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The study of fuzzy context sequences

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Abstract

In some cases, the relationship between an object set X and an attribute set Y is set up by means of a fuzzy context sequence. A particular case of this situation appears when we want to study the evolution of an L-fuzzy context in time.
In this work, we analyze these situations. First we introduce the fuzzy context sequence definition and remind the main results about OWA operators. With the aid of these operators, we propose an exhaustive study of the different contexts values of the sequence using some new relations.
In the second part, we also study the fuzzy context sequences establishing tendencies and temporal patterns.
Finally, we illustrate all the results by means of examples.

Keywords: L-fuzzy context, L-fuzzy concept, fuzzy context sequences, OWA operators.

1. Introduction

The L-Fuzzy Concept Analysis analyzes the information from an L-fuzzy context by means of the L-fuzzy concepts. These L-fuzzy contexts are tuples (L,X,Y,R), with L a complete lattice, X and Y sets of objects and attributes, and $R \in L^{X\times Y}$ an L-fuzzy relation between the objects and the attributes.

In some situations, we have several relations between the object set X and the attribute set Y, making up what we are going to say a fuzzy context sequence. When this sequence represents an evolution in time we can be more ambitious and try to predict future tendencies besides studying past behaviors. The study of this fuzzy context sequences will be the main target of this work.

We take as starting point a sequence formed by the L-fuzzy contexts $(L,X,Y,R_i)_{i\in I}$, with $I \subseteq \mathbb{N}$ a finite set, where X and Y are the sets of objects and attributes respectively, and R_i represents the ith relation between the objects of X and the attributes of Y.

The final goal is the study of the fuzzy context sequence and the derived information by the L-fuzzy concepts. To do this, we analyze two different situations: in the first one, we study the values that emphasize in the L-fuzzy contexts regardless of the context in which they are, and in the second one, it is important to maintain the order of the contexts since they represent an evolution in time.

In this second case, it will be of special interest the study of the evolution of the attributes by means of the search of patterns. Works in this line to analyze the course of time in a Formal context can be found in [1,2,3].

In [2,3] K.E. Wolff defines the Temporal Concept Analysis where a Conceptual Time System is introduced such that the state and phase spaces are defined as concept lattices which represent the meaning of the states with respect to the chosen time description. On the other hand, the authors define the hidden evolution patterns in [1,4] using temporal matching in the case of Formal Concept Analysis.

In this paper, we show a new method for L-Fuzzy Contexts with quantitative data that allows the detection of some kind of regularity.

There are many applications of this technique. For instance, we can think of a fuzzy context sequence that shows the monthly sports articles sales in certain shops throughout a period of time. This study will allow to establish tendencies and patterns on which we can base to make decisions.

Firstly, we will see some important results in the L-Fuzzy Concept Analysis.

2. L-fuzzy contexts

The Formal Concept Analysis of R. Wille [5] extracts information from a binary table that represents a Formal context (X,Y,R) with X and Y finite sets of objects and attributes respectively and $R \subseteq X \times Y$. The hidden information consists of pairs (A,B) with $A \subseteq X$ and $B \subseteq Y$, called Formal concepts, verifying $A^* = B$ and $B^* = A$, where $(\cdot)^*$ is a derivation operator that associates the attributes related to the elements of A to every object set A, and the objects related to the attributes of B to every attribute set B. These Formal Concepts can be interpreted as a group of objects A that shares the attributes of B.

In previous works [7,8] we have defined the L-fuzzy contexts (L,X,Y,R), with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X\times Y}$ a fuzzy relation between the objects and the attributes. This is an extension of Wille's Formal contexts to the fuzzy case when we want to study the relations between the objects and the attributes with values in a complete lattice L, instead of binary values.

In our case, to work with these L-fuzzy contexts, we have defined the derivation operators 1 and 2 given by means of these expressions:

$\forall A \in L^X, \forall B \in L^Y$

$$A_1(y) = \inf_{x\in X}\{\mathscr{I}(A(x),R(x,y))\}$$

$$B_2(x) = \inf_{y\in Y}\{\mathscr{I}(B(y),R(x,y))\}$$

with $\mathscr{I}$ a fuzzy implication operator defined in the lattice $(L,\leqslant)$.

Some authors use a residuated implication operator in their definitions of derivation operators [11,13,16].

The information stored in the context is visualized by means of the L-fuzzy concepts that are pairs $(M,M_1) \in (L^X,L^Y)$ with $M \in fix(\varphi)$, set of fixed points of the operator φ, being defined from the derivation operators 1 and 2 as $\varphi(M) = (M_1)_2 = M_{12}$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a group of objects that share a group of attributes in a fuzzy way.

Using the usual order relation between fuzzy sets, that is,

$$\forall M,N \in L^X, \quad M \leqslant N \Longleftrightarrow M(x) \leqslant N(x) \quad \forall x \in X,$$

we define the set $\mathscr{L} = \{(M,M_1)/M \in fix(\varphi)\}$ with the order relation $\preceq$ defined as:

$\forall (M,M_1),(N,N_1) \in \mathscr{L}$,

$$(M,M_1) \preceq (N,N_1) \text{ if } M \leqslant N (\text{ or } N_1 \leqslant M_1)$$

As φ is an order preserving operator, by the theorem of Tarski [6], the set $fix(\varphi)$ is a complete lattice and then $(\mathscr{L},\preceq)$ is also a complete lattice that is said to be [7,8] the L-fuzzy concept lattice.

