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A theoretical development on the entropy of interval-valued intuitionistic fuzzy soft sets based on the distance measure

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Abstract

In this work, the axiomatical definition of similarity measure, distance measure and inclusion measure for interval-valued intuitionistic fuzzy soft set ($IVIFSSs$) are given. An axiomatical definition of entropy measure for $IVIFSSs$ based on distance is firstly proposed, which is consistent with the axiomatical definition of fuzzy entropy of fuzzy sets introduced by De Luca and Termini. By different compositions of aggregation operators and a fuzzy negation operator, we obtain eight general formulae to calculate the distance measures of $IVIFSSs$ based on fuzzy equivalences. Then we discuss the relationships among entropy measures, distance measures, similarity measures and inclusion measures of $IVIFSSs$. We prove that the presented entropy measures can be transformed into the similarity measures and the inclusion measures of $IVIFSSs$ based on fuzzy equivalences.

Keywords: interval-valued intuitionistic fuzzy soft set; entropy; similarity measures; inclusion measures; fuzzy equivalences.

1. Introduction

Many new set theories treating imprecision and uncertainty have been proposed since fuzzy sets were introduced by Zadeh[1]. Atanassov's intuitionistic fuzzy sets[3] ($IFSs$), vague sets[4] and interval-valued fuzzy sets[20,21] ($IVFSs$), as extensions of classic fuzzy set theory, are proved to be useful in dealing with imprecision and uncertainty. As a combining concept of $IFSs$ and $IVFSs$, interval-valued intuitionistic fuzzy sets ($IVIFSs$) introduced by Atanassov [5] greatly furnishes the additional capability to model non-statistical uncertainty by providing a membership interval and a non-membership interval. Therefore, $IVIFSs$ play a significant role in the uncertain system and receives much attention. The concept of soft set theory, which can be used as a general mathematical tool for dealing with uncertainty, is initiated by Molodtsov[6] in 1999. Since it has been pointed out that classical soft sets are not appropriate to deal with imprecise and fuzzy parameters, some fuzzy (or intuitionistic fuzzy, interval-valued fuzzy) extensions of soft set theory , yielding fuzzy (or intuitionistic fuzzy, interval-valued fuzzy) soft set theory [6,7,8,9,10,11] has been presented to deal with imprecise and fuzzy parameters. Recently, by combining the interval-valued intuitionistic fuzzy sets and soft sets, Jiang et al [12] propose a new soft set model: interval-valued intuitionistic fuzzy soft sets ($IVIFSSs$). Intuitively,

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interval-valued intuitionistic fuzzy soft set can be regarded as an interval-valued fuzzy extension of the intuitionistic fuzzy soft set [8,9,10] or an intuitionistic fuzzy extension of the interval-valued fuzzy soft set [11].

Some scholars have already noticed and studied entropy measures based on distance for fuzzy sets and extensions of fuzzy sets. Mi[13] extended De Lucas axioms [2] to introduce an entropy of fuzzy set based on fuzzy distance. Later, Farhadinia[16] propose a class of entropies of $IVFSs$ based on the distance measure and investigate the relationship between the entropy measure and the similarity measure. Zhang et al[17] propose an axiomatical definition of entropy measure for $IVIFSs$ based on distances and discuss the relationship between entropy with similarity and inclusion measure. However, few scholars have paid attention to the entropy measures based on distance for fuzzy (or intuitionistic fuzzy, interval-valued fuzzy, interval-valued intuitionistic fuzzy) extensions of soft sets yet. In this work, we provide an axiomatic definition of entropy based on distance for $IVIFSSs$ and discuss the relationship between entropy measure with similarity, distance and inclusion measures for $IVIFSSs$. There are several reasons that motivate us to do this research. Firstly, although there are a number of researches regarding entropy measures for hybrid fuzzy set theory, few literatures studied the entropy measure of $IVIFSSs$; Secondly, the uncertain measures of $IVIFSSs$ have great application potential in many fields such as uncertain system control, decision-making and pattern recognition; Thirdly, the study of relationships between different measure benefits us in achieving as more information as possible through each measure. This new extension not only provides a significant addition to existing theories for handling uncertainties, but also leads to potential areas of further research field and pertinent applications. It is worth noticing that we give a method to construct the distance measures of $IVIFSSs$ by aggregating fuzzy equivalencies and prove that the presented entropy measures can be transformed into the similarity measures and the inclusion measures of $IVIFSSs$ based on fuzzy equivalences.

The structure of this paper is as follows. Section 2 reviews some concepts which are necessary for our paper. Section 3 provides the axiomatic definitions of similarity measure, distance measure and inclusion measure of $IVIFSSs$, an information entropy based on distance is also introduced to estimate uncertainty in $IVIFSSs$. Corresponding calculate formulae or construction methods of these measures are also given. In section 4, we investigate the relationship between the entropy measure and other uncertain measures of $IVIFSSs$, prove that both the similarity measures and the inclusion measures of $IVIFSSs$ can be constructed by entropy measures of $IVIFSSs$. In section 5, an application of the entropy and the distance measure of $IVIFSSs$ is given. This paper is concluded in Section 6.

2. Preliminaries

In this section, we shall recall several definitions which are necessary for our paper.

Let U be the universe of discourse and P be the set of all possible parameters related to the objects in U. In the following discussion, we assume that both U and P are nonempty finite sets.

Definition 1. [6] Let $\mathscr{P}(U)$ be the power set of U, a pair (F,A) is called a soft set in the universe U, where $A \in P$ and F is a mapping given by

$$F : A \longrightarrow \mathscr{P}(U)$$

In other words, the soft set is not a kind of set in ordinary sense, but a parameterized family of subsets of the set U. For any parameter $e_i \in A, F(e_i) \subseteq U$ may be considered as the set of e_i- approximate elements of the soft set (F,A).

Interval-valued intuitionistic fuzzy set was first introduced by Atanassov and Gargov [18]. It is characterized by an interval-valued membership degree and an interval-valued non-membership degree.

Definition 2. [5,18] An interval-valued intuitionistic fuzzy set on a universe U is an object of the form $A = \{(x, u_A(x), v_A(x))/x \in U\}$, where $u_A : U \longrightarrow Int([0,1])$ and $v_A : U \longrightarrow Int([0,1])$ satisfy the following condition: $\forall x \in U,\ sup(u_A(x)) +$

$sup(v_A(x)) \leqslant 1$. ($Int([0,1])$ stands for the set of all closed subintervals of $[0,1]$).

The class of all interval-valued intuitionistic fuzzy sets ($IVIFSs$) on U will be denoted by $IVIFS(U)$.

For an arbitrary set $A \subseteq [0,1]$, define $\overline{A} = supA$ and $\underline{A} = infA$. The interval-valued intuitionistic fuzzy set A can be written as

$$A = \{(x, [\underline{u}_A(x), \overline{u}_A(x))], [\underline{v}_A(x), \overline{v}_A(x))]/x \in U\}$$

with the condition: $0 \leqslant \overline{u}_A(x) + \overline{v}_A(x) \leqslant 1$ for all $x \in U$.

The union, intersection and complement of the interval-valued intuitionistic fuzzy sets are defined as follows: let $A, B \in IVIFS(U)$, then
1) the union of A and B is denoted by $A \cup B$ where

$$A \cup B = \{\langle x, [sup(\underline{u}_A(x), \underline{u}_B(x)), sup(\overline{u}_A(x), \overline{u}_B(x))], [inf(\underline{v}_A(x), \underline{v}_B(x)), inf(\overline{v}_A(x), \overline{v}_B(x))]\rangle | x \in U\}.$$

2)the intersection of A and B is denoted by $A \cap B$ where

$$A \cap B = \{\langle x, [inf(\underline{u}_A(x), \underline{u}_B(x)), inf(\overline{u}_A(x), \overline{u}_B(x)], [sup(\underline{v}_A(x), \underline{v}_B(x)), sup(\overline{v}_A(x), \overline{v}_B(x))]\rangle | x \in U\}.$$

3) the complement of A is denoted by A^C where

$$A^C = \{\langle x, v_A(x), u_A(x)\rangle\}.$$

Atanassov[5] shows that $A \cup B$, $A \cap B$ and A^C are again interval-valued intuitionistic fuzzy sets.

