukuhara difference is defined by:

Definition 2.2 Let \bar{a} and \bar{b} be two interval numbers, if there exists an interval number \bar{c} such that $\bar{a} = \bar{b} + \bar{c}$, then \bar{c} is called the Hukuhara difference between \bar{a} and \bar{b} , denoted by $\bar{a} -_H \bar{b} = [a^- - b^-, a^+ - b^+]$.

For example, let $\bar{a} = [2, 5]$ and $\bar{b} = [4, 6]$, since there exists the interval number $\bar{c} = [-2, -1]$ such that $[2, 5] = [4, 6] + [-2, -1]$. We get $\bar{a} -_H \bar{b} = [-2, -1]$, which is obviously different to $\bar{a} - \bar{b} = [-4, 1]$.

Definition 2.3 Let $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ be two interval numbers, \bar{a} is said to be an “imaginary” interval number if $a^- > a^+$, and $\bar{a} -_H \bar{b}$ is said to the “imaginary” Hukuhara difference between \bar{a} and \bar{b} if $a^- - b^- > a^+ - b^+$.

For example, $\bar{a} = [5, 3]$ is an “imaginary” interval number. Let $\bar{a} = [2, 5]$ and $\bar{b} = [4, 8]$. Since there does not exist an interval number \bar{c} such that $\bar{a} = \bar{b} + \bar{c}$, we can not use the Hukuhara difference. If we do not consider the condition given in Definition 2.2, and adopt the Hukuhara difference between \bar{a} and \bar{b} . Then we get $\bar{a} -_H \bar{b} = [-2, -3]$, which is a so-called “imaginary” interval number. And the Hukuhara difference between \bar{a} and \bar{b} is said to the “imaginary” Hukuhara difference

Definition 2.4 Let $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ be two interval numbers, then $\bar{a} -_{eH} \bar{b}$ is called the extension Hukuhara difference between \bar{a} and \bar{b} , where $\bar{a} -_{eH} \bar{b} = [a^- - b^-, a^+ - b^+]$.

From Definition 2.4, we know when there exists an interval number \bar{c} such that $\bar{a} = \bar{b} + \bar{c}$, then their extension Hukuhara difference is equal to their Hukuhara difference. Otherwise, the extension Hukuhara difference between \bar{a} and \bar{b} is equal to their “imaginary” Hukuhara difference.

For convenience, in this paper, the sets of positive interval numbers and positive imaginary interval numbers are denoted by $\bar{\mathbb{R}}_+$. Moreover, the difference between interval numbers adopts the extension Hukuhara difference. If there is no fear of confliction, we still use $\bar{a} -_H \bar{b}$ to denote the extension Hukuhara difference between the interval numbers in $\bar{\mathbb{R}}_+$.

Definition 2.5 Let \bar{a} and \bar{b} be two interval numbers, we write

- (1) $\bar{a} \geq \bar{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$;
- (2) $\bar{a} = \bar{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

2.2. Some Basic Concepts on Crisp Games with a Coalition Structure

Let $N=\{1,2,\dots,n\}$ be the set of players. The crisp coalitions in N are denoted by S_0, \dots , where $S_0 = \{i \mid i \in S_0 \text{ s.t. } i \in N\}$, i.e., S_0 is formed by the players belong to it. The power set of all crisp subsets in N is denoted by $P(N)$. For any $S_0 \in P(N)$, the cardinality of S_0 is denoted by low case s . A function $v_0 : P(N) \rightarrow \mathbb{R}_+$, satisfying $v_0(\emptyset) = 0$, is called a set function. By $G_0(N)$, we denote the set of all set functions in N .

A crisp coalition structure $\Pi = \{B_0^1, B_0^2, \dots, B_0^m\}$ in N is a partition of N , i.e., $\cup_{1 \leq j \leq m} B_0^j = N$ and $B_0^i \cap B_0^j = \emptyset$ for all $i, j \in M = \{1, 2, \dots, m\}$ with $i \neq j$. A crisp coalition structure in N is denoted by (N, Π) . For any $S_0 \in P(N, \Pi)$, S_0 is called a feasible coalition, where $P(N, \Pi)$ denotes the set of all feasible coalitions in (N, Π) .

Owen [15] gave the Owen value for crisp games with a coalition structure as follows:

$$\beta_i(N, v_0, \Pi) = \sum_{R \subseteq M \setminus k} \sum_{S_0 \subseteq B_0^k \setminus i} \alpha_m^r \beta_{b_k}^s (v_0(S_0 \cup Q_0 \cup i) - v_0(S_0 \cup Q_0)) \quad \forall i \in N, \quad (1)$$

where $\alpha_m^r = \frac{r!(m-r-1)!}{m!}$, $\beta_{b_k}^s = \frac{s!(b_k-s-1)!}{b_k!}$, m and r

denote the cardinalities of M and R , respectively, $Q_0 = \cup_{p \in R} B_0^p$ and $\Pi = \{B_0^1, B_0^2, \dots, B_0^m\}$.

2.3. Some Basic Concepts for Fuzzy Games with a Coalition Structure and Interval Payoffs

The set of all fuzzy coalitions in N is denoted by $L(N)$. The fuzzy coalitions in $L(N)$ are denoted by S, \dots , i.g., $S = \{S(i) \mid i \in N\}$, where $S(i)$ indicates the membership grade of i in S , i.e., the rate of the i th player in S . For any $S \in L(N)$, the support is denoted by $\text{Supp} S = \{i \in N \mid S(i) > 0\}$, and the cardinality is denoted by $|\text{Supp} S|$. We use the notation $S \subseteq T$ if and only if $S(i) = T(i)$ or $S(i) = 0$ for all $i \in N$. For all $S, T \in L(N)$, $S \vee T$ denotes the union of fuzzy coalitions S and T , namely, $i \in \text{Supp}(S \vee T)$ if and only if $i \in \text{Supp} S \cup \text{Supp} T$ and $(S \vee T)(i) = S(i) \vee T(i)$; $S \wedge T$ denotes the intersection of fuzzy coalitions S and T , namely, $i \in \text{Supp}(S \wedge T)$ if and only if $i \in \text{Supp} S \cap \text{Supp} T$ and $(S \wedge T)(i) = S(i) \wedge T(i)$.

In the following, we will use $S = \{S(i_1), S(i_2), \dots, S(i_n)\}$ to denote the fuzzy coalition $S \in L(N)$. Furthermore, the braces will be omitted for singletons, e.g. by writing S ,

$S \vee (\wedge) T, U(i)$ instead of $\{S\}, \{S\} \vee (\wedge) \{T\}, \{U(i)\}$ for any $\{S\}, \{T\}, \{U(i)\} \in L(N)$, where U is a fuzzy coalition in N .

Similar to the definition of the coalition structure for crisp case introduced by Owen [15], we give the definition of the coalition structure for fuzzy coalitions as follows:

Definition 2.6 Let $U \in L(N)$, Γ is said to be a coalition structure on U , if Γ satisfies

- (1) $\vee_{k=1}^m B_k = U$,
- (2) $\text{Supp} B_p \cap \text{Supp} B_q = \emptyset \quad \forall p, q \in \{1, 2, \dots, m\} \text{ s.t. } p \neq q$,

where $\Gamma = \{B_1, B_2, \dots, B_m\}$ and $1 \leq m \leq |\text{Supp} U|$, denoted by (U, Γ) .