On the other hand, given $A \in L^X$, (or $B \in L^Y$) we can obtain the associated L-fuzzy concept applying twice the derivation operators. In the case of using a residuated implication, as we do in this work, the associated L-fuzzy concept is (A_{12}, A_1) (or (B_2, B_{21})).

Other important results about this theory are in [9,10,11,12,13,14,15,16].

3. Fuzzy context sequences

In this section we are interested in the study of the fuzzy context sequences. We are going to see the formal definition:

Definition 1. A fuzzy context sequence is a tuple $(L, X, Y, R_i)_{i \in I}$ with $L = [0,1]$ a complete lattice, X and Y sets of objects and attributes respectively and $R_i \in L^{X \times Y}, \forall i \in I$, with $I \subseteq \mathbb{N}$ a finite set.

In the case that we want to define a new L-fuzzy context that summarizes the information of the different contexts of the sequence, we have to aggregate the observations of the relations R_i. Thus, we can use the average (with or without weight), obtain the intervals whose lower bound is the minimum of the observations and the upper one the maximum of them, obtaining an interval-valued L-fuzzy context, or working with multivalued contexts. We have developed these ideas in previous works [17,18].

The use of weighted averages [19,20] to summarize the information stored in the different relations allows us to associate different weights to the L-fuzzy contexts highlighting some of them. Thus, the new relation R is defined as:

$$R(x,y) = \sum_{i \in I} w_i . R_i(x,y), \forall x \in X, y \in Y$$

verifying, as is required by the definition, that $\sum_{i \in I} w_i = 1$, $\forall (w_i)_{i \in I}$,

However, it is possible that some observations of an L-fuzzy context of the sequence are interesting whereas others not so much. For instance, as we studied in [21], the used methods for obtaining the L-fuzzy concepts do not give good results when we have very low values in some relations.

On the other hand, to study similar situations by means of multivalued contexts in [17] we used multisets and expertons. In that case, all the observations were analyzed globally without the establishment of different studies based on different exigency levels. This is one of the new contributions of this work.

Let us see the following example.

Example 1. Let $(L, X, Y, R_i)_{i \in I}$ be a fuzzy context sequence that represents the sales of sports articles (X) in some establishments (Y) throughout a period of time (I), and we want to study the places where the main sales hold taking into account that there are seasonal sporting goods (for instance skies, bathing suits) and of a certain zone (it is more possible to sale skies in Colorado than in Florida).

In this case, the weighted average model is not valid since it is very difficult to associate a weight to an L-fuzzy context (in some months more bath suits are sold whereas, in others, skies are).

To analyze this situation, it could be interesting the use of the OWA[22,23] operators with the most of the weights near the largest values. In this way, we give more relevance to the largest observations, independently of the moment when they have taken place and, on the other hand, we would avoid some small values in the resulting relations (that can give problems in the calculation of the L-fuzzy concepts as has been already studied in [21]).

These are the definitions of these operators given by Yager [22]:

Definition 2. A mapping F from $L^n \longrightarrow L$, where $L = [0,1]$ is called an OWA operator of dimension n if associated with F is a weighting n-tuple $W = (w_1, w_2 \dots w_n)$ such that $w_i \in [0,1]$ and $\sum_{1 \leqslant i \leqslant n} w_i = 1$, where $F(a_1, a_2, \dots a_n) = w_1.b_1 + w_2.b_2 + \cdots + w_n.b_n$, with b_i the ith largest element in the collection $a_1, a_2, \dots a_n$.

There are two particular cases of special interest: W_* defined by the weighting n-tuple with $w_n = 1$ and $w_j = 0, \forall j \neq n$, and W^* defined by the weighting n-tuple such that $w_1 = 1$ and $w_j = 0, \forall j \neq 1$.

It is proved that $F_*(a_1, a_2, \dots a_n) = \min_j(a_j)$ and $F^*(a_1, a_2, \dots a_n) = \max_j(a_j)$. These operators are said to be *and* and *or*, respectively.

In order to do a more general study of the fuzzy context sequence, we are interested in the use of operators close to *or*. To measure this proximity we can use the orness degree definition given by [22]:

Definition 3. Let F be an OWA aggregation operator with an n-tuple of weights $W = (w_1, w_2, \dots w_n)$. The orness degree associated with this operator is defined as:

$$orness(W) = (1/n-1)\sum_{i=1}^{n}((n-i).w_i)$$

Example 2. If we take the n-tuple of weights $W = (1, 0, 0, \dots 0)$, then $orness(W) = 1$, and if $W = (0, 0, 0, \dots 1)$, then $orness(W) = 0$.

Modifying the weights of these OWA operators we can go from the minimum (for all) to the maximum (exists) although neither of the bounds are the most interesting in our opinion because both represent a biased information. Really, by means of these OWA operators, we are representing quantifiers (most, at least the half, etc.)

Returning to the initial situation and using these OWA operators, we can give the following definition that summarizes the information stored in the fuzzy context sequence:

Definition 4. Let $(L, X, Y, R_i)_{i \in I}$ be the fuzzy context sequence and F an OWA aggregation operator. We can define an L-fuzzy relation R_F that aggregates the information of the different L-fuzzy contexts, in the case that we want to study the largest values, by means of this expression:

$$\begin{aligned} R_F(x,y) =& F(R_1(x,y), R_2(x,y) \dots R_{|I|}(x,y)) = \\ =& w_1.b_1 + w_2.b_2 + \dots + w_{|I|}.b_{|I|}, \\ & \forall x \in X, y \in Y \end{aligned}$$

where $W = (w_1, w_2, \dots w_{|I|})$ is the weighting tuple associated with F.