Jiang et al.[12] define interval-valued intuitionistic fuzzy soft sets ($IVIFSSs$) by combining interval-valued intuitionistic fuzzy sets and soft sets, and then give some operations on $IVIFSSs$.

Definition 3. [12] A pair (F,A) is an interval-valued intuitionistic fuzzy soft set over U, where $A \in P$ and F is a mapping given by

$$F : A \longrightarrow IVIFS(U)$$

The class of all interval-valued intuitionistic fuzzy soft sets over U will be denoted by $IVIFSS(U)$.

An interval-valued intuitionistic fuzzy soft set is a parameterized family of interval-valued intuitionistic fuzzy subsets of U, thus, its universe is the set of all interval-valued intuitionistic fuzzy sets of U, i.e., $IVIFS(U)$. For any parameter $e_i \in A$, $F(e_i)$ is referred as the interval-valued intuitionistic fuzzy value set of parameter e_i, it can be written as:

$$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$$

with the condition $0 \leqslant \overline{u}_{F(e_i)}(x_j) + \overline{v}_{F(e_i)}(x_j) \leqslant 1$. Here, $u_{F(e_i)}(x_j)$ is the interval-valued fuzzy membership degree that object x_j holds on parameter e_i, $v_{F(e_i)}(x_j)$ is the interval-valued fuzzy non-membership degree that object x_j holds on parameter e_i.

Definition 4. [19] Let $[a_1, b_1], [a_2, b_2] \in Int([0,1])$, we define
$[a_1, b_1] \leqslant [a_2, b_2]$; iff $a_1 \leqslant a_2$; $b_1 \leqslant b_2$;
$[a_1, b_1] \preceq [a_2, b_2]$; iff $a_1 \leqslant a_2$; $b_1 \geqslant b_2$;
$[a_1, b_1] = [a_2, b_2]$; iff $a_1 = a_2$; $b_1 = b_2$.

Definition 5. [12] Let U be an initial universe and P be a set of parameters. Suppose that $A, B \subseteq P$, (F,A) and (G,B) are two interval-valued intuitionistic fuzzy soft sets, we say that (F,A) is an interval-valued intuitionistic fuzzy soft subset of (G,B) if and only if

(1) $A \subseteq B$;

(2) $\forall e_i \in A$, $F(e_i)$ is an interval-valued intuitionistic fuzzy subset of $G(e_i)$, that is, $[\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)] \leqslant [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)]$ and $[\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)] \geqslant [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]$ for all $x_j \in U$, $e_i \in A$.

This relationship is denoted by $(F,A) \subseteq (G,B)$. (F,A) and (G,B) are said to be intuitionistic equal if and only if $(F,A) \supseteq (G,B)$ and $(F,A) \subseteq (G,B)$ at the same time, we write $(F,A) = (G,B)$.

The union and intersection of the interval-valued intuitionistic fuzzy soft sets are defined[12] as follows: let (F,A), $(G,B) \in IVIFSS(U)$, then
1) The union of (F,A) and (G,B) is an interval-valued intuitionistic fuzzy soft set (H,C), where

$C = A \cup B$ and $e_i \in C$.
$u_{H(e_i)}(x_j) = u_{F(e_i)}(x_j)$, $v_{H(e_i)}(x_j) = v_{F(e_i)}(x_j)$, if $e_i \in A \setminus B, x_j \in U$;
$u_{H(e_i)}(x_j) = u_{G(e_i)}(x_j)$, $v_{H(e_i)}(x_j) = v_{G(e_i)}(x_j)$, if $e_i \in B \setminus A, x_j \in U$;
$u_{H(e_i)}(x_j) = [sup(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), sup(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))]$,
$v_{H(e_i)}(x_j) = [inf(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)), inf(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))]$ if $e_i \in A \cap B, x_j \in U$.
We denote it by $(F,A) \cup (G,B) = (H,C)$.
2) The intersection of (F,A) and (G,B) is an interval-valued intuitionistic fuzzy soft set (H,C), where $C = A \cup B$ and $e_i \in C$.
$u_{H(e_i)}(x_j) = u_{F(e_i)}(x_j)$, $v_{H(e_i)}(x_j) = v_{F(e_i)}(x_j)$, if $e_i \in A \setminus B, x_j \in U$;
$u_{H(e_i)}(x_j) = u_{G(e_i)}(x_j)$, $v_{H(e_i)}(x_j) = v_{G(e_i)}(x_j)$, if $e_i \in B \setminus A, x_j \in U$;
$u_{H(e_i)}(x_j) = [inf(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), inf(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))]$,
$v_{H(e_i)}(x_j) = [sup(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)), sup(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))]$,
if $e_i \in A \cap B, x_j \in U$.
We denote it by $(F,A) \cap (G,B) = (H,C)$.

Definition 6. The relative complement of an interval-valued intuitionistic fuzzy soft set (F,A) is denoted by $(F,A)^C$ and is defined by $(F,A)^C = (F^C, A)$, where $F^C : A \longrightarrow IVIFS(U)$ is a mapping given by $F^C(e_i) = \{\langle x_j, v_{F(e_i)}(x_j), u_{F(e_i)}(x_j)\rangle | x_j \in U\}$ for all $e_i \in A$.

Definition 7. [12] An interval-valued intuitionistic fuzzy soft set (F,A) over U is said to be a null interval-valued intuitionistic fuzzy soft set denoted by $(\emptyset, A)$, if $u_{F(e_i)}(x_j) = [0,0], v_{F(e_i)}(x_j) = [1,1]$ for all $e_i \in A$, $x_j \in U$.

Definition 8. [12] An interval-valued intuitionistic fuzzy soft set (F,A) over U is said to be an absolute interval-valued intu- itionistic fuzzy soft set denoted by (U,A), if $u_{F(e_i)}(x_j) = [1,1], v_{F(e_i)}(x_j) = [0,0]$ for all $e_i \in A$, $x_j \in U$.

3. The distance, similarity, inclusion measure and entropy of *IVIFSSs*

3.1. Axiomatic definitions

In this subsection, we extend the axiomatic definitions of the distance, similarity, inclusion measure and entropy of *IVIFSs* in Ref. [17] to *IVIFSSs*.

Definition 9. Let (F,P), (G,P) and (H,P) be interval-valued intuitionistic fuzzy soft sets over U, i.e., $(F,P),(G,P),(H,P) \in IVIFSS(U)$. Let D be a mapping $D : IVIFSS(U) \times IVIFSS(U) \longrightarrow [0,1]$. If $D((F,P),(G,P))$ satisfies the following properties ((1)-(4)):

(1) $D((F,P),(F,P)^C) = 1$, if (F,P) is a classical soft set;

(2) $D((F,P),(G,P)) = 0$, iff $(F,P) = (G,P)$;

(3) $D((F,P),(G,P)) = D((G,P),(F,P))$;

(4) $D((F,P),(H,P)) \geqslant D((F,P),(G,P))$ and $D((F,P),(H,P)) \geqslant D((G,P),(H,P))$, if $(F,P) \subseteq (G,P) \subseteq (H,P)$.

Then $D((F,P),(G,P))$ is a distance measure between interval-valued intuitionistic fuzzy soft sets (F,P) and (G,P).