For any (U, Γ) , the set of the feasible coalitions in (U, Γ) is denoted by $L(U, \Gamma)$.

Example 2.1 Let $N = \{1, 2, 3, 4\}$, and $U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in $L(N)$ such that $U(i) > 0$ for any $i \in N$. $\Gamma = \{B_1, B_2\}$ is a fuzzy coalition structure in U , where $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$, then $L(U, \Gamma) = \{\emptyset, \{U(i)\}_{i \in N}, \{U(1), U(2)\}, \{U(3), U(4)\}, \{U(1), U(2), U(3)\}, \{U(1), U(2), U(4)\}, \{U(1), U(3), U(4)\}, \{U(2), U(3), U(4)\}, U\}$.

A function $\bar{v} : L(U, \Gamma) \rightarrow \bar{\mathbb{R}}_+$, satisfying $\bar{v}(\emptyset) = [0, 0]$, is called an interval set function. All interval set functions on $L(U, \Gamma)$ are denoted by $IG(U, \Gamma)$.

Example 2.2 In Example 2.1, if $U(1) = 0.3, U(2) = 0.8, U(3) = 0.5$ and $U(4) = 0.6$, and the interval set function \bar{v} is defined as follows:

$$\bar{v}(\emptyset) = [0, 0], \bar{v}(U(1)) = [2, 3], \bar{v}(U(2)) = [1, 2], \bar{v}(U(3)) = [2, 4], \bar{v}(U(4)) = [3, 6], \bar{v}(U(1), U(2)) = [5, 7], \bar{v}(U(3), U(4)) = [9, 12], \bar{v}(U(1), U(2), U(3)) = [10, 13], \bar{v}(U(1), U(2), U(4)) = [12, 15], \bar{v}(U(1), U(3), U(4)) = [13, 15], \bar{v}(U) = [20, 22].$$

Then it is an interval set function belongs to $IG(U, \Gamma)$.

Definition 2.7 Let $\bar{v} \in IG(U, \Gamma)$, if $\bar{v}^B(R_0) = \bar{v}(\vee_{r \in R_0} B_r)$ for any $R_0 \subseteq M_0$, then \bar{v}^B is said to be an interval quotient game in (U, Γ) , where $\Gamma = \{B_1, B_2, \dots, B_m\}$ and $M_0 = \{1, 2, \dots, m\}$, denoted by (M_0, \bar{v}^B) .

Definition 2.8 Let $\bar{v} \in IG(U, \Gamma)$ and $T \in L(U, \Gamma)$, if we

have $\bar{v}(T \wedge S) = \bar{v}(S)$ for any $S \in L(U, \Gamma)$, then T is called a carrier for \bar{v} in $L(U, \Gamma)$.

3. The Interval Owen Value for Fuzzy Games with a Coalition Structure and Interval Payoffs

For a given $\bar{v} \in IG(U, \Gamma)$. According to the work of Owen [15], we define the function $\varphi : IG(U, \Gamma) \rightarrow \overline{\mathbb{R}}_+^{|\text{Supp}U|}$ as follows:

$$\varphi_i(U, \bar{v}, \Gamma) = \sum_{R_0 \subseteq M_0 \setminus k} \sum_{S \subseteq B_k, i \in \text{Supp}S} \alpha_m^r \beta_{B_k}^S (\bar{v}(S \vee Q) - \bar{v}((S \setminus U(i)) \vee Q)) \quad \forall i \in \text{Supp}U, \quad (2)$$

where $\beta_{B_k}^S = (|\text{Supp}S| - 1)! (|\text{Supp}B_k| - |\text{Supp}S|)! / |\text{Supp}B_k|!$ and $\alpha_m^r = r!(m - r - 1)! / m!$, m and r respectively denote the cardinalities of M_0 and R_0 , $Q = \bigvee_{i \in R_0} B_i$.

If every player's participation level is 1, and the coalition values are real numbers, then Eq.(2) degenerates to be Eq.(1). Furthermore, if there is only one coalition in Γ , and every player's participation level is 1, then Eq.(2) degenerates to be the Shapley value given by Alparslan Gök et al.[12].

Example 3.1 Let $N = \{1, 2, 3, 4\}$, $U = \{U(i)\}_{i \in N}$ with $U(i) = 0.3$ ($i \in N$), and $\Gamma = \{B_1, B_2\}$, where $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$. The fuzzy coalition values are given by

$$\begin{aligned} \bar{v}(U(i)) &= [2, 4], \quad \bar{v}(U(1), U(2)) = \bar{v}(U(3), U(4)) = [8, 9], \\ \bar{v}(U(i), U(j), U(k)) &= [15, 18] \quad (i, j, k \in \{1, 2, 3, 4\}), \\ \bar{v}(U) &= [25, 30]. \end{aligned}$$

Since there does not exist interval number \bar{c} such that $\bar{v}(U(1)) + \bar{c} = \bar{v}(U(1), U(2))$, we can not use the Hukuhara difference in this game. If we adopt the extension Hukuhara difference between interval numbers, by Eq.(2) we have $\varphi_i(U, \bar{v}, \Gamma) = [6.25, 7.25]$ ($i \in N$). Since this is a symmetry game and every player can get more payoffs than that by himself, we know this cooperative game can be formed. Furthermore, it is not difficult to get the vector $(\varphi_i(U, \bar{v}, \Gamma))_{i \in N}$ is an imputation, and an element in the core of this game.

Although the extension Hukuhara difference between interval numbers can be used to situations than the Hukuhara difference, there exist games with interval numbers which can not use the extension Hukuhara difference.

Example 3.2 In Example 3.1, if we take the values of the fuzzy coalitions as listed in table 1.

Table 1 (The interval values of the fuzzy coalitions)

S	$\bar{v}(S)$	S	$\bar{v}(S)$
$\{U(1)\}$	[2,10]	$\{U(1), U(2), U(3)\}$	[28,30]
$\{U(2)\}$	[4,6]	$\{U(1), U(3), U(4)\}$	[28,30]
$\{U(3)\}$	[2,10]	$\{U(1), U(2), U(4)\}$	[24,26]
$\{U(4)\}$	[4,6]	$\{U(2), U(3), U(4)\}$	[24,26]
$\{U(1), U(2)\}$	[16,18]	U	[38,40]
$\{U(3), U(4)\}$	[16,18]		

From Eq.(2), we have

$$\varphi_2(U, \bar{v}, \Gamma) = \varphi_4(U, \bar{v}, \Gamma) = [9, 8]$$

and

$$\varphi_1(U, \bar{v}, \Gamma) = \varphi_3(U, \bar{v}, \Gamma) = [10, 12].$$

Since the payoffs of the players 2 and 4 are "imaginary" interval numbers, which makes no sense.

In order to avoid the above situation, in this paper, for any $\bar{v} \in IG(U, \Gamma)$ we always mean every player's payoff obtained by Eq.(2) is an interval number.