There are two special interesting cases:

- W verifying that $orness(W)$ is larger than a threshold that we want to establish.
- W such that $w_i = 1/k$, if $i \leqslant k$ and $w_i = 0$, if $i > k$. That is, the average of the k largest values (with $k \in \mathbb{N}, k \leqslant |I|$)

In the next section we apply these OWA operators to the L-fuzzy contexts to study the values that stand out in the L-fuzzy contexts and to analyze tendencies when the sequence represents the evolution in time.

3.1. The fuzzy context sequence general study

For a more exhaustive study of the fuzzy context sequence, we can define $|I|$ relations associated with the different demand levels using OWA operators where the weighting tuple W has just one non-null value $w_k = 1$, for a certain $k \leqslant |I|$.

Definition 5. Given a fuzzy context sequence $(L, X, Y, R_i)_{i \in I}$ with X and Y sets of objects and attributes respectively and $R_i \in L^{X \times Y}, \forall i \in I$, and given a certain $k \in \mathbb{N}, k \leqslant |I|$, we define the relation $R^{(k)}$ using an OWA operator F_k with the weighting tuple W such that $w_k = 1$ and $w_i = 0, \forall i \neq k$.

$$\begin{aligned} & R_{F_k}(x,y) = F_k(R_1(x,y), R_2(x,y) \dots R_{|I|}(x,y)), \\ & \forall x \in X, y \in Y. \end{aligned}$$

To simplify the following notations, we will denote by $R^{(k)}$ this relation R_{F_k}.

Another way to express this definition is:

$$R^{(k)}(x,y) = \min J_{xy}^k$$

where J_{xy}^k is the set formed by the k largest values associated with the pair (x,y) in the R_i relations.

In this way, we are saying that there are at least k observations larger than or equal to the values of the relation $R^{(k)}$. So, this relation measures the degree in which x is at least k times related to y.

We have chosen the minimum OWA operator in this definition, but it could be possible to use another one if we want to be less demanding.

Example 3. We come back to the fuzzy context sequence $(L, X, Y, R_i)_{i \in I}$ of Example 1 that represents the sports articles sales $X = \{x_1, x_2, x_3\}$ in some establishment $Y = \{y_1, y_2, y_3\}$ during a period of time. In the following relations $R_i, i \in I$, that have values in $L = [0,1]$, the percentage of product sales in each establishment based on the stock during the last 5 months are gathered.

$$R_1 = \begin{pmatrix} 0.7 & 1 & 0.8 \\ 0 & 0.1 & 0.1 \\ 0 & 0.1 & 0 \end{pmatrix} R_2 = \begin{pmatrix} 1 & 0.8 & 1 \\ 0.2 & 0.4 & 0.1 \\ 0 & 0 & 0.2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0.6 & 0.5 & 0.7 \\ 0 & 0.1 & 0.2 \end{pmatrix} R_4 = \begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.1 & 0.5 & 0.3 \\ 0.6 & 0.8 & 0.8 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.8 & 1 & 0.9 \end{pmatrix}$$

First, by means of the L-Fuzzy Concept Analysis, we want to study in what establishments there are greater sales of each product without mattering when the sale has been carried out.

As we have expressed before, there are seasonal sporting goods that are sold in certain periods of time and not in others (skies, bathing suits ...). Therefore, try to summarize the information of the family of L-fuzzy concepts by means of the average, for instance, would not give good results (if a product is only sold during a pair of months in the year, the average with the other months would give a value close to 0 and we would not obtain good results applying the L-Fuzzy Concept Analysis).

On the other hand, if we fix the demand level for instance to $k = 2$ and use Definition 5, then we have the following relation:

$$R^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 0.2 & 0.5 & 0.3 \\ 0.6 & 0.8 & 0.8 \end{pmatrix}$$

Now, we take the L-fuzzy context $(L, X, Y, R^{(2)})$ and obtain the L-fuzzy concepts associated with the crisp singletons $\{x_1\}$ and $\{x_3\}$ using the Lukasiewicz implication operator ($\mathscr{I}(a,b) = \min(1, 1-a+b)$):

$\{x_1\} \longrightarrow (\{x_1/1, x_2/0.2, x_3/0.6\}, \{y_1/1, y_2/1, y_3/1\})$
$\{x_3\} \longrightarrow (\{x_1/1, x_2/0.5, x_3/1\}, \{y_1/0.6, y_2/0.8, y_3/0.8\})$

In this case, we can say that article x_1 has been successfully sold in the three establishments, at least during two months, and that there are, at least in two months, high sales of articles x_1 and x_3, more in the establishments y_2 and y_3.

As the chosen implication operator is the Lukasiewicz one, the membership degree of the fuzzy intension of the L-fuzzy concepts is coincident with the rows of the L-fuzzy relation. This coincidence does not hold when we are using a different implication operator. Moreover, in all the cases we obtain a more complete information by means of the fuzzy extension.

Analogously, we can take other different k levels.

In particular, the computation of the L-fuzzy concepts associated with $R^{(1)}$ allows to analyze in what stores the main sales of each article during a month (independent of the month) have taken place. If we take relation $R^{(k)}$ with $k > 1$ we are relaxing the exigency taking the k greater sales for our study.

These studies allow us to ignore the small values of the relations (the sales of a non-seasonal sporting goods are close to 0) since, in this case, if we take the average of the relations, the results will be biased.