Definition 10. Let (F,P), (G,P) and (H,P) be interval-valued intuitionistic fuzzy soft sets over U, i.e., $(F,P),(G,P),(H,P) \in IVIFSS(U)$. Let S be a mapping $S : IVIFSS(U) \times IVIFSS(U) \longrightarrow [0,1]$. If $S((F,P),(G,P))$ satisfies the following properties ((1)-(4)):

(1) $S((F,P),(F,P)^C) = 0$, if (F,P) is a classical soft set;

(2) $S((F,P),(G,P)) = 1$, iff $(F,P) = (G,P)$;

(3) $S((F,P),(G,P)) = S((G,P),(F,P))$;

(4) $S((F,P),(H,P)) \leqslant S((F,P),(G,P))$ and $S((F,P),(H,P)) \leqslant S((G,P),(H,P))$, if $(F,P) \subseteq (G,P) \subseteq (H,P)$.

Then $S((F,P),(G,P))$ is a similarity measure between interval-valued intuitionistic fuzzy soft sets (F,P) and (G,P).

Definition 11. A real function $J : IVIFSS(U) \times IVIFSS(U) \longrightarrow [0,1]$ is named as the inclusion measure of interval-valued intuitionistic fuzzy soft sets, if J has the following properties:

(1) If $(F,P) = (U,P), (G,P) = (\emptyset,P)$, then $J((F,P),(G,P)) = 0$;

(2) $J((F,P),(G,P)) = 1$, iff $(F,P) \subseteq (G,P)$;

(3) If $(F,P) \subseteq (G,P) \subseteq (H,P)$, then $J((H,P),(F,P)) \leqslant J((G,P),(F,P))$ and $J((H,P),(F,P)) \leqslant J((H,P),(G,P))$.

Then $J((F,P),(G,P))$ is called an inclusion measure of interval-valued intuitionistic fuzzy soft sets.

Definition 12. Let (Q,P) be an interval-valued intuitionistic fuzzy soft set on U, s.t. for $\forall e_i \in P$, $Q(e_i) = \{\langle x_j, [1/2,1/2], [1/2,1/2]\rangle | x_j \in U\}$. A real function $I : IVIFSS(U) \longrightarrow [0,1]$ is called an entropy for interval-valued intuitionistic fuzzy soft sets, if I has the following properties:

(1) $I((F,P)) = 0$ if (F,P) is a classical soft set;

(2) $I((F,P)) = 1$ iff $u_{F(e_i)}(x_j) = v_{F(e_i)}(x_j) = [1/2,1/2], \forall e_i \in P, x_j \in U$;

(3) $I((F,P)) = I((F,P)^C)$;

(4) $I((F,P)) \leqslant I((G,P))$, if $D((F,P),(Q,P)) \geqslant D((G,P),(Q,P))$.

Here, the requirement (2) implies that entropy of (F,P) will be maximum if (F,P) is equal to (Q,P); the requirement (4) implies that the closer an interval-valued intuitiionistic fuzzy soft set (F,P) is to (Q,P), the more entropy of (F,P) should decrease.

3.2. *Some general formulae to construct the distance measure of* $IVIFSSs$

Before giving some general formulae to construct the distance measure of $IVIFSSs$, we review the notions of aggregation operators and equivalence operators.

Definition 13. [14] A function $M : \bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$ is an aggregation operator if it satisfies the following properties: for each $n \in N$ and $x_i, y_i \in [0,1]$,

(1) $M(x_i) = x_i$.

(2) $M(\underbrace{0,0,\ldots,0}_{\text{n times}}) = 0$.

(3) $M(\underbrace{1,1,\ldots,1}_{\text{n times}}) = 1$.

(4) $M(x_1,x_2,...x_n) \leqslant M(y_1,y_2,...y_n)$ whenever $x_i \leqslant y_i, \forall i \in \{1,2,...n\}$.

severe

This definition allows us to introduce the following notions:

An aggregation operator $M : \bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$ is called a severe-aggregation operator if it satisfies properties: for each $n \in N$ and $x_i \in [0,1](i = \{1,2,\ldots,n\})$,

(5) $M(x_1,x_2,...x_n) < 1$ if $x_i < 1, \forall i \in \{1,2,...n\}$.

(6) $M(x_1,x_2,...x_n) > 0$ if $x_i > 0, \forall i \in \{1,2,...n\}$.

An aggregation operator $M : \bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$ is called a top-aggregation operator if it satisfies property: for each $n \in N$ and $x_i \in [0,1](i = \{1,2,\ldots,n\})$,

(7) $M(x_1,x_2,...x_n) = 1 \Leftrightarrow x_i = 1, \forall i \in \{1,2,...n\}$.

An aggregation operator $M : \bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$ is called a bottom-aggregation operator if it satisfies property: for each $n \in N$ and $x_i \in [0,1](i = \{1,2,\ldots,n\})$,

(8) $M(x_1,x_2,...x_n) = 0 \Leftrightarrow x_i = 0, \forall i \in \{1,2,...n\}$.

An aggregation operator $M : \bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$ is called an idempotent-aggregation operator if it satisfies property: for each $n \in N$ and $x \in [0,1]$,

(9) $M(\underbrace{x,x,\ldots,x}_{\text{n times}}) = x$ for $\forall x \in [0,1]$.

Example 1. As examples of the severe-aggregation operators, we take: for each $n \in N$ and $x_i \in [0,1](i = \{1,2,\ldots,n\})$,

(1) $M(x_1,x_2,...x_n)=\frac{1}{n}\sum_{i=1}^{n}x_i$.

(2) $M(x_1,x_2,...x_n)=\lambda min(x_1,x_2,...,x_n)+(1-\lambda)$ $max(x_1,x_2,...,x_n)$ with $\lambda\in[0,1]$.

(3) $M(x_1,x_2,...x_n)=max(x_1,x_2,...,x_n)/(max(x_1,$ $x_2,...,x_n)+max(1-x_1,1-x_2,...,1-x_n))$.

As examples of the top-aggregation operators, we take: for each $n\in N$ and $x_i\in[0,1](i=\{1,2,\dots,n\})$,

(1) $M(x_1,x_2,...,x_n)=(\frac{x_1^p+x_2^p+...+x_n^p}{n})^{\frac{1}{p}}$, $p\geqslant 1$.

(2) $M(x_1,x_2,...,x_n)=x_1^p\wedge x_2^p\wedge...\wedge x_n^p$, $p\geqslant 1$.

As examples of the bottom-aggregation operators, we take: for each $n\in N$ and $x_i\in[0,1](i=\{1,2,\dots,n\})$,

(1) $M(x_1,x_2,...,x_n)=(\frac{x_1^p+x_2^p+...+x_n^p}{n})^{\frac{1}{p}}$, $p\geqslant 1$.

(2) $M(x_1,x_2,...,x_n)=x_1^p\vee x_2^p\vee...\vee x_n^p$, $p\geqslant 1$.

As examples of the idempotent-aggregation operators, we take: for each $n\in N$ and $x_i\in[0,1](i=\{1,2,\dots,n\})$,

(1) $M(x_1,x_2,...,x_n)=(\frac{x_1^p+x_2^p+...+x_n^p}{n})^{\frac{1}{p}}$, $p\geqslant 1$.

(2) $M(x_1,x_2,...x_n)=\lambda min(x_1,x_2,...,x_n)+(1-\lambda)$ $max(x_1,x_2,...,x_n)$ with $\lambda\in[0,1]$.

(3) $M(x_1,x_2,...,x_n)=x_1\wedge x_2\wedge...\wedge x_n$.

(4) $M(x_1,x_2,...,x_n)=x_1\vee x_2\vee...\vee x_n$.