Definition 3.1 Let $\bar{v} \in IG(U, \Gamma)$, a function $f : IG(U, \Gamma) \rightarrow \overline{\mathbb{R}}_+^{|\text{Supp}U|}$ is said to be an interval Owen value for \bar{v} , if it satisfies the following axioms:

Axiom 1: If T is a carrier in U for $\bar{v} \in IG(U, \Gamma)$, then

$$\bar{v}(T) = \sum_{i \in \text{Supp}T} f_i(U, \bar{v}, \Gamma);$$

Axiom 2: Let $B_k \in \Gamma$ and $i, j \in \text{Supp}B_k$, if we have $v(T \vee U(i)) = v(T \vee U(j))$ for any $T \in L(U, \Gamma)$ with $i, j \notin \text{Supp}T$, then $f_i(U, v, \Gamma) = f_j(U, v, \Gamma)$.

Axiom 3: Let $k, s \in M_0 = \{1, 2, \dots, m\}$, if we have $\bar{v}^B(R_0 \cup k) = \bar{v}^B(R_0 \cup s)$ for any $R_0 \subseteq M_0 \setminus \{k, s\}$, then

$$\sum_{i \in \text{Supp}B_k} f_i(U, \bar{v}, \Gamma) = \sum_{j \in \text{Supp}B_s} f_j(U, \bar{v}, \Gamma);$$

Axiom 4: Let $\bar{v}, \bar{w} \in IG(U, \Gamma)$ and all $\alpha, \beta \in \mathbb{R}_+$, we have $f(U, \alpha\bar{v} + \beta\bar{w}, \Gamma) = \alpha f(U, \bar{v}, \Gamma) + \beta f(U, \bar{w}, \Gamma)$.

Theorem 3.1 Let $\bar{v} \in IG(U, \Gamma)$, the function $\varphi : IG(U, \Gamma) \rightarrow \overline{\mathbb{R}}_+^{|\text{Supp}U|}$, defined by Eq.(2), is the unique interval Owen value for \bar{v} in $L(U, \Gamma)$.

Proof. From the properties of the extension Hukuhara difference between interval numbers, we have

$$\bar{a} -_H \bar{b} + \bar{b} = \bar{a}$$

and

$$(\lambda - \beta)\bar{a} = \lambda\bar{a} -_H \beta\bar{a}$$

for all $\bar{a}, \bar{b} \in \bar{\mathbb{R}}_+$ and all $\lambda, \beta \in \mathbb{R}_+$.

According to the proof of the Owen value in crisp case, we can easily get the existence. In the following, we shall show the uniqueness.

For any $\bar{v} \in IG(U, \Gamma)$, we first show \bar{v} can be expressed by

$$\bar{v} = \sum_{\emptyset \neq T \in L(U, \Gamma)} c_T u_T, \tag{3}$$

where $c_T = \sum_{P_0 \subseteq R_0} (-1)_H^{r-p} \left(\sum_{A \subseteq S} (-1)_H^{|\text{Supp}S| - |\text{Supp}A|} \bar{v}(S \vee Q) \right)$,

$T = S \vee_{i \in R_0} B_i$ such that $S \subseteq B_k$ for some $k \in M_0$,

$$R_0 \subseteq M_0 \setminus \{k\} \text{ and } u_T(W) = \begin{cases} 1 & T \subseteq W \\ 0 & \text{otherwise} \end{cases}$$

For any $C \in L(U, \Gamma)$, without loss of generality, suppose $C = D \vee_{i \in E_0} B_i$, where $D \subseteq B_k$ and $E_0 \subseteq M_0$.

$$\begin{aligned} & \left(\sum_{\emptyset \neq T \in L(U, \Gamma)} c_T u_T \right)(C) \\ &= \sum_{\emptyset \neq T \in L(U, \Gamma)} c_T u_T(C) \\ &= \sum_{\emptyset \neq T \in L(U, \Gamma), T \subseteq C} c_T \\ &= \sum_{\emptyset \neq T \in L(U, \Gamma), T \subseteq C} \sum_{P_0 \subseteq R_0} (-1)_H^{r-p} \left(\sum_{A \subseteq S} (-1)_H^{|\text{Supp}S| - |\text{Supp}A|} \bar{v}(S \vee Q) \right) \\ &= \sum_{R_0 \subseteq E_0} \sum_{P_0 \subseteq R_0} (-1)_H^{r-p} \left(\sum_{S \subseteq D} \sum_{A \subseteq S} (-1)_H^{|\text{Supp}S| - |\text{Supp}A|} \bar{v}(S \vee Q) \right) \\ &= \sum_{R_0 \subseteq E_0} \sum_{P_0 \subseteq R_0} (-1)_H^{r-p} \left(\sum_{S \subseteq D} \sum_{t=|\text{Supp}S|}^{|\text{Supp}D|} (-1)_H^{t-|\text{Supp}S|} \right). \end{aligned}$$

Since $\sum_{t=|\text{Supp}S|}^{|\text{Supp}D|} (-1)_H^{t-|\text{Supp}S|} \binom{|\text{Supp}D| - |\text{Supp}S|}{t - |\text{Supp}S|} = 0$ for any

$S \subseteq D$, we get

$$\begin{aligned} & \left(\sum_{\emptyset \neq T \in L(U, \Gamma)} c_T u_T \right)(C) \\ &= \sum_{R_0 \subseteq E_0} \sum_{P_0 \subseteq R_0} (-1)_H^{r-p} \bar{v}(D \vee Q) \\ &= \sum_{R_0 \subseteq E_0} \sum_{t=p}^r (-1)_H^{t-p} \binom{r-p}{t-p} \bar{v}(D \vee Q), \end{aligned}$$

where r and p denote the cardinalities of R_0 and P_0 , respectively.

Since $\sum_{t=p}^r (-1)_H^{t-p} \binom{r-p}{t-p} = 0$ for any $R_0 \subset E_0$, we get

$$\left(\sum_{\emptyset \neq T \in L(U, \Gamma)} c_T u_T \right)(C) = \bar{v}(C).$$

Thus, Eq.(3) holds.

From *Axiom 4*, we only need to prove the uniqueness of Eq.(2) on u_T for any $T \in L(U, \Gamma)$.

Let

$$M'_0 = \{j \in M_0 : \text{Supp}B_j \cap \text{Supp}T \neq \emptyset\}$$

and

$$\text{Supp}B'_j = \text{Supp}B_j \cap \text{Supp}T.$$

Define the unanimity game on (M, u_T^B) as follows:

$$u_T^B(R_0) = \begin{cases} 1 & M'_0 \subseteq R_0 \\ 0 & M'_0 \not\subseteq R_0 \end{cases},$$

where $R_0 \subseteq M_0$.

Let f be a solution on (U, u_T, Γ) that satisfies the above axioms. From *Axiom 1*, we have $f_i(U, u_T, \Gamma) = 0$ for any $i \notin \text{Supp}T$. Furthermore, from *Axioms 1* and *3*, we obtain

$$\sum_{i \in \text{Supp}B_k} f_i(U, u_T, \Gamma) = \begin{cases} 0 & k \notin M'_0 \\ \frac{c_T}{|M'_0|} & k \in M'_0 \end{cases},$$

where $|M'_0|$ denotes the cardinality of M'_0 .