The observation of these L-fuzzy concepts gives the idea for the following propositions:

Proposition 1. *Consider $k \in \mathbb{N}$, with $k \leqslant |I|$. If (A,B) is an L-fuzzy concept of the L-fuzzy context $(L, X, Y, R^{(k)})$, then $\forall h \in \mathbb{N}, h \leqslant k$, there exists an L-fuzzy concept (C,D) of the L-fuzzy context $(L, X, Y, R^{(h)})$ such that $A \leqslant C$ and $B \leqslant D$.*

Proof. If $k = h$, then it is obvious.

Otherwise, when $h < k$, $R^{(k)}(x,y) \leqslant R^{(h)}(x,y) \quad \forall (x,y) \in X \times Y$. That is, $R^{(k)} \leqslant R^{(h)}$. Thus, the L-fuzzy set B derived from A in $(L, X, Y, R^{(k)})$ is a subset of the L-fuzzy set D derived from A in $(L, X, Y, R^{(h)})$. Therefore, $B \leqslant D$.

Now, we derive again D in $(L, X, Y, R^{(h)})$, obtaining the set C (C=D_2) and, applying the properties of this closure operator formed by the composition of the derivation operators [11]: $A \leqslant A_{12} = D_2 = C$.

Therefore, the other inequality also holds. Moreover, it is obvious that if we use a residuated implication operator the obtained pair (C,D) is an L-fuzzy concept. □

The following result sets up relations between the L-fuzzy concepts associated with the same starting set (see section 2) in the different L-fuzzy contexts.

Proposition 2. *Consider $k,h \in \mathbb{N}$, with $k,h \leqslant |I|$ and consider $A \in L^X$. (A^k,B^k) and (A^h,B^h) are the L-fuzzy concepts associated with A in the L-fuzzy contexts $(L,X,Y,R^{(k)})$ and $(L,X,Y,R^{(h)})$ respectively. If $k \leqslant h$ then $B^k \geqslant B^h$.*

Moreover, if $\mathscr{I}$ is a residuated implication operator and the set A is the crisp singleton $\{x_i\}$:

$$A(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

then, $A^k(x_i) = A^h(x_i) = 1$.

A similar result is obtained taking as a starting point an L-fuzzy set of attributes $B \in L^Y$.

Proof. Consider $A \in L^X$. Unfolding the fuzzy extensions of both L-fuzzy concepts, and taking into account that a fuzzy implication operator is increasing on its second argument:

$$\begin{aligned} B^k(y) &= \inf_{x\in X}\{\mathscr{I}(A(x),R^{(k)}(x,y))\} \\ &\geqslant \inf_{x\in X}\{\mathscr{I}(A(x),R^{(h)}(x,y))\} = B^h(y) \end{aligned}$$

This result holds for every A and for every implication operator.

On the other hand, if we take a crisp singleton:

$$A(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

and a residuated implication, then the membership degree of x_i in the fuzzy extension of the L-fuzzy concepts is equal to 1:

$$B^k(y) = \inf_{x\in X}\{\mathscr{I}(A(x),R^{(k)}(x,y))\} = R^{(k)}(x_i,y)$$

$$\begin{aligned} A^k(x) &= \inf_{y\in Y}\{\mathscr{I}(B^k(y),R^{(k)}(x,y))\} \\ &= \inf_{y\in Y}\{\mathscr{I}(R^{(k)}(x_i,y),R^{(k)}(x,y))\}. \end{aligned}$$

Therefore, $A^k(x_i) = 1$.

Similarly, the result for the other L-fuzzy set can be proved. □

However, the inequality $A^k \leqslant A^h$ does not always hold, as can be seen if we come back to the previous example and we compare the fuzzy extension $A(x) = \{x_1/1, x_2/0, x_3/0\}$ of the derived L-fuzzy concept in the L-fuzzy contexts $(L,X,Y,R^{(2)})$ and $(L,X,Y,R^{(4)})$:

In $(L,X,Y,R^{(2)})$, the result is $A^2 = \{x_1/1, x_2/0.2, x_3/0.6\}$ whereas in $(L,X,Y,R^{(4)})$ we get $A^4 = \{x_1/1, x_2/0.5, x_3/0.5\}$.

In the following section, we introduce the variable time in our study.

3.2. Temporal analysis of the fuzzy context sequence

Fixed $k \in I$, and a pair (x,y), with $x \in X$ and $y \in Y$, Definition 5 uses the minimum of the k largest observations $R_i(x,y), i \in I$, of the fuzzy context sequence, but does not allow to make an analysis of their evolution in time.

In this section, we approach this subject by means of studies that analyze tendencies as well as patterns.

3.2.1. Temporal trends

The following definition takes the minimum value of the relations between each object and each attribute from an instant h.

Definition 6. Let $(L,X,Y,R_i)_{i\in I}$ be a fuzzy context sequence with X and Y sets of objects and attributes respectively and $R_i \in L^{X\times Y}$. We define an L-fuzzy relation $\overline{R}^{(h)}$ (with the notation adopted in Definition 5), using an OWA operator F with a weighting tuple W of dimension $k = |I| - h + 1$ with $w_k = 1$ the only non-null value:

$$\overline{R}^{(h)}(x,y) = F(R_h(x,y), R_{h+1}(x,y) \dots R_{|I|}(x,y)),$$
$$\forall x \in X, y \in Y.$$

In other words:

$$\overline{R}^{(h)}(x,y) = \min_{i \geqslant h}\{R_i(x,y)\}, \forall x \in X, y \in Y$$

As in the previous section, instead of the minimum (that is very demanding) we can take the maximum, the average or other aggregation operators changing the weighting tuple W of the OWA operator.