Definition 14. [15] A function $E:[0,1]^2\longrightarrow[0,1]$ is called a fuzzy equivalence if it satisfies the following properties:

(1) $E(x,y)=E(y,x)$ for all $x,y\in[0,1]$.

(2) $E(x,x)=1$ for all $x\in[0,1]$.

(3) $E(0,1)=E(1,0)=0$.

(4) For all $x,y,x',y'\in[0,1]$, if $x\leqslant x'\leqslant y'\leqslant y$, then $E(x,y)\leqslant E(x',y')$.

In this article, we strength condition (2) to (2′):

(2′) For all $x,y\in[0,1]$, $E(x,y)=1$ iff $x=y$.

Definition 15. [22] If a decreasing function $n:[0,1]\longrightarrow[0,1]$ satisfies the boundary conditions $n(0)=1$ and $n(1)=0$, then n is called a fuzzy negation.

If a fuzzy negation $n:[0,1]\longrightarrow[0,1]$ is a strictly decreasing function, it is called a strict fuzzy negation in this work.

By the compositions of three severe-aggregation operators and a strict fuzzy negation operator, we obtain eight general formulae to calculate the distance measures of $IVIFSSs$ based on fuzzy equivalencies.

Definition 16. Given $U=\{x_1,x_2,...,x_n\}$ and $P=\{e_1,e_2,...,e_m\}$. Let M_k $(k=1,2,3)$ be severe-aggregation operators. Let E_l $(l=1,2,3,4)$ be fuzzy equivalence operators and f be a strict fuzzy negation. Suppose $D_q(q=1,2,...,8):IVIFSS(U)\times IVIFSS(U)\longrightarrow[0,1]$ are functions defined for all $(F,P),(G,P)\in IVIFSS(U)$ as follows: for any $e_i\in P,x_j\in U$,

$$D_1((F,P),(G,P))=\overset{n}{\underset{j=1}{M_1}}(\overset{m}{\underset{i=1}{M_2}}(M_3(f(E_1(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),f(E_2(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),f(E_3(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)),f(E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))))))).\tag{1}$$

$$D_2((F,P),(G,P))=\overset{m}{\underset{i=1}{M_2}}(\overset{n}{\underset{j=1}{M_1}}(M_3(f(E_1(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),f(E_2(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),f(E_3(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)),f(E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))))))).\tag{2}$$

$$D_3((F,P),(G,P))=\overset{n}{\underset{j=1}{M_1}}(\overset{m}{\underset{i=1}{M_2}}(f(M_3(E_1(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_2(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_3(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)),E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))))))).\tag{3}$$

$$D_4((F,P),(G,P))=\overset{m}{\underset{i=1}{M_2}}(\overset{n}{\underset{j=1}{M_1}}(f(M_3(E_1(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_2(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_3(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)),E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))))))).\tag{4}$$

$$D_5((F,P),(G,P)) = \overset{n}{\underset{j=1}{M_1}}(f(\overset{m}{\underset{i=1}{M_2}}(M_3(E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)), E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)), E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)))))). \tag{5}$$

$$D_6((F,P),(G,P)) = \overset{m}{\underset{i=1}{M_2}}(f(\overset{n}{\underset{j=1}{M_1}}(M_3(E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)), E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)), E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)))))). \tag{6}$$

$$D_7((F,P),(G,P)) = f(\overset{n}{\underset{j=1}{M_1}}(\overset{m}{\underset{i=1}{M_2}}(M_3(E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)), E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)), E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)))))). \tag{7}$$

$$D_8((F,P),(G,P)) = f(\overset{m}{\underset{i=1}{M_2}}(\overset{n}{\underset{j=1}{M_1}}(M_3(E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)), E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)), E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)))))). \tag{8}$$

Theorem 1. *$D_q((F,P),(G,P))(q \in \{1,2,...8\})$ in Definition 16 are distance measures between interval-valued intuitionistic fuzzy soft sets (F,P) and (G,P).*

Proof. (1) If (F,P) is a classical soft set, we have
$[\underline{u}_{F(e_i)}(x_j), \bar{u}_{F(e_i)}(x_j)] = [1,1]$, $[\underline{v}_{F(e_i)}(x_j), \bar{v}_{F(e_i)}(x_j)] = [0,0]$ or
$[\underline{u}_{F(e_i)}(x_j), \bar{u}_{F(e_i)}(x_j)] = [0,0]$, $[\underline{v}_{F(e_i)}(x_j), \bar{v}_{F(e_i)}(x_j)] = [1,1]$, $\forall e_i \in P$, $x_j \in U$.
Then we get
$[\underline{u}_{F^C(e_i)}(x_j), \bar{u}_{F^C(e_i)}(x_j)] = [0,0]$, $[\underline{v}_{F^C(e_i)}(x_j), \bar{v}_{F^C(e_i)}(x_j)] = [1,1]$ or
$[\underline{u}_{F^C(e_i)}(x_j), \bar{u}_{F^C(e_i)}(x_j)] = [1,1]$, $[\underline{v}_{F^C(e_i)}(x_j), \bar{v}_{F^C(e_i)}(x_j)] = [0,0]$, $\forall e_i \in P$, $x_j \in U$.
By property (3) of fuzzy equivalence operators, we have
$E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{F^C(e_i)}(x_j)) = E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{F^C(e_i)}(x_j)) = E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{F^C(e_i)}(x_j)) = E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{F^C(e_i)}(x_j)) = 0$, $\forall e_i \in P$, $x_j \in U$.
Thus, we have
$D_q((F,P),(G,P)) = 1$ $(q \in \{1,2,...8\})$.
(2) If $(F,P) = (G,P)$, it is obviously that $D_q((F,P),(G,P)) = 0$ $(q \in \{1,2,...8\})$.
For $q \in \{1,2,...8\}$, assume that $D_q((F,P),(G,P)) = 0$, if there exists a $e_i \in P$, and a $x_j \in U$, s.t.
$E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)) < 1$ or
$E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)) < 1$ or
$E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)) < 1$ or
$E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))) < 1$,
since f is a strict fuzzy negation, we get $D_q((F,P),(G,P)) > 0$. It is a contradiction.
So, we have
$E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)) = E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)) = E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)) = E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))) = 1$, $\forall e_i \in P, x_j \in U$.
Thus, we have for any $e_i \in P, x_j \in U$,
$\bar{u}_{F(e_i)}(x_j) = \bar{u}_{G(e_i)}(x_j)$, $\underline{u}_{F(e_i)}(x_j) = \underline{u}_{G(e_i)}(x_j)$,
$\bar{v}_{F(e_i)}(x_j) = \bar{v}_{G(e_i)}(x_j)$, $\underline{v}_{F(e_i)}(x_j) = \underline{v}_{G(e_i)}(x_j)$,
that is, $(F,P) = (G,P)$.
(3) By the commutative law of the fuzzy equivalence operators, we can easily get that
$D_q((F,P),(G,P)) = D_q((G,P),(F,P))(q \in \{1,2,...8\})$.
(4) Since $(F,P) \subseteq (G,P) \subseteq (H,P)$, we have for any $e_i \in P, x_j \in U$,
$[\underline{u}_{F(e_i)}(x_j), \bar{u}_{F(e_i)}(x_j)] \leqslant [\underline{u}_{G(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j)] \leqslant [\underline{u}_{H(e_i)}(x_j), \bar{u}_{H(e_i)}(x_j)]$,
$[\underline{v}_{F(e_i)}(x_j), \bar{v}_{F(e_i)}(x_j)] \geqslant [\underline{v}_{G(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j)] \geqslant [\underline{v}_{H(e_i)}(x_j), \bar{v}_{H(e_i)}(x_j)]$.
By the property of fuzzy equivalence operators, we get for any $e_i \in P$, $x_j \in U$,
$E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{H(e_i)}(x_j)) \leqslant E_1(\bar{u}_{F(e_i)}(x_j), \bar{u}_{G(e_i)}(x_j))$,
$E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j)) \leqslant E_2(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))$,
$E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{H(e_i)}(x_j)) \leqslant E_3(\bar{v}_{F(e_i)}(x_j), \bar{v}_{G(e_i)}(x_j))$,
$E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j)) \leqslant E_4(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))$.
Thus, $D_q((F,P),(H,P)) \geqslant D_q((F,P),(G,P))$ $(q \in \{1,2,...8\})$. □

Remark 1. All of the distance measures for $IVIFSSs$ are discussed on discrete universes here, the cases for continuous universes can be researched similarly.