By *Axiom 2*, we obtain

$$f_i(U, u_T, \Gamma) = \begin{cases} 0 & i \notin \text{Supp}T \\ \frac{1}{|M'_0| \|\text{Supp}B'_k|} & i \in \text{Supp}B'_k \end{cases}.$$

On the other hand, by Eq.(2) we have $\varphi_i(U, u_T, \Gamma) = 0$ for any $i \notin \text{Supp}T$. If $i \in \text{Supp}T$, then there exists $k \in M_0$ such that $i \in \text{Supp}B'_k = \text{Supp}B_k \cap \text{Supp}T$, and

$$u_T(Q \vee S \vee U(i)) - u_T(Q \vee S) = 1$$

if and only if

$$Q = \vee_{r \in R_0} B_r,$$

where $M'_0 \setminus k \subseteq R_0 \subseteq M_0 \setminus k$ and $\text{Supp}B'_k \setminus i \subseteq \text{Supp}S \subseteq \text{Supp}B_k \setminus i$.

If $i \in \text{Supp}B'_k$, we get

$$\begin{aligned} \varphi_i(U, u_T, \Gamma) &= \sum_{M'_0 \setminus k \subseteq R_0 \subseteq M_0 \setminus k} \sum_{\substack{\text{Supp}B'_k \setminus i \subseteq \\ \text{Supp}D \subseteq \text{Supp}B_k \setminus i}} \alpha_m^r \beta_{B_k}^S \\ &= \frac{1}{|M'_0| \|\text{Supp}B'_k|}. \end{aligned}$$

Namely, φ and f coincide on u_T . \square

Definition 3.2 Let $\bar{v} \in IG(U, \Gamma)$, \bar{v} is said to be convex if

$$\bar{v}(S \vee T) + \bar{v}(S \wedge T) \geq \bar{v}(T) + \bar{v}(S)$$

for all $S, T \in L(U, \Gamma)$.

Definition 3.3 Let $\bar{v} \in IG(U, \Gamma)$, the vector $\bar{x} = (\bar{x}_i)_{i \in \text{Supp}U}$ is said to be an IPMAF for \bar{v} in $L(U, \Gamma)$, if

- 1) $\sum_{i \in \text{Supp}S} \bar{x}_i(S) = \bar{v}(S) \quad \forall S \in L(U, \Gamma)$,
- 2) $\bar{x}_i(S) \leq \bar{x}_i(T) \quad \forall i \in \text{Supp}S' \quad \forall S, T \in L(U, \Gamma)$
 $s.t. S \subseteq T$,

where $S' \subseteq B_k$ ($k \in M_0$), $S = S' \vee_{r \in R_0} B_r$ and $R_0 \subseteq M_0 \setminus \{k\}$.

Theorem 3.2 Let $\bar{v} \in IG(U, \Gamma)$, if \bar{v} is convex, then $(\varphi_i(U, \bar{v}, \Gamma))_{i \in \text{Supp}U}$ is an IPMAF for \bar{v} in $L(U, \Gamma)$.

Proof. From Theorem 3.1, we only need to show the second condition in Definition 3.3. For all $S, T \in L(U, \Gamma)$, without loss of generality, suppose $S = S' \vee_{r \in R_0^S} B_r$ and $T = T' \vee_{r \in R_0^T} B_r$, where $S' \subseteq T' \subseteq B_k$ and $R_0^S \subseteq R_0^T \subseteq M_0 \setminus \{k\}$.

From the convexity of $\bar{v} \in IG(U, \Gamma)$, we have

$$\bar{v}(T) -_H \bar{v}(T \setminus U(i)) \geq \bar{v}(S) -_H \bar{v}(S \setminus U(i))$$

for any $i \in \text{Supp}S'$.

Case (1). When $R_0^S = R_0^T$ and $|\text{Supp}S'| + 1 = |\text{Supp}T'|$, for any $W \subseteq S'$ we have

$$\beta_{S'}^W = \sum_{K \subseteq T' \setminus S'} \beta_{T'}^{W \vee K},$$

where

$$\beta_{S'}^W = \frac{(|\text{Supp}W| - 1)! (|\text{Supp}S'| - |\text{Supp}W|)!}{|\text{Supp}S'|!}$$

and

$$\beta_{T'}^{W \vee K} = \frac{(|\text{Supp}(W \vee K)| - 1)! (|\text{Supp}T'| - |\text{Supp}(W \vee K)|)!}{|\text{Supp}T'|!}.$$

From Eq.(2), we get

$$\begin{aligned} & \varphi_i(S, \bar{v}, \Gamma_S) \\ &= \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \alpha_{R_0^S}^{H_0} \beta_{S'}^W (\bar{v}(W \vee Q) -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee Q)) \\ & \leq \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \alpha_{R_0^S}^{H_0} \beta_{S'}^W (\bar{v}(W \vee K \vee Q) -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee K \vee Q)) \\ &= \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \alpha_{R_0^S}^{H_0} \sum_{K \subseteq T' \setminus S'} \beta_{T'}^{W \vee K} (\bar{v}(W \vee K \vee Q) -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee K \vee Q)) \end{aligned}$$

$$\begin{aligned} &= \sum_{H_0 \subseteq R_0^S} \alpha_{R_0^S}^{H_0} \sum_{W \subseteq S', i \in \text{Supp}W} \sum_{K \subseteq T' \setminus S'} \beta_{T'}^{W \vee K} (\bar{v}(W \vee K \vee Q) -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee K \vee Q)) \\ &= \sum_{H_0 \subseteq R_0^T} \alpha_{R_0^T}^{H_0} \sum_{W \subseteq T', i \in \text{Supp}W} \beta_{T'}^W (\bar{v}(W \vee Q) -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee Q)) \\ &= \varphi_i(T, \bar{v}, \Gamma_T), \end{aligned}$$

where $\alpha_{R_0^S}^{H_0} = \alpha_{R_0^T}^{H_0} = h!(r^S - h)! / (r^S + 1)!$, $Q = \vee_{l \in H_0} B_l$, r^S and h respectively denote the cardinalities of R_0^S and H_0 , Γ_S and Γ_T respectively denote the restriction of Γ on S and T .

From recursion method, we have

$$\varphi_i(S, \bar{v}, \Gamma_S) \leq \varphi_i(T, \bar{v}, \Gamma_T)$$

for any $S' \subseteq T' \subseteq B_k$ and $R_0^S = R_0^T$.

Case (2). If $|\text{Supp}S'| = |\text{Supp}T'|$ and $r^S + 1 = r^T$, where r^T denotes the cardinality of R_0^T .

For any $H_0 \subseteq R_0^S$, we obtain

$$\alpha_{R_0^S}^{H_0} = \sum_{P_0 \subseteq R_0^T \setminus R_0^S} \alpha_{R_0^S}^{H_0 \cup P_0},$$

where $\alpha_{R_0^S}^{H_0 \cup P_0} = (h + p)!(r^T - h - p)! / (r^T + 1)!$. p denotes the cardinality of P_0 .