Example 4. If we come back to the previous example and we want to study tendencies of the sequence, we can take a value h and analyze the L-fuzzy concepts.

For instance, if $h = 4$, we have the L-fuzzy relation:

$$\overline{R}^{(4)} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.6 & 0.8 & 0.8 \end{pmatrix}$$

and, taking as L-fuzzy context $(L, X, Y, \overline{R}^{(4)})$ and using the Lukasiewicz implication to obtain the L-fuzzy concepts associated with the crisp singletons, we have the following results:

$\{x_1\} \longrightarrow (\{x_1/1, x_2/0.9, x_3/1\}, \{y_1/0.1, y_2/0, y_3/0\})$
$\{x_2\} \longrightarrow (\{x_1/0.9, x_2/1, x_3/1\}, \{y_1/0, y_2/0.1, y_3/0\})$
$\{x_3\} \longrightarrow (\{x_1/0.2, x_2/0.2, x_3/1\}, \{y_1/0.6, y_2/0.8, y_3/0.8\})$

We can say that the future tendency is that only article x_3 will have good sales in all the establishments whereas x_1 and x_2 will not be sold much and always associated with x_3, the first one in the establishment y_1 essentially, and the second one in y_2.

Obviously, the smaller is the value of h, the safer will be the prediction that we do.

Moreover, we can establish comparisons between the different L-fuzzy concepts obtained from the different relations $\overline{R}^{(i)}, \forall i \in I$.

Proposition 3. *Consider $A \in L^X$. Let $(\overline{A}^k, \overline{B}^k)$ and $(\overline{A}^h, \overline{B}^h)$ be the L-fuzzy concepts associated with A in the L-fuzzy contexts $(L, X, Y, \overline{R}^{(k)})$ and $(L, X, Y, \overline{R}^{(h)})$ respectively, with $k, h \leqslant |I|$. If $k \leqslant h$ then $\overline{B}^k \leqslant \overline{B}^h$.*

Moreover, if we use a residuated implication operator $\mathscr{I}$ and a crisp singleton A, then

$$\overline{A}^k(x_i) = \overline{A}^h(x_i) = 1$$

with x_i the element of X where the crisp singleton A takes value 1.

A similar result is obtained taking as a starting point an L-fuzzy set of attributes $B \in L^Y$.

Proof. Similar to Proposition 2 taking into account that, in this case, if $k \leqslant h$ then $\overline{R}^{(k)} \leqslant \overline{R}^{(h)}$.

□

The meaning of this result is that if we look at the fuzzy intensions obtained for the different L-fuzzy contexts of the sequence, then they form a non-decreasing chain $\forall y \in Y$.

Example 5. In our example, the L-fuzzy concepts obtained taking as a starting point the crisp singleton $\{x_3\}$ in the L-fuzzy contexts $(L, X, Y, \overline{R}^{(4)})$ and $(L, X, Y, \overline{R}^{(5)})$, using the Lukasiewicz implication operator, are:

$\overline{R}^{(4)} : (\{x_1/0.2, x_2/0.2, x_3/1\}, \{y_1/0.6, y_2/0.8, y_3/0.9\})$
$\overline{R}^{(5)} : (\{x_1/0, x_2/0.1, x_3/1\}, \{y_1/0.8, y_2/1, y_3/0.9\})$

verifying the previous proposition.

On the other hand, since if an object and an attribute are related from instant h, they are related at least $|I| - h + 1$ times, hence a similar result between the L-fuzzy concepts obtained using Definition 5 and 6 can be seen.

Proposition 4. *If we take as starting point $A \in L^X$, then for any $h \in I$, the fuzzy intension $\overline{B}^h$ of the L-fuzzy concept $(\overline{A}^h, \overline{B}^h)$ obtained in $(L, X, Y, \overline{R}^{(h)})$ is included in the fuzzy intension B^k of the L-fuzzy concept (A^k, B^k) obtained in $(L, X, Y, R^{(k)})$ with $k = |I| - h + 1$. That is,*

$$\overline{B}^h(y) \leqslant B^k(y), \quad \forall y \in Y$$

We have also a similar result from $B \in L^Y$.

Proof. Immediate using the previous proposition proof and the inequality $\overline{R}^{(h)} \leqslant R^{(k)}$ with $k = |I| - h + 1$. □

Example 6. If we take $h = 4$ and $k = 2$, and the L-fuzzy contexts $(L,X,Y,\overline{R}^{(4)})$ and $(L,X,Y,R^{(2)})$, then taking as a starting point the object x_1 and using the Lukasiewicz implication operator, the following L-fuzzy concepts are obtained:

$$\overline{R}^{(4)} : (\{x_1/1, x_2/0.9, x_3/1\}, \{y_1/0.1, y_2/0, y_3/0\})$$
$$R^{(2)} : (\{x_1/1, x_2/0.2, x_3/0.6\}, \{y_1/1, y_2/1, y_3/1\})$$

And the previous proposition holds.