Remark 2. If the $IVIFSSs$ degenerate to $IVIFSs$, the distance measures of $IVIFSSs$ degenerate to the corresponding distance measures of $IVIFSs$.

Example 2. Considering $(F,P),(G,P) \in$

$IVIFSS(U)$, let

(1) $M_1(x_1,x_2,...x_n)=M_2(x_1,x_2,...x_n)=\frac{1}{n}\sum_{i=1}^{n}x_i$, $x_i\in[0,1]$, $\forall n\in N$;

(2) $E_1(x_1,x_2)=E_2(x_1,x_2)=E_3(x_1,x_2)=E_4(x_1,x_2)=1-|x_1-x_2|$, for any $x_1,x_2\in[0,1]$;

(3) $f(x)=1-x$, $\forall x\in[0,1]$,

then, we may construct the following distance measures for $IVIFSSs$ by Eq.(2) in Definition 16.
(1) Let $M_3(x_1,x_2,x_3,x_4)=[\frac{1}{4}(x_1^2+x_2^2+x_3^2+x_4^2)]^{\frac{1}{2}}$, then we get the Normalized Euclidean distance

$$d_1((F,P),(G,P))=\{\frac{1}{4mn}\sum_{i=1}^{m}\sum_{j=1}^{n}[(\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j))^2+(\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j))^2+(\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j))^2+(\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j))^2]\}^{\frac{1}{2}}.$$

(2) Let $M_3(x_1,x_2,x_3,x_4)=\frac{1}{4}(x_1+x_2+x_3+x_4)$, then we get the Normalized hamming distance

$$d_2((F,P),(G,P))=\frac{1}{4mn}\sum_{j=1}^{n}\sum_{i=1}^{m}[|\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)|+|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)|+|\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|+|\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j))|].$$

(3) Let $M_3(x_1,x_2,x_3,x_4)=\frac{1}{2}(x_1\vee x_2+x_3\vee x_4)$, then we get the Normalized hamming distance measure induced by Hausdorff metric

$$d_3((F,P),(G,P))=\frac{1}{2mn}\sum_{j=1}^{n}\sum_{i=1}^{m}[(|\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)|\vee|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)|)+(|\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|\vee|\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j))|)].$$

(4) Let $M_3(x_1,x_2,x_3,x_4)=x_1^2\vee x_2^2\vee x_3^2\vee x_4^2$, then we get the fourth distance

$$d_4((F,P),(G,P))=\frac{1}{mn}\sum_{j=1}^{n}\sum_{i=1}^{m}[(\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j))^2\vee(\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j))^2\vee(\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j))^2\vee(\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j))^2].$$

(5) Let $M_3(x_1,x_2,x_3,x_4)=[\frac{1}{4}(x_1^3+x_2^3+x_3^3+x_4^3)]^{\frac{1}{3}}$, then we get the fifth distance

$$d_5((F,P),(G,P))=\{\frac{1}{4mn}\sum_{i=1}^{m}\sum_{j=1}^{n}[(\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j))^3+(\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j))^3+(\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j))^3+(\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j))^3]\}^{\frac{1}{3}}.$$

If $(F,P),(G,P)\in IVIFSS(U)$ are reduced to $F,G\in IVIFS(U)$, we get the following distance measures of $IVIFSs$. Note that the similarity measures $d_1'-d_3'$ of $IVIFSs$ have been proposed in Ref. [17], whereas $d_4'-d_5'$ are new for $IVIFSs$.
(1) The Normalized Euclidean distance

$$d_1'(F,G)=\{\frac{1}{4n}\sum_{j=1}^{n}[(\overline{u}_F(x_j)-\overline{u}_G(x_j))^2+(\underline{u}_F(x_j)-\underline{u}_G(x_j))^2+(\overline{v}_F(x_j)-\overline{v}_G(x_j))^2+(\underline{v}_F(x_j)-\underline{v}_G(x_j))^2]\}^{\frac{1}{2}}.$$

(2) The Normalized hamming distance

$$d_2'(F,G)=\frac{1}{4n}\sum_{j=1}^{n}[|\overline{u}_F(x_j)-\overline{u}_G(x_j)|+|\underline{u}_F(x_j)-\underline{u}_G(x_j)|+|\overline{v}_F(x_j)-\overline{v}_G(x_j)|+|\underline{v}_F(x_j)-\underline{v}_G(x_j))|].$$

(3) The Normalized hamming distance measure induced by Hausdorff metric

$$d_3'(F,G)=\frac{1}{2n}\sum_{j=1}^{n}[(|\overline{u}_F(x_j)-\overline{u}_G(x_j)|\vee|\underline{u}_F(x_j)-\underline{u}_G(x_j)|)+(|\overline{v}_F(x_j)-\overline{v}_G(x_j)|\vee|\underline{v}_F(x_j)-\underline{v}_G(x_j))|)].$$

(4) Let $M_3(x_1,x_2,x_3,x_4)=x_1^2\vee x_2^2\vee x_3^2\vee x_4^2$, then we get the fourth distance

$$d_4'(F,G)=\frac{1}{n}\sum_{j=1}^{n}[(\overline{u}_F(x_j)-\overline{u}_G(x_j))^2\vee(\underline{u}_F(x_j)-\underline{u}_G(x_j))^2\vee(\overline{v}_F(x_j)-\overline{v}_G(x_j))^2\vee(\underline{v}_F(x_j)-\underline{v}_G(x_j))^2].$$

(5) Let $M_3(x_1,x_2,x_3,x_4)=[\frac{1}{4}(x_1^3+x_2^3+x_3^3+x_4^3)]^{\frac{1}{3}}$, then we get the fifth distance

$$d_5'(F,G))=\{\frac{1}{4n}\sum_{j=1}^{n}[(\overline{u}_F(x_j)-\overline{u}_G(x_j)^3+(\underline{u}_F(x_j)-\underline{u}_G(x_j)^3+(\overline{v}_F(x_j)-\overline{v}_G(x_j))^3+(\underline{v}_F(x_j)-\underline{v}_G(x_j))^3]\}^{\frac{1}{3}}.$$

Example 3. Considering $(F,P),(G,P) \in IVIFSS(U)$, let
(1)$M_1(x_1,x_2,...,x_n) = \lambda min(x_1,x_2,...,x_n) + (1-\lambda)max(x_1,x_2,...,x_n)$ with $\lambda \in [0,1]$,
$M_2(x_1,x_2,...,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$,
$M_3(x_1,x_2,...,x_n) = x_1 \vee x_2,..., \vee x_n$,
for each $n \in N$ and $x_i \in [0,1]$, $i \in \{1,2,...,n\}$.
(2)$E_1(x_1,x_2) = E_2(x_1,x_2) = 1 - |x_1^2 - x_2^2|$, $E_3(x_1,x_2) = E_4(x_1,x_2) = \frac{2x_1x_2}{x_1^2+x_2^2}$ for any $x_1,x_2 \in [0,1]$.
(3) $f(x) = 1-x$, for any $x \in [0,1]$.
We may construct the distance measure for $IVIFSSs$ by Eq.(3) in Definition 16 as follows.
$d_6((F,P),(G,P)) = \lambda min(\alpha_1,\alpha_2,...,\alpha_n) + (1-\lambda)max(\alpha_1,\alpha_2,...,\alpha_n)$,
where $\lambda \in [0,1]$ and $\alpha_j = \frac{1}{m}\sum_{i=1}^{m}\{1 - [(1 - |\overline{u}_{F(e_i)}(x_j)^2 - \overline{u}_{G(e_i)}(x_j)^2|) \vee (1 - |\underline{u}_{F(e_i)}(x_j)^2 - \underline{u}_{G(e_i)}(x_j)^2|) \vee \frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2} \vee \frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2}]\}$,
$(j = 1,2,...,n)$.