From Eq.(2), we have

$$\begin{aligned} & \varphi_i(S, \bar{v}, \Gamma_S) \\ &= \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \alpha_{R_0^S}^{H_0} \beta_{S'}^W (\bar{v}(W \vee Q) -_H \bar{v}((W \setminus U(i)) \vee Q)) \\ & \leq \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \alpha_{R_0^S}^{H_0} \beta_{S'}^W (\bar{v}(W \vee Q \vee Q') -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee Q \vee Q')) \\ &= \sum_{H_0 \subseteq R_0^S} \sum_{W \subseteq S', i \in \text{Supp}W} \sum_{P_0 \subseteq R_0^T \setminus R_0^S} \alpha_{R_0^S}^{H_0 \cup P_0} \beta_{S'}^W (\bar{v}(W \vee Q \vee Q') -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee Q \vee Q')) \\ &= \sum_{H_0 \subseteq R_0^S} \sum_{P_0 \subseteq R_0^T \setminus R_0^S} \alpha_{R_0^S}^{H_0 \cup P_0} \sum_{W \subseteq T', i \in \text{Supp}W} \beta_{T'}^W (\bar{v}(W \vee Q \vee Q') -_H \\ & \quad \bar{v}((W \setminus U(i)) \vee Q \vee Q')) \\ &= \sum_{H_0 \subseteq R_0^T} \alpha_{R_0^T}^{E_0} \sum_{W \subseteq T', i \in \text{Supp}W} \beta_{T'}^W (\bar{v}(D \vee Q) -_H \bar{v}((D \setminus U(i)) \vee Q)) \\ &= \varphi_i(T, \bar{v}, \Gamma_T), \end{aligned}$$

where $Q' = \vee_{l \in P_0} B_l$.

From recursion method, we have

$$\varphi_i(S, \bar{v}, \Gamma_S) \leq \varphi_i(T, \bar{v}, \Gamma_T)$$

for any $S' = T' \subseteq B_k$ and $R_0^S \subseteq R_0^T$.

By cases (1) and (2), the conclusion is obtained. \square

Definition 3.4 Let $\bar{v} \in IG(U, \Gamma)$, the core $IC(U, \bar{v}, \Gamma)$ of \bar{v} is defined by

$$IC(U, \bar{v}, \Gamma) = \{ \bar{x} = \{ \bar{x}_i \}_{i \in \text{Supp} U} \mid \sum_{i \in \text{Supp} U} \bar{x}_i = \bar{v}(U), \\ \sum_{i \in \text{Supp} S} \bar{x}_i \geq \bar{v}(S), \forall S \in L(U, \Gamma) \}.$$

Theorem 3.3 Let $\bar{v} \in IG(U, \Gamma)$, if \bar{v} is convex, then $(\varphi_i(U, \bar{v}, \Gamma))_{i \in \text{Supp} U} \in IC(U, \bar{v}, \Gamma)$.

Proof. From Theorem 3.1, we get

$$\sum_{i \in \text{Supp} U} \varphi_i(U, \bar{v}, \Gamma) = \bar{v}(U).$$

By the convexity of \bar{v} and Theorem 3.2, we have

$$\bar{v}(S) = \sum_{i \in \text{Supp} S} \varphi_i(S, \bar{v}, \Gamma) \leq \sum_{i \in \text{Supp} S} \varphi_i(U, \bar{v}, \Gamma)$$

for any $S \in L(U, \Gamma)$.

Namely, $(\varphi_i(U, \bar{v}, \Gamma))_{i \in \text{Supp} U} \in IC(U, \bar{v}, \Gamma)$. \square

If there is only one coalition in Γ , and the coalition values are real numbers, then IPMAF degenerates to be fuzzy population monotonic allocation function introduced by Tsurumi et al. [4].

Furthermore, if there is only one coalition in Γ , and the coalition values are real numbers, then the core $IC(U, \Gamma)$ degenerates to be the core of fuzzy games presented by Yu and Zhang [25].

4. A Special Kind of Fuzzy Games with a Coalition Structure and Interval Payoffs

In this section, we shall discuss a special kind of fuzzy games with a coalition structure and interval payoffs, which can be seen as an extension of fuzzy games presented by Tsurumi et al. [4]. Similar to Tsurumi et al. [4], we give the fuzzy coalition value as follows:

$$\bar{v}_C(S) = \sum_{l=1}^{q(S)} \bar{v}_0([S]_{h_l})(h_l - h_{l-1}), \quad (4)$$

where S is a fuzzy coalition as usual. $Q(S) = \{U(i) \mid U(i) > 0, i \in \text{Supp} S\}$ and $[S]_{h_l} = \{i \in \text{Supp} S \mid U(i) \geq h_l\}$. $q(S)$ denote s the cardinality of $Q(S)$, and the elements in $Q(S)$ are written in the increasing order as $0 = h_0 \leq h_1 \leq \dots \leq h_{q(S)}$. \bar{v}_0 is the associated game of $\bar{v} \in IG(U, \Gamma)$ defined on $[U]_{h_l}$ for any $l \in \{1, 2, \dots, q(U)\}$.

From Eq.(4), we know it is equivalent to

$$\bar{v}_C(S) = \sum_{l=1}^{q(S)} (\bar{v}_0([S]_{h_l}) - {}_H \bar{v}_0([S]_{h_{l+1}})) h_l.$$

Let $IG_C(U, \Gamma)$ be the set of this kind of fuzzy games on (U, Γ) with interval payoffs.

Similar to Definition 3.1, we give the following definition for the interval Owen value in $IG_C(U, \Gamma)$.

Definition 4.1 Let $\bar{v}_C \in IG_C(U, \Gamma)$, a function $f : IG_C(U, \Gamma) \rightarrow \bar{\mathbb{R}}_+^{|\text{Supp} U|}$ is said to be an interval Owen value for \bar{v}_C , if it satisfies the following axioms:

Axiom 1: If T is a carrier for \bar{v}_C in U , then

$$\bar{v}_C(T) = \sum_{i \in \text{Supp} T} f_i(U, \bar{v}_C, \Gamma);$$

Axiom 2: Let $B_k \in \Gamma$ and $i, j \in [B_k]_{h_l}$, where $l \in \{1, 2, \dots, q(U)\}$.

If we have $\bar{v}_0(S_0 \cup i) = \bar{v}_0(S_0 \cup j)$ for any $S_0 \in L([U]_{h_l}, \Gamma)$ with $i, j \notin S_0$, then

$$f_i([U]_{h_l}, \bar{v}_0, \Gamma) = f_j([U]_{h_l}, \bar{v}_0, \Gamma);$$

Axiom 3: Let $l \in \{1, 2, \dots, q(U)\}$, $M_0^h = \{j \in M_0 \mid [B_j]_{h_l} \neq \emptyset\}$

and all $k, s \in M_0^h$ with $k \neq s$. If $\bar{v}_0^{[B_k]_{h_l}}(R_0 \cup k) = \bar{v}_0^{[B_s]_{h_l}}(R_0 \cup s)$ for all $R_0 \subseteq M_0^h \setminus \{k, s\}$, then

$$\sum_{i \in [B_k]_{h_l}} f_i([U]_{h_l}, \bar{v}_0, \Gamma) = \sum_{j \in [B_s]_{h_l}} f_j([U]_{h_l}, \bar{v}_0, \Gamma);$$

Axiom 4: Let $\bar{w}_C, \bar{v}_C \in IG_C(U, \Gamma)$, then

$$f(U, \bar{v}_C + \bar{w}_C, \Gamma) = f(U, \bar{v}_C, \Gamma) + f(U, \bar{w}_C, \Gamma),$$

where $([U]_{h_l}, \Gamma)$ is a coalition structure in $[U]_{h_l}$ w.r.t. (U, Γ) , and $L([U]_{h_l}, \Gamma)$ denotes the set of all feasible coalitions in $([U]_{h_l}, \Gamma)$. \bar{v}_0 is the associated game of $\bar{v}_C \in IG_C(U, \Gamma)$ defined in $[U]_{h_l}$ for any $l \in \{1, 2, \dots, q(U)\}$.