An important result is the one that allows the study of the attributes associated with some elements of X from an instant h in two different ways:

Theorem 5. *Given A a crisp subset of X, and $\mathscr{I}$ a residuated implication.*

The fuzzy intension $\overline{B}^h \in L^Y$ of the L-fuzzy concept derived from A in $(L,X,Y,\overline{R}^{(h)})$ is equal to the intersection of the fuzzy intensions B_i of the L-fuzzy concepts obtained in the L-fuzzy contexts (L,X,Y,R_i) with $i \geqslant h$. That is,

$$\overline{B}^h(y) = \min_{i \geqslant h} B_i(y), \quad \forall y \in Y$$

Proof. If we use a residuated implication operator $\mathscr{I}$, then we have that $\forall y \in Y$:

$$\overline{B}^h(y) = \inf_{x \in X}\{\mathscr{I}(A(x), \overline{R}^{(h)}(x,y))\} = \min_{x \in X/A(x)=1} \overline{R}^{(h)}(x,y)$$

By the definition of $\overline{R}^h(x,y)$ we can say that:

$$\overline{B}^h(y) = \min_{x \in X/A(x)=1} \{\min_{i \geqslant h}\{R_i(x,y)\}\} =$$
$$= \min_{i \geqslant h}\{\min_{x \in X/A(x)=1}\{R_i(x,y)\}\} = \min_{i \geqslant h} B_i(y).$$

□

This result can be generalized replacing the minimum by any OWA operator in the proposition and in the definition of the relations $\overline{R}^{(h)}$.

Remark 1. This proposal justifies the utility of the defined relations $\overline{R}^{(h)}$ since allows the study of the attributes associated with some objects from an instant h looking only at the L-fuzzy context $(L,X,Y,\overline{R}^{(h)})$ instead of all the L-fuzzy contexts of the sequence.

3.2.2. Temporal patterns

In this section, we want to study temporal patterns in the sense of [1] to identify the evolution of a set of contexts with time.

Our interest is the study of tendencies of the attributes with respect to one or several objects by means of new definitions (these attributes do not necessarily share those objects in a certain L-fuzzy concept).

We will also use here residuated implication operators in the calculus of the L-fuzzy concepts associated with determined objects.

We are going to see a first definition.

Definition 7. Consider $x_0 \in X$ and $A \in L^X$ the singleton $\{x_0\}$. Consider (A_i, B_i) the L-fuzzy concepts associated with A in the fuzzy context sequence (L,X,Y,R_i) with $i \in I$.

The attribute set whose membership degrees in the different L-fuzzy concepts (A_i, B_i) are non-decreasing, $\forall i \in I$, is said to be $Trend(x_0)$:

$$Trend(x_0) = \{y \in Y \;/ B_i(y) \leqslant B_{i+1}(y), \forall i < |I|\}$$

Example 7. If we come back to Example 3 and obtain the fuzzy intensions of the L-fuzzy concepts derived from x_1, x_2 and x_3, then we have:

$Trend(x_1) = \emptyset$
$Trend(x_2) = \emptyset$
$Trend(x_3) = \{y_1, y_3\}$

This is a very demanding definition but it allows to establish patterns with a high degree of fulfillment.

We can extend this definition to any crisp subset of X:

Definition 8. For all $Z \subseteq X$, we define:

$$Trend(Z) = \{y \in Y \;/ B_{x_i}(y) \leqslant B_{x_{i+1}}(y), \forall i < |I|, \forall x \in Z\}$$

where B_{x_i} is the fuzzy intension of the L-fuzzy concept obtained taking the crisp singleton associated with $x \in Z$ in the L-fuzzy context (L,X,Y,R_i).

As a particular case, we have the set $Trend(X)$ for which the following result is provided:

Proposition 6. *It is verified that $Trend(X) = Y$ if and only if the L-fuzzy relations $R^{(k)}$ and $\overline{R}^{(h)}$ defined in Definitions 5 and 6, and R_h given in the fuzzy context sequence definition, are coincident for all $k,h \in I$ such that $k = |I| - h + 1$.*

Proof. If $\forall y \in Y$, it is verified that $y \in Trend(x) \forall x \in X$, then this means that $\forall y \in Y, \forall x \in X, B_{x_i}(y) \leqslant B_{x_{i+1}}(y), \forall i < |I|$ where B_{x_i} is the fuzzy intension of the L-fuzzy context derived from the crisp singleton associated with $x \in X$ in the L-fuzzy context (L,X,Y,R_i).

This is equivalent to
$R_i(x,y) \leqslant R_{i+1}(x,y), \forall x \in X, \forall y \in Y, \forall i < |I| \iff R^{(k)}(x,y) = R_{|I|-k+1}(x,y) = \min\limits_{i \geqslant h}\{R(x,y)\} =$
$= \overline{R}^{(h)}(x,y)$.

□

Remark 2. This is a particular but very interesting case for some practical situations. In our example, as the L-fuzzy contexts store sales, then we are saying that the sales are always increasing.

In most of the cases, this result only holds for some values of x and y.

As particular cases, we have $Trend(x_0) = Y$ and $Trend(X) = y_0$. In the first case, all the L-fuzzy relations of the sequences are non-decreasing for row x_0, that is, the membership degrees of the attributes are non-decreasing for all the L-fuzzy concepts associated with x_0 in the different L-fuzzy contexts of the sequence. In the second case, the same is verified for attribute y_0.

Next, by means of the following result, we prove that if the attributes are $Trend$, then the membership degrees of the L-fuzzy concepts obtained from R_i and $\overline{R}^{(i)}$ are coincident for a crisp singleton. This result does not hold if the attributes are not $Trend$, therefore, the study of the L-fuzzy relations $\overline{R}^{(i)}$ when we want to study future tendencies will be very important in that case.