4. Relationships between distance, similarity, inclusion measures and entropy for $IVIFSSs$

4.1. Transformation of distance measures into similarity measures for $IVIFSSs$

Theorem 2. *Let f' be a strict fuzzy negation and D be a distance measure of interval-valued intuitionistic fuzzy soft sets. Then a similarity measure S of interval-valued intuitionistic fuzzy soft sets can be deduced from the distance measure D as follows:*

$$S((F,P)(G,P)) = f'(D((F,P),(G,P)))$$

Remark 3. If we take the strict fuzzy negation $f'(x) = 1-x$ for all $x \in [0,1]$, by the distance measures $D_i((F,P),(G,P))(1 \leqslant i \leqslant 8)$ given in Definition 16, we can generate the corresponding similarity measures of interval-valued intuitionistic fuzzy soft sets as $S_i((F,P),(G,P)) = 1 - D_i((F,P),(G,P)), (1 \leqslant i \leqslant 8)$.

Example 4. Considering the distance measure given in Example 3, take $f'(x) = 1-x$, one can get a similarity measure of $IVIFSSs$ as follows.
$S((F,P),(G,P)) = 1 - [\lambda min(\alpha_1,\alpha_2,...,\alpha_n) + (1-\lambda)max(\alpha_1,\alpha_2,...,\alpha_n)]$,
where $\lambda \in [0,1]$ and $\alpha_j = \frac{1}{m}\sum_{i=1}^{m}\{1 - [(1 - |\overline{u}_{F(e_i)}(x_j)^2 - \overline{u}_{G(e_i)}(x_j)^2|) \vee (1 - |\underline{u}_{F(e_i)}(x_j)^2 - \underline{u}_{G(e_i)}(x_j)^2|) \vee \frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2} \vee \frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2}]\}$,
$(j = 1,2,...,n)$.

4.2. Transformation of distance measures into entropies for $IVIFSSs$

Now we present a transformation method for constructing entropy of $IVIFSSs$ based on the distance measure of $IVIFSSs$ as follows.

Theorem 3. *Let (Q,P) be an interval-valued intuitionistic fuzzy soft set on U, s.t. for any $e_i \in P$, $Q(e_i) = \{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle | x_j \in U\}$. Suppose that*

(1) for each $p \in \{1,2,3\}$, M_p is both a bottom-aggregation operator and an idempotent-aggregation operator;

(2) $M_3(x_1,x_2,x_3,x_4) = M_3(x_3,x_4,x_1,x_2)$ for $x_1,x_2,x_3,x_4 \in [0,1]$;

(3) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$;

(4) $f(x) = 1-x$, for any $x \in [0,1]$;

(5) $D_1((F,P),(Q,P))$ and $D_2((F,P),(Q,P))$ are distance measures between (F,P) and (Q,P) constructed by Eq.(1) and Eq.(2) in Definition 16, respectively;

(6) f' is a strict fuzzy negation,

then for any $(F,P) \in IVIFSS(U)$,

$$I_q((F,P)) = f'(2D_q((F,P),(Q,P)))(q = 1,2)$$

are entropies for interval-valued intuitionistic fuzzy soft sets.

Proof. It is sufficient to show that $I((F,P))$ satisfies the requirements (1)-(4) listed in Definition 12.
(1) If (F,P) is a classical soft set, we have
$[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)] = [1,1]$, $[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)] = [0,0]$ or
$[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)] = [0,0]$, $[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)] =$

$[1,1]$, $\forall e_i \in P$, $x_j \in U$.
Since $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$, and $f(x) = 1-x$ for any $x \in [0,1]$,
we have $f(E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})) = \frac{1}{2}$, $\forall e_i \in P, x_j \in U$.
Since $M_p(p=1,2,3)$ is an idempotent-aggregation operator, we have $D_q((F,P),(Q,P)) = \frac{1}{2}$ $(q=1,2)$, i.e., $2D_q((F,P),(Q,P)) = 1$ $(q=1,2)$.
Thus, we get $I_q((F,P)) = f'(1) = 0$ $(q=1,2)$.
(2) Since $M_p(p=1,2,3)$ is a bottom-aggregation operator and f' is a strict fuzzy negation, we get
$I_q((F,P)) = 1$ $(q=1,2)$
$\Leftrightarrow 2D_q((F,P),(Q,P)) = 0$ $(q=1,2)$
$\Leftrightarrow D_q((F,P),(Q,P)) = 0$ $(q=1,2)$
$\Leftrightarrow f(E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})) = f(E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})) = 0$, $\forall e_i \in P, x_j \in U$.
$\Leftrightarrow E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2}) = E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2}) = E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2}) = E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2}) = 1$, $\forall e_i \in P$, $x_j \in U$.
$\Leftrightarrow u_{F(e_i)}(x_j) = v_{F(e_i)}(x_j) = [\frac{1}{2},\frac{1}{2}]$, $\forall e_i \in P, x_j \in U$.
(3) For any $e_i \in P$,
if $F(e_i) = \langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle$, $\forall x_j \in U$,
then $F^C(e_i) = \langle x_j, v_{F(e_i)}(x_j), u_{F(e_i)}(x_j)\rangle$, $\forall x_j \in U$.
Since $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2)$ for any $x_1,x_2 \in [0,1]$, we have
$E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2}) = E_3(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})$,
$E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2}) = E_4(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})$,
$E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2}) = E_1(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})$,
$E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2}) = E_2(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})$,
$\forall e_i \in P, x_j \in U$.
Since $M_3(x_1,x_2,x_3,x_4) = M_3(x_3,x_4,x_1,x_2)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$, we have
$M_3(f(E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})), f(E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})), f(E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})), f(E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})))$
$= M_3(f(E_3(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})), f(E_4(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})), f(E_1(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})), f(E_2(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})))$
$= M_3(f(E_1(\overline{v}_{F(e_i)}(x_j),\frac{1}{2})), f(E_2(\underline{v}_{F(e_i)}(x_j),\frac{1}{2})), f(E_3(\overline{u}_{F(e_i)}(x_j),\frac{1}{2})), f(E_4(\underline{u}_{F(e_i)}(x_j),\frac{1}{2})))$,
$\forall e_i \in P, x_j \in U$.
By Definition 16 we get
$D_q((F,P),(Q,P)) = D_q((F^C,P),(Q,P))$ $(q=1,2)$,
Thus, $I((F,P)) = I((F,P)^C)$.
(4) Since f' is a fuzzy negation,
if $D_q((F,P),(Q,P)) \geqslant D_q((G,P),(Q,P))$ $(q=1,2)$,
then $f'(2D_q((F,P),(Q,P))) \leqslant f'(2D_q((G,P),(Q,P)))$ $(q=1,2)$, i.e., $I_q((F,P)) \leqslant I_q((G,P))$ $(q=1,2)$. □

Example 5. Now we list some aggregation operators M_3 which satisfy the conditions in Theorem 3: for any $x_1,x_2,x_3,x_4 \in [0,1]$,

(1) $M_3(x_1,x_2,x_3,x_4) = (\frac{(x_1\vee x_2)^p+(x_3\vee x_4)^p}{2})^{\frac{1}{p}}$, $p \geqslant 1$.