Theorem 4.1 Let $\bar{v}_C \in IG_C(U, \Gamma)$, the function $\varphi^C : IG_C(U, \Gamma) \rightarrow \bar{\mathbb{R}}_+^{|\text{Supp} U|}$, defined by

$$\varphi_i^C(U, \bar{v}_C, \Gamma) = \sum_{l=1}^{q(U)} \varphi_i([U]_{h_l}, \bar{v}_0, \Gamma)(h_l - h_{l-1}) \quad \forall i \in \text{Supp} U, \quad (5)$$

where

$$\varphi_i([U]_{h_l}, \bar{v}_0, \Gamma) = \sum_{R_0 \subseteq M_0^h \setminus k} \sum_{i \in S_0, S_0 \subseteq [B_k]_{h_l}} \alpha_{m_{h_l}}^r \beta_{[B_k]_{h_l}}^{S_0} (\bar{v}_0(S_0 \cup Q_0) - {}_H \bar{v}_0((S_0 \cup Q_0) \setminus i)), \\ \alpha_{m_{h_l}}^r = r!(m_{h_l} - r - 1)! / m_{h_l}!, \beta_{[B_k]_{h_l}}^{S_0} = (s - 1)! (|[B_k]_{h_l} - s|)! / |[B_k]_{h_l}|!, \\ Q_0 = \bigcup_{r \in R_0} [B_r]_{h_l} \text{ and } M_0^h = \{j \in M_0 \mid [B_j]_{h_l} \neq \emptyset\} \quad . \quad m_{h_l} \text{ and } |[B_k]_{h_l}|$$

respectively denote the cardinalities of $M_0^{h_l}$ and $[B_k]_{h_l}$, and \bar{v}_0 is the associated game of \bar{v}_C . Then φ^C is the unique interval Owen value for \bar{v}_C in $L(U, \Gamma)$.

From Eq.(4) and Definition 2.6, we know $\{[B_j]_{h_l}\}_{j \in M_0}$ is a coalition structure in $[U]_{h_l}$, and the set of all feasible coalitions in $([U]_{h_l}, \Gamma)$ is different w.r.t. the different $l \in \{1, 2, \dots, q(U)\}$.

If there is only one coalition in Γ , and the coalition values are real numbers, then Eq.(5) degenerates to be the Shapley value proposed by Tsurumi et al.[4].

Proof. *Axiom 1.* From Theorem 4 in [4], we know T is a carrier in U for \bar{v}_C if and only if $[T]_{h_l}$ is a carrier in $[U]_{h_l}$ for \bar{v}_0 , where $l=1, 2, \dots, q(U)$.

For any $S_0 \subseteq [B_k]_{h_l}$, let

$$\bar{v}_0^{Q_0}(S_0) = \bar{v}_0(S_0 \cup Q_0) -_H \bar{v}_0(Q_0).$$

From Eq.(5), we get

$$\begin{aligned} & \sum_{i \in \text{Supp} T} \varphi_i^C(U, \bar{v}_C, \Gamma) \\ &= \sum_{i \in \text{Supp} U} \varphi_i^C(U, \bar{v}_C, \Gamma) \\ &= \sum_{i \in \text{Supp} U} \sum_{l=1}^{q(U)} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \sum_{i \in S_0, S_0 \subseteq [B_k]_{h_l}} \alpha_{m_{h_l}}^r \beta_{[B_k]_{h_l}}^{S_0} (\bar{v}_0^{Q_0}(S_0) -_H \bar{v}_0^{Q_0}(S_0 \setminus i))(h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r \sum_{i \in \text{Supp} U} \sum_{i \in S_0, S_0 \subseteq [B_k]_{h_l}} \beta_{[B_k]_{h_l}}^{S_0} (\bar{v}_0^{Q_0}(S_0) -_H \bar{v}_0^{Q_0}(S_0 \setminus i))(h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r \bar{v}_0^{Q_0}([B_k]_{h_l})(h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r (\bar{v}_0([B_k]_{h_l} \cup Q_0) -_H \bar{v}_0([B_k]_{h_l})) \times (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r (\bar{v}_0^{[B]_{h_l}}(R_0 \cup k) -_H \bar{v}_0^{[B]_{h_l}}(R_0)) \times (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \bar{v}_0^{[B]_{h_l}}(M_0^{h_l})(h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \bar{v}_0([U]_{h_l})(h_l - h_{l-1}) \end{aligned}$$

$$\begin{aligned} &= \sum_{l=1}^{q(U)} \bar{v}_0^{[B]_{h_l}}([T]_{h_l})(h_l - h_{l-1}) \\ &= \bar{v}_C(T), \end{aligned}$$

where $\bar{v}_0^{[B]_{h_l}}$ is an interval quotient game on $M_0^{h_l}$ w.r.t. $([U]_{h_l}, \Gamma)$;

From Eq.(5), we have

$$\begin{aligned} \varphi_i([U]_{h_l}, \bar{v}_0, \Gamma) &= \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \sum_{i \in S_0, S_0 \subseteq [B_k]_{h_l}} \alpha_{m_{h_l}}^r \beta_{[B_k]_{h_l}}^{S_0} \\ &\quad \times (\bar{v}_0(S_0 \cup Q_0) -_H \bar{v}_0((S_0 \cup Q_0) \setminus i))(h_l - h_{l-1}) \end{aligned}$$

Thus, *Axiom 2* holds;

By Eq.(5), we get

$$\begin{aligned} & \sum_{i \in [B_k]_{h_l}} \varphi_i([U]_{h_l}, \bar{v}_0, \Gamma) \\ &= \sum_{i \in [B_k]_{h_l}} \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \sum_{i \in S_0, S_0 \subseteq [B_k]_{h_l}} \alpha_{m_{h_l}}^r \beta_{[B_k]_{h_l}}^{S_0} (\bar{v}_0(S_0 \cup Q_0) -_H \bar{v}_0((S_0 \cup Q_0) \setminus i))(h_l - h_{l-1}) \\ &= \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r (\bar{v}_0([B_k]_{h_l} \cup Q_0) -_H \bar{v}_0(Q_0))(h_l - h_{l-1}) \\ &= \sum_{R_0 \subseteq M_0^{h_l} \setminus k} \alpha_{m_{h_l}}^r (\bar{v}_0^{[B]_{h_l}}(R_0 \cup k) -_H \bar{v}_0^{[B]_{h_l}}(R_0))(h_l - h_{l-1}) \\ &= \sum_{R_0 \subseteq M_0^{h_l} \setminus s} \alpha_{m_{h_l}}^r (\bar{v}_0^{[B]_{h_l}}(R_0 \cup s) -_H \bar{v}_0^{[B]_{h_l}}(R_0))(h_l - h_{l-1}) \\ &= \sum_{j \in [B_k]_{h_l}} \varphi_j([U]_{h_l}, \bar{v}_0, \Gamma); \end{aligned}$$

Thus, *Axiom 3* holds.