Proposition 7. *Given $x_0 \in X$ and $A \in L^X$ the singleton $\{x_0\}$. Consider (A_i, B_i) and $(\overline{A}^i, \overline{B}^i), \forall i \in I$, the L-fuzzy concepts associated with A in the L-fuzzy contexts (L,X,Y,R_i) and $(L,X,Y,\overline{R}^{(i)})$, respectively. The attribute $y \in Trend(x_0)$ if and only if $B_i(y) = \overline{B}^i(y), \forall i < |I|$.*

Proof. $\Longrightarrow]$ If $y \in Trend(x_0)$, then $B_i(y) \leqslant B_{i+1}(y), \forall i < |I|$. Moreover $B_i(y) = R_i(x_0,y), \forall i \in I$ then, we can say that $R_i(x_0,y) \leqslant R_{i+1}(x_0,y), \forall i < |I|$. So, $\forall i \in I$, $\overline{B}^i(y) = \overline{R}^{(i)}(x_0,y) = \min\limits_{k \geqslant i}\{R_k(x_0,y)\} = R_i(x_0,y) = B_i(y)$.

$\Longleftarrow]$ If $B_i(y) = \overline{B}^i(y), \forall i < |I|$, as by definition $\overline{B}^i(y) = \overline{R}^{(i)}(x_0,y)$ and the L-fuzzy relations $\overline{R}^{(i)}(x_0,y) \leqslant \overline{R}^{(i+1)}(x_0,y), \forall i \in |I|$, we can prove that $B_i(y) \leqslant B_{i+1}(y), \forall i < |I|$. Therefore, $y \in Trend(x_0)$.

□

As we can see in the following example, the equality does not hold if $y \notin Trend(x_0)$.

Example 8. If we take, for instance, x_3 and we calculate the fuzzy intensions of the L-fuzzy concepts for the different L-fuzzy relations $R_i, i \in I$, we obtain the following results:

$$B_1 = \{y_1/0, y_2/0.1, y_3/0\}$$
$$B_2 = \{y_1/0, y_2/0, y_3/0.2\}$$
$$B_3 = \{y_1/0, y_2/0.1, y_3/0.2\}$$
$$B_4 = \{y_1/0.6, y_2/0.8, y_3/0.8\}$$
$$B_5 = \{y_1/0.8, y_2/1, y_3/0.9\}$$

In this case, y_1 and y_3 are $Trend$ attributes for x_3 whereas y_2 not.

Now, if we obtain the fuzzy intensions in the L-fuzzy contexts associated with the L-fuzzy relations $\overline{R}^{(i)}$ we have:

$$\overline{B}^1 = \{y_1/0, y_2/0, y_3/0\}$$
$$\overline{B}^2 = \{y_1/0, y_2/0, y_3/0.2\}$$
$$\overline{B}^3 = \{y_1/0, y_2/0.1, y_3/0.2\}$$
$$\overline{B}^4 = \{y_1/0.6, y_2/0.8, y_3/0.8\}$$
$$\overline{B}^5 = \{y_1/0.8, y_2/1, y_3/0.9\}$$

We can see that

$$B_i(y_1) = \overline{B}^i(y_1), B_i(y_3) = \overline{B}^i(y_3), \forall i < |I|,$$

but $B_1(y_2) \neq \overline{B}^1(y_2)$.

In same cases, for instance when $|I|$ is very large, a partial study throughout a time interval can be interesting:

Definition 9. Consider $x_0 \in X$ and $i_1, i_2 \in I$ such that $i_1 \leqslant i_2$.

$$Trend_{[i_1,i_2]}(x_0) = \{y \in Y \,/ B_i(y) \leqslant B_{i+1}(y), \forall i, i_1 \leqslant i < i_2\}$$

That is, this is the set of attributes whose membership degrees are non-decreasing in the fuzzy intensions of the L-fuzzy concepts derived from the L-fuzzy contexts (L,X,Y,R_i), with $i_1 \leqslant i \leqslant i_2$.

Example 9. If we fix $[i_1,i_2] = [1,3]$ in the previous example, we have:
$Trend_{[1,3]}(x_1) = \{y_1, y_3\}$
$Trend_{[1,3]}(x_2) = \{y_1, y_2, y_3\}$
$Trend_{[1,3]}(x_3) = \{y_1, y_3\}$
Therefore:
$Trend_{[1,3]}(x_1,x_2,x_3) = \{y_1, y_3\}$

As a consequence of Proposition 7 we can establish the following corollary:

Corollary 8. *Given $x_0 \in X$ and $A \in L^X$ the singleton $\{x_0\}$. Consider (A_i,B_i) and $(\overline{A}^i, \overline{B}^i), \forall i \in I, i_1 \leqslant i \leqslant i_2$ the L-fuzzy concepts associated with A in the L-fuzzy contexts (L,X,Y,R_i) and $(L,X,Y,\overline{R}^{(i)})$, respectively. The attribute $y \in Trend_{[i_1,i_2]}(x_0)$ if and only if $B_i(y) = \overline{B}^i(y), \forall i \in I, i_1 \leqslant i < i_2$.*

The intervals where $i_2 = |I|$ are of special interest since we are studying future tendencies in that case, what will allow to establish a relationship with relations $\overline{R}^{(i)}$.