(2) $M_3(x_1,x_2,x_3,x_4) = (\frac{(x_1+x_2)^p\vee(x_3+x_4)^p}{2})^{\frac{1}{p}}$, $p \geqslant 1$.

(3) $M_3(x_1,x_2,x_3,x_4) = (\frac{x_1^p+x_2^p+x_3^p+x_4^p}{4})^{\frac{1}{p}}$, $p \geqslant 1$.

Theorem 4. *Let (Q,P) be an interval-valued intuitionistic fuzzy soft set on U, s.t. for any $e_i \in P$, $Q(e_i) = \{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle | x_j \in U\}$. Suppose that*

(1) for each $p \in \{1,2\}$, M_p is both a bottom-aggregation operator and an idempotent-aggregation operator;

(2) M_3 is both a top-aggregation operator and an idempotent-aggregation operator;

(3) $M_3(x_1,x_2,x_3,x_4) = M_3(x_3,x_4,x_1,x_2)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(4) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$;

(5) $f(x) = 1-x$, for any $x \in [0,1]$;

(6) $D_3((F,P),(Q,P))$ and $D_4((F,P),(Q,P))$ are distance measures between (F,P) and (Q,P) given by Eq.(3) and Eq.(4) in Definition 16, respectively;

(7) f' is a strict fuzzy negation,

then for any $(F,P) \in IVIFSS(U)$, $I_q((F,P)) = f'(2D_q((F,P),(Q,P)))$ $(q=3,4)$, is an entropy for interval-valued intuitionistic fuzzy soft sets based on the corresponding distance D_q $(q=3,4)$.

Theorem 5. *Let (Q,P) be an interval-valued intuitionistic fuzzy soft set on U, s.t. for any $e_i \in P$, $Q(e_i) = \{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle | x_j \in U\}$. Suppose that*

(1) M_1 is both a bottom-aggregation operator and an idempotent-aggregation operator;

(2) for each $p \in \{2,3\}$, M_p is both a top-aggregation operator and an idempotent-aggregation operator;

(3) $M_3(x_1,x_2,x_3,x_4) = M_3(x_3,x_4,x_1,x_2)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(4) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$;

(5) $f(x) = 1-x$, for any $x \in [0,1]$;

(6) $D_5((F,P),(Q,P))$ and $D_6((F,P),(Q,P))$ are distance measures between (F,P) and (Q,P) given by Eq.(5) and Eq.(6) in Definition 16, respectively;

(7) f' is a strict fuzzy negation,

then for any $(F,P) \in IVIFSS(U)$,
$I_q((F,P)) = f'(2D_q((F,P),(Q,P)))$ $(q=5,6)$,
is an entropy for interval-valued intuitionistic fuzzy soft sets based on the corresponding distance D_q $(q=5,6)$.

Theorem 6. *Let (Q,P) be an interval-valued intuitionistic fuzzy soft set on U, s.t. for any $e_i \in P$, $Q(e_i) = \{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle | x_j \in U\}$. Suppose that*

(1) for each $p \in \{1,2,3\}$, M_p is both an idempotent-aggregation operator and a top-aggregation operator;

(2) $M_3(x_1,x_2,x_3,x_4) = M_3(x_3,x_4,x_1,x_2)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(3) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$;

(4) $f(x) = 1-x$, for any $x \in [0,1]$;

(5) $D_q((F,P),(Q,P))(q=7,8)$ are distance measures between (F,P) and (Q,P) given by Eq.(7) and Eq.(8) in Definition 16, respectively;

(6) f' is a strict fuzzy negation,

then for any $(F,P) \in IVIFS(U)$,
$I_q((F,P)) = f'(2D_q((F,P),(Q,P)))$ $(q=7,8)$,
is an entropy for interval-valued intuitiionistic fuzzy soft sets based on the corresponding distance D_q $(q=7,8)$.

Example 6. Now we list some aggregation operators M_3 which satisfies the conditions in Theorem 4-6: for any $x_1,x_2,x_3,x_4 \in [0,1]$,
(1) $M_3(x_1,x_2,x_3,x_4) = (\frac{(x_1\wedge x_2)^p+(x_3\wedge x_4)^p}{2})^{\frac{1}{p}}$, $p \geqslant 1$.
(2) $M_3(x_1,x_2,x_3,x_4) = (\frac{(x_1+x_2)^p\wedge(x_3+x_4)^p}{2})^{\frac{1}{p}}$, $p \geqslant 1$.
(3) $M_3(x_1,x_2,x_3,x_4) = (\frac{x_1^p+x_2^p+x_3^p+x_4^p}{4})^{\frac{1}{p}}$, $p \geqslant 1$.

Remark 4. We can easily obtain a large number of distances by Definition 16, employing different aggregation operators. Furthermore, we can easily obtain a large number of entropies by Theorem 3-6, employing different distances.

4.3. Transformation of entropies into similarity measures for *IVIFSSs*

Next, we provide a transformational method of constructing similarity measure of *IVIFSSs* based on the entropy of *IVIFSSs* as below.

Definition 17. Let $(F,P),(G,P) \in IVIFSS(U)$, assume that: for any $e_i \in P$,

$$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)],[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\},$$

$$G(e_i) = \{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}.$$

Suppose that

(1) M_1 is a bottom-aggregation operator,

(2) $M_1(x_1,x_2,x_3,x_4) \geqslant M_2(x_1,x_2,x_3,x_4)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$,

(3) f is a strict fuzzy negation,

(4) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for any $\alpha \in [1,+\infty), \beta \in [1,+\infty)$, we can define a new interval-valued intuitionistic fuzzy set $(\psi_1(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i \in P, x_j \in U$,

$$\underline{u}_{\psi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{1/\alpha}\};$$

$$\overline{u}_{\psi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]\};$$

$$\underline{v}_{\psi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\beta}\};$$

$$\overline{v}_{\psi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]\}.$$

Theorem 7. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P) \in IVIFSS(U)$, then $I((\psi_1(F,G),P))$ is a similarity measure of (F,P) and (G,P).*

Proof. We only need to prove that all the properties in Definition 10 hold.

(1)If (F,P) is a classical soft set, then for all $e_i \in P, x_j \in U$, we know
$u_{F(e_i)}(x_j) = [1,1]$, $v_{F(e_i)}(x_j) = [0,0]$, $u_{F^C(e_i)}(x_j) = [0,0]$, $v_{F^C(e_i)}(x_j) = [1,1]$, or
$u_{F(e_i)}(x_j) = [0,0]$, $v_{F(e_i)}(x_j) = [1,1]$, $u_{F^C(e_i)}(x_j) = [1,1]$, $v_{F^C(e_i)}(x_j) = [0,0]$,
then we have
$E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F^C(e_i)}(x_j)) = E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F^C(e_i)}(x_j)) = E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F^C(e_i)}(x_j)) = E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F^C(e_i)}(x_j)) = 0$.
Since M_1, M_2 are aggregation operators, we get
$M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F^C(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F^C(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F^C(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F^C(e_i)}(x_j)))) = 1$,
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F^C(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F^C(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F^C(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F^C(e_i)}(x_j)))) = 1$,
$\forall e_i \in P, x_j \in U$.
hence we get
$\underline{u}_{\psi_1(F,F^C)(e_i)}(x_j) = \overline{u}_{\psi_1(F,F^C)(e_i)}(x_j) = 0$,
$\underline{v}_{\psi_1(F,F^C)(e_i)}(x_j) = \overline{v}_{\psi_1(F,F^C)(e_i)}(x_j) = 1$,
$\forall e_i \in P, x_j \in U$.
So, $(\psi_1(F,F^C),P)$ is crisp soft set in U.
By Definition 12 of entropy for $IVIFSSs$, we have $S((F,P),(F^C,P)) = I((\psi_1(F,F^C),P)) = 0$.