From Eq.(5), one can easily get *Axiom 4*;

(Uniqueness) From Theorem 3.1, we know $\bar{v}_0 \in IG_0([U]_{h_l}, \Gamma)$ can be uniquely expressed by

$$\bar{v}_0 = \sum_{\emptyset \neq T_0 \in L([U]_{h_l}, \Gamma)} c_{T_0} u_{T_0}$$

for any $l \in \{1, 2, \dots, q(U)\}$, where

$$c_{T_0} = \sum_{R_0 \subseteq R_0} (-1)^{r-p} \left(\sum_{A_0 \subseteq S_0} (-1)^{s-a} \bar{v}_0(A_0 \vee Q_0) \right),$$

$T_0 = S_0 \cup_{l \in R_0} [B_l]_{h_l}$ and $S_0 \subseteq [B_k]_{h_l}$ for some $k \in M_0^{h_l}$,

$$u_{T_0}(W_0) = \begin{cases} 1 & T_0 \subseteq W_0 \text{ and } R_0 \subseteq M_0^{h_l} \setminus \{k\}. \\ 0 & \text{otherwise} \end{cases}$$

From Eq.(4) and *Axiom 4*, we only need to prove the uniqueness of Eq.(5) on game $c_{S_0} u_{S_0}$ for any $S_0 \in L([U]_{h_l}, \Gamma)$. Similar to Theorem 3.1, we have

$$\varphi_i([U]_{h_l}, c_{S_0} u_{S_0}, \Gamma) = \begin{cases} 0 & \text{otherwise} \\ \frac{c_{S_0}}{|M_{h_l}'| \cdot |[B_k]_{h_l}|} & i \in [B_k]_{h_l}, k \in M_0^{h_l}, \end{cases}$$

where $M_0^{h_i} = \{j \in M_0^h : [B_j]_{h_i} \cap S_0 \neq \emptyset\}$ and $[B_j]_{h_i}' = [B_j]_{h_i} \cap S_0$. \square

Theorem 4.2 Let $\bar{v}_C \in IG_C(U, \Gamma)$, the function $\phi^C : IG_C(U, \Gamma) \rightarrow \mathbb{R}_+^{[SuppU]}$, defined by

$$\phi_i^C(U, \bar{v}_C, \Gamma) = \sum_{l=1}^{q(U)} (\phi_i([U]_{h_l}, \bar{v}_0, \Gamma) -_H \phi_i([U]_{h_{l+1}}, \bar{v}_0, \Gamma)) h_l \quad \forall i \in SuppU, \quad (6)$$

where

$$\phi_i([U]_{h_l}, \bar{v}_0, \Gamma) = \sum_{R_0 \subseteq M_0^h \setminus k} \sum_{i \in S_0 \subseteq [B_k]_{h_l}} \alpha_{m_{h_l}}^r \beta_{[B_k]_{h_l}}^{S_0} (\bar{v}_0(S_0 \cup Q_0) -_H \bar{v}_0((S_0 \cup Q_0) \setminus i)), \alpha_{m_{h_l}}^r \text{ and } \beta_{[B_k]_{h_l}}^{S_0} \text{ as given in Theorem 4.1. Then } \phi^C \text{ is equivalent to } \phi^C.$$

Proof. From Eq.(6), we have

$$\begin{aligned} & \phi_i^C(U, \bar{v}_C, \Gamma) \\ &= \sum_{l=1}^{q(U)} (\phi_i([U]_{h_l}, \bar{v}_0, \Gamma) -_H \phi_i([U]_{h_{l+1}}, \bar{v}_0, \Gamma)) h_l \\ &= (\phi_i([U]_{h_1}, \bar{v}_0, \Gamma) -_H \phi_i([U]_{h_2}, \bar{v}_0, \Gamma)) h_1 + \\ & \quad (\phi_i([U]_{h_2}, \bar{v}_0, \Gamma) -_H \phi_i([U]_{h_3}, \bar{v}_0, \Gamma)) h_2 + \\ & \quad \dots + \phi_i([U]_{h_{q(U)}}, \bar{v}_0, \Gamma) h_{q(U)} \\ &= (\phi_i([U]_{h_1}, \bar{v}_0, \Gamma) h_1 -_H \phi_i([U]_{h_2}, \bar{v}_0, \Gamma) h_1) + \\ & \quad (\phi_i([U]_{h_2}, \bar{v}_0, \Gamma) h_2 -_H \phi_i([U]_{h_3}, \bar{v}_0, \Gamma) h_2) + \\ & \quad \dots + \phi_i([U]_{h_{q(U)}}, \bar{v}_0, \Gamma) h_{q(U)} \\ &= \phi_i([U]_{h_1}, \bar{v}_0, \Gamma) (h_1 - h_0) + \phi_i([U]_{h_2}, \bar{v}_0, \Gamma) (h_2 - h_1) \\ & \quad + \dots + \phi_i([U]_{h_{q(U)}}, \bar{v}_0, \Gamma) (h_{q(U)} - h_{q(U)-1}) \\ &= \sum_{l=1}^{q(U)} \phi_i([U]_{h_l}, \bar{v}_0, \Gamma) (h_l - h_{l-1}) \\ &= \phi_i^C(U, \bar{v}, \Gamma). \quad \square \end{aligned}$$

Definition 4.2 Let $\bar{v}_C \in IG_C(U, \Gamma)$, the vector $\bar{x} = \{\bar{x}_i\}_{i \in SuppU}$ is said to be an IPMAF for \bar{v}_C in $L(U, \Gamma)$, if \bar{x} satisfies

- 1) $\sum_{i \in SuppS} \bar{x}_i(S) = \sum_{l=1}^{q(S)} \bar{v}_0([S]_{h_l})(h_l - h_{l-1}) \quad \forall S \in L(U, \Gamma),$
- 2) $\bar{x}_i(S) \leq \bar{x}_i(T), \quad \forall i \in SuppS', \quad \forall S, T \in L(U, \Gamma)$
s.t. $S \subseteq T,$

where $S' \subseteq B_k (k \in M_0^h), S = S' \setminus_{q \in R} B_q$ and $R \subseteq M \setminus k.$

Theorem 4.3 Let $\bar{v}_C \in IG_C(U, \Gamma)$, if the associated game $\bar{v}_0 \in IG_0([U]_{h_l}, \Gamma)$ of \bar{v}_C is convex, then $(\phi_i^C(U, \bar{v}_C, \Gamma))_{i \in SuppU}$ is an IPMAF for \bar{v}_C in $L(U, \Gamma).$

Proof. From Theorem 4.1, we know the first condition holds. In the following, we shall show the second condition. From $S \subseteq T$, we know $[S]_{h_l} \subseteq [T]_{h_l}$ for any $l \in \{1, 2, \dots, q(T)\}.$