As the definition of *Trend* is very demanding, we can give a second one:

Definition 10. Consider $x_0 \in X$ and $A \in L^X$ the singleton $\{x_0\}$. Let (A_i,B_i) be the L-fuzzy concepts associated with A in the fuzzy context sequence (L,X,Y,R_i) with $i \in I$:

$$Persistent(x_0) = \{y \in Y \,/ B_i(y) \geqslant B_1(y), \forall i, 1 < i \leqslant |I|\}$$

is the set of attributes whose membership degrees in the fuzzy intensions of the L-fuzzy concepts (A_i,B_i) with $i \in I$ are bigger than or equal to the values of the L-fuzzy concept (A_1,B_1).

In the same conditions:

$$Transient(x_0) = \{y \in Y \,/ y \notin Persistent(x_0)\}.$$

Example 10. In our example:
$Persistent(x_1) = \emptyset$, $Transient(x_1) = \{y_1, y_2, y_3\}$
$Persistent(x_2) = \{y_1, y_2\}$, $Transient(x_2) = \{y_3\}$
$Persistent(x_3) = \{y_1, y_3\}$, $Transient(x_3) = \{y_2\}$

As can be seen, the main difference between *Trend* and *Persistent* for this example is in object x_2 (we remind that $Trend(x_2) = \emptyset$). That is, there is no establishment with increasing sales although these sales are bigger than the ones of the first month.

With this definition, Propositions 6 and 7 are not necessarily verified, as can be seen in the following example:

Example 11. Let us suppose that we have a fuzzy context sequence (L,X,Y,R_i) with $i = 1,2,3$ associated with the L-fuzzy relations:

$$R_1 = \begin{pmatrix} 0.3 & 0.2 \\ 0.8 & 0.5 \end{pmatrix} R_2 = \begin{pmatrix} 0.5 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0.4 & 0.9 \\ 1 & 0.7 \end{pmatrix}$$

In this case, we can say that $Persistent(X) = Y$ since the fuzzy intensions of the L-fuzzy concepts associated with x_1 and x_2 in R_2 and R_3 are bigger than or equal to the R_1 ones. However, if we take $\overline{R}^{(2)}$ and $R^{(2)}$, we have the following result:

$$\overline{R}^{(2)} = \begin{pmatrix} 0.4 & 0.4 \\ 0.8 & 0.6 \end{pmatrix} R^{(2)} = \begin{pmatrix} 0.4 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}$$

that is, they are not equal to R_2. Therefore, the *Persistent* definition does not verify Proposition 6.

Neither Proposition 7 is fulfilled as we are going to see:

Consider $A = \{x_1/1, x_2/0\}$. Let $B_2 = \{y_1/0.5, y_2/0.4\}$ and $\overline{B}^{(2)} = \{y_1/0.4, y_2/0.4\}$ be the fuzzy intensions of the L-fuzzy concepts associated with A in the L-fuzzy contexts (L,X,Y,R_2)

and $(L,X,Y,\overline{R}^{(2)})$. Then, $y_1 \in Persistent(x_1)$ but $B_2(y_1) \neq \overline{B}^{(2)}(y_1)$.

Also in this case we can establish definitions in a certain interval:

Definition 11. Consider $x_0 \in X$ and $i_1, i_2 \in I$ such that $i_1 \leqslant i_2$.

$$Persistent_{[i_1,i_2]}(x_0) = \{y \in Y \,/ B_i(y) \geqslant B_{i_1}(y), \forall i, i_1 < i \leqslant i_2\}$$

The *Persistent* definition is related to Definitions 5 and 6, verifying the following result:

Proposition 9. *Given $i_0 \in I$, $Persistent_{[i_0,|I|]}(X) = Y$, if and only if the L-fuzzy relations $R^{(k)}$ and $\overline{R}^{(i_0)}$ defined in 5 and 6 and R_{i_0} given in the fuzzy context sequence definition, are coincident for $k = |I| - i_0 + 1$.*

Proof. If $\forall y \in Y, y \in Persistent_{[i_0,|I|]}(x) \forall x \in X$, then $\forall y \in Y, \forall x \in X, B_{x_i}(y) \leqslant B_{x_{i_0}}(y), \forall i < |I|, i \geqslant i_0$, where B_{x_i} is the fuzzy intension of the L-fuzzy concept derived from the crisp singleton associated with $x \in X$ in the L-fuzzy context (L,X,Y,R_i).

That is equivalent to $R_i(x,y) \geqslant R_{i_0}(x,y), \forall x \in X, \forall y \in Y, \forall i \in |I|, i \geqslant i_0 \iff$ given $k = |I| - i_0 + 1, R^{(k)}(x,y) = R_{|I|-k+1}(x,y) = R_{i_0} = \min_{i \geqslant i_0}\{R(x,y)\} = \overline{R}^{(i_0)}(x,y)$. □

In particular, when $i_0 = 1$ we have that $Persistent(X) = Y \iff R^{(|I|)} = \overline{R}^{(1)} = R_1$.

Finally, we have to say that neither in the definition of $Trend(x_0)$ nor in the $Persistent(x_0)$ one, a high membership degree (close to 1) for the attribute in the L-fuzzy concepts is demanded, which would assure somehow that this is an attribute associated with the object x_0.

4. Conclusions and future work

In this work, we have used OWA operators to study the fuzzy context sequence and the derived information by means of the L-fuzzy contexts.

After that, we have studied tendencies and patterns that we find when the sequence represents the evolution in the course of time of an L-fuzzy context. In the future we want to study if there are more suitable fuzzy definitions for this situation.

On the other hand, these L-fuzzy contexts that evolve with time can be generalize if we study L-fuzzy contexts where the observations are other L-fuzzy contexts. This is the task that we will study in the future.

Finally we will try to apply these results to the interval-valued fuzzy context sequences.

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