(2)$S((F,P),(G,P)) = I((\psi_1(F,G),P)) = 1$.
$\Leftrightarrow u_{\psi_1(F,G)(e_i)}(x_j) = v_{\psi_1(F,G)(e_i)}(x_j) = [\frac{1}{2},\frac{1}{2}]$, $\forall e_i \in P, x_j \in U$.
$\Leftrightarrow M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)))) = 0$ and
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)))) = 0$,
$\forall e_i \in P, \forall x_j \in U$.
$\Leftrightarrow f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j))) = 0$,
$f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j))) = 0$,
$f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j))) = 0$,
$f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))) = 0$,
$\forall e_i \in P, \forall x_j \in U$.
$\Leftrightarrow E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)) = E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)) = E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)) = E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)) = 1$, $\forall e_i \in P, \forall x_j \in U$.
$\Leftrightarrow$ $\underline{u}_{F(e_i)}(x_j) = \underline{u}_{G(e_i)}(x_j)$, $\overline{u}_{F(e_i)}(x_j) = \overline{u}_{G(e_i)}(x_j)$, $\underline{v}_{F(e_i)}(x_j) = \underline{v}_{G(e_i)}(x_j)$, and $\overline{v}_{F(e_i)}(x_j) = \overline{v}_{G(e_i)}(x_j)$, $\forall e_i \in P, \forall x_j \in U$.
$\Leftrightarrow (F,P) = (G,P)$.

(3)From the definition of $(\psi_1(F,G),E)$, we know for any $e_i \in P, x_j \in U$,
$u_{\psi_1(F,G)(e_i)}(x_j) = u_{\psi_1(G,F)(e_i)}(x_j)$,
$v_{\psi_1(F,G)(e_i)}(x_j) = v_{\psi_1(G,F)(e_i)}(x_j)$,
that is, $(\psi_1(F,G),P) = (\psi_1(G,F),P)$,
then we get $I((\psi_1(F,G),P)) = I((\psi_1(G,F),P))$
$\Leftrightarrow S((F,P),(G,P)) = S((G,P),(F,P))$.

(4)If $(F,P) \subseteq (G,P) \subseteq (H,P)$, we know for any $e_i \in P, x_j \in U$,
$\underline{u}_{F(e_i)}(x_j) \leqslant \underline{u}_{G(e_i)}(x_j) \leqslant \underline{u}_{H(e_i)}(x_j)$,
$\overline{u}_{F(e_i)}(x_j) \leqslant \overline{u}_{G(e_i)}(x_j) \leqslant \overline{u}_{H(e_i)}(x_j)$,
$\underline{v}_{F(e_i)}(x_j) \geqslant \underline{v}_{G(e_i)}(x_j) \geqslant \underline{v}_{H(e_i)}(x_j)$,
$\overline{v}_{F(e_i)}(x_j) \geqslant \overline{v}_{G(e_i)}(x_j) \geqslant \overline{v}_{H(e_i)}(x_j)$,

hence,
$E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j)) \leqslant E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))$,
$E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{H(e_i)}(x_j)) \leqslant E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))$,
$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j)) \leqslant E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))$,
$E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{H(e_i)}(x_j)) \leqslant E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))$,
from properties of aggregation operators and decreasing monotone property of f we have
$M_1(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{H(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{H(e_i)}(x_j)))) \geqslant$
$M_1(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))$,
and
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{H(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{H(e_i)}(x_j)))) \geqslant$
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))$.
Thus, we get
$u_{\psi_1(F,H)(e_i)}(x_j) \leqslant u_{\psi_1(F,G)(e_i)}(x_j) \leqslant [\frac{1}{2}, \frac{1}{2}]$,
$v_{\psi_1(F,H)(e_i)}(x_j) \geqslant v_{\psi_1(F,G)(e_i)}(x_j) \geqslant [\frac{1}{2}, \frac{1}{2}]$,
$\forall e_i \in P, \forall x_j \in U$.
Let $(Q,P) \in IVIFSS(U)$ and $Q(e_i) = \{\langle x_j, [1/2, 1/2]), [1/2, 1/2]\rangle | x_j \in U\}$ for any $e_i \in P$, then we get
$(\psi_1(F,H),P) \subseteq (\psi_1(F,G),P) \subseteq (Q,P)$.
Similarly, we get
$(\psi_1(F,H),P) \subseteq (\psi_1(G,H),P) \subseteq (Q,P)$.
By Definition 9 of distance measure for $IVIFSSs$, we know
$D((\psi_1(F,G),P),(Q,P)) \leqslant D((\psi_1(F,H),P),(Q,P))$,
$D((\psi_1(G,H),P),(Q,P)) \leqslant D((\psi_1(F,H),P),(Q,P))$.
By Definition 12 of entropy for $IVIFSSs$, we conclude that
$I((\psi_1(F,H),P)) \leqslant I((\psi_1(F,G),P))$,
$I((\psi_1(F,H),P)) \leqslant I((\psi_1(G,H),P))$.
Hence,
$I((\psi_1(F,H),P)) \leqslant I((\psi_1(F,G),P)) \wedge I((\psi_1(G,H),P))$,
that is,
$S((F,P),(H,P)) \leqslant S((F,P),(G,P)) \wedge S((G,P),(H,P))$.
□

Example 7. Let I be an entropy measure of interval-valued intuitionistic fuzzy soft sets. For $(F,P),(G,P) \in IVIFSS(U)$, assume that: for any $e_i \in P$,

$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$,

$G(e_i) = \{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}$.

let

(1) $M_1(x_1,x_2,x_3,x_4) = (x_1 \vee x_2) \vee (x_3 \vee x_4)$,

(2) $M_2(x_1,x_2,x_3,x_4) = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$,

(3) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1 - |x_1 - x_2|$,

(4) $\alpha = \beta = 2$,

(5) $f(x) = 1 - x$,

we get an interval-valued intuitionistic fuzzy soft set $(\psi_1'(F,G),P)$ from (F,P) and (G,P) by Definition 17 as follows: for any $e_i \in P, x_j \in U$,

$$\underline{u}_{\psi_1'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [max((|\underline{u}_{F(e_i)}(x_j) - \underline{u}_{G(e_i)}(x_j)| \vee |\overline{u}_{F(e_i)}(x_j) - \overline{u}_{G(e_i)}(x_j)|), (|\underline{v}_{F(e_i)}(x_j) - \underline{v}_{G(e_i)}(x_j)| \vee |\overline{v}_{F(e_i)}(x_j) - \overline{v}_{G(e_i)}(x_j)|))]^{1/2}\};$$

$$\overline{u}_{\psi_1'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [max((|\underline{u}_{F(e_i)}(x_j) - \underline{u}_{G(e_i)}(x_j)| \vee |\overline{u}_{F(e_i)}(x_j) - \overline{u}_{G(e_i)}(x_j)|), (|\underline{v}_{F(e_i)}(x_j) - \underline{v}_{G(e_i)}(x_j)| \vee |\overline{v}_{F(e_i)}(x_j) - \overline{v}_{G(e_i)}(x_j)|))]\};$$

$$\underline{v}_{\psi_1'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 + [min((|\underline{u}_{F(e_i)}(x_j) - \underline{u}_{G(e_i)}(x_j)| \vee |\overline{