By Theorem 3.2, we have

$$\phi_i([S]_{h_l}, \bar{v}_0, \Gamma) \leq \phi_i([T]_{h_l}, \bar{v}_0, \Gamma)$$

for any $i \in [S]_{h_l}$ and any $l \in \{1, 2, \dots, q(T)\}.$

Thus,

$$\begin{aligned} \phi_i^C(S, \bar{v}_C, \Gamma) &= \sum_{l=1}^{q(S)} \phi_i([S]_{h_l}, \bar{v}_0, \Gamma) (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(T)} \phi_i([S]_{h_l}, \bar{v}_0, \Gamma) (h_l - h_{l-1}) \\ &\leq \sum_{l=1}^{q(T)} \phi_i([T]_{h_l}, \bar{v}_0, \Gamma) (h_l - h_{l-1}) \\ &= \phi_i^C(T, \bar{v}_C, \Gamma) \end{aligned}$$

for any $i \in SuppS'.$ \square

From Definition 3.4, we know the core $IC(U, \bar{v}_C, \Gamma)$ of $\bar{v}_C \in IG_C(U, \Gamma)$ can be expressed by

$$\begin{aligned} IC(U, \bar{v}_C, \Gamma) &= \{\bar{x} = \{\bar{x}_i\}_{i \in SuppU} \mid \sum_{i \in SuppU} \bar{x}_i = \\ & \sum_{l=1}^{q(U)} \bar{v}_0([U]_{h_l})(h_l - h_{l-1}), \sum_{i \in SuppS} \bar{x}_i \geq \\ & \sum_{l=1}^{q(S)} \bar{v}_0([S]_{h_l})(h_l - h_{l-1}), \forall S \in L(U, \Gamma)\}. \end{aligned}$$

Theorem 4.4 Let $\bar{v}_C \in IG_C(U, \Gamma)$, if the associated game $\bar{v}_0 \in IG_0([U]_{h_l}, \Gamma)$ of \bar{v}_C is convex, then

$$(\phi_i^C(U, \bar{v}_C, \Gamma))_{i \in SuppU} \in IC(U, \bar{v}_C, \Gamma).$$

Proof. From Theorems 4.1 and 4.3, one can easily get the conclusion. \square

Theorem 4.5 Let $\bar{v}_C \in IG_C(U, \Gamma)$, if the associated game $\bar{v}_0 \in IG_0([U]_{h_l}, \Gamma)$ of \bar{v}_C is convex, then $IC(U, \bar{v}_C, \Gamma) \neq \emptyset,$ and can be expressed by

$$C(U, \bar{v}_C, \Gamma) = \left\{ \bar{x} = \{\bar{x}_i\}_{i \in SuppU} \mid \sum_{i \in SuppU} \bar{x}_i = \sum_{l=1}^{q(U)} \bar{v}_0([U]_{h_l})(h_l - h_{l-1}) \right\}$$

$$\sum_{i \in [U]_{h_t}} \bar{y}_i^{[U]_{h_t}} (h_t - h_{t-1}), \forall \bar{y} = \{\bar{y}_i^{[U]_{h_t}}\}_{i \in [U]_{h_t}} \in C([U]_{h_t}, \bar{v}_0, \Gamma), \forall l = \{1, 2, \dots, q(U)\}, \quad (7)$$

where $C([U]_{h_t}, \bar{v}_0, \Gamma)$ denotes the core of \bar{v}_0 in $L([U]_{h_t}, \Gamma)$.

Proof. From Theorem 4.3 in [25], it is not difficult to get the conclusion. \square

If there is only one coalition in Γ , and the coalition values are real numbers, then Eq.(7) degenerates to be the core given by Yu and Zhang [25].

Example 4.1 There are 5 companies that cooperate to develop a new product. Namely, the set of players $N = \{1, 2, 3, 4, 5\}$. Since there exist many uncertainty factors in the process of development. The players only know the possible payoffs of the coalitions formed by applying all their resources as given in Table 2. In order to reduce risk, the players are not willing to put all their resources into this cooperation, but partial participation. Furthermore, since their capabilities are different, not all coalitions can be formed. If the players' participation levels in fuzzy coalition U are given by $U(1) = 0.3$, $U(2) = 0.6$, $U(3) = U(4) = 0.8$ and $U(5) = 0.9$.

All the formed coalitions have the coalition structure $\Gamma = \{B_1, B_2\}$, where $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4), U(5)\}$.

Then this is a fuzzy game with a coalition structure and interval payoffs. If the fuzzy coalition values and that of their associated crisp coalitions have the relationship as given in Eq.(4). Then, this is fuzzy game in $IG_C(U, \Gamma)$.

Table 2 (The interval values of the crisp coalitions)

S_0	$\bar{v}_0(S_0)$	S_0	$\bar{v}_0(S_0)$
{1}	[1,2]	{1,2,4}	[15,16]
{2}	[1,3]	{1,2,5}	[12,15]
{3}	[2,3]	{2,3,4}	[12,15]
{4}	[2,3]	{2,3,5}	[13,15]
{5}	[1,4]	{2,4,5}	[13,15]
{1,2}	[7,8]	{3,4,5}	[16,18]
{2,3}	[4,6]	{1,3,4,5}	[20,22]
{2,4}	[4,6]	{2,3,4,5}	[23,25]
{2,5}	[5,7]	{1,2,3,4}	[22,25]
{3,4}	[5,8]	{1,2,3,5}	[26,28]
{3,5}	[4,7]	{1,2,4,5}	[26,28]
{4,5}	[4,7]	{1,2,3,4,5}	[36,40]
{1,2,3}	[15,16]		

From Eq.(5), we have

$$\begin{aligned} \varphi_1^C(U, \bar{v}_C, \Gamma) &= [1.8, 1.95], \\ \varphi_2^C(U, \bar{v}_C, \Gamma) &= [3.45, 4.05], \\ \varphi_3^C(U, \bar{v}_C, \Gamma) &= \varphi_4^C(U, \bar{v}_C, \Gamma) = [5.305, 5.625], \\ \varphi_5^C(U, \bar{v}_C, \Gamma) &= [5.13, 6.25]. \end{aligned}$$

According to Table 2, we know the associated game \bar{v}_0 is convex, so $(\varphi_i^C(U, \bar{v}_C, \Gamma))_{i=\{1,2,\dots,5\}}$ is an IPMAF for \bar{v}_C in $L(U, \Gamma)$, and an element in the core.

Since the players 3 and 4 are symmetric in this game, they get the same interval payoffs. Furthermore, the Hukuhara difference between interval numbers can not be used in this example, since there does not exist interval number \bar{c} such that $\bar{v}_0(5) + \bar{c} = \bar{v}_0(2, 5)$.

5. Conclusion

Based on the extension Hukuhara difference between interval numbers, we have researched the interval Owen value for fuzzy games with a coalition structure and interval payoffs, which extends the researching scope of game theory. Meantime, some properties of the given interval Owen value are studied. In order to better understand this kind of fuzzy games, we further discuss a special kind of fuzzy games with a coalition structure and interval payoffs, which can be seen as an extension of fuzzy games given by Tsurumi et al. [4]. However, we only study the interval Owen value for this kind of fuzzy games, and it will be interesting to discuss other payoff indices on it

As a special kind of fuzzy numbers, the researching for fuzzy games with coalition structure and interval

payoffs provides reference to fuzzy games with a coalition structure and fuzzy payoffs.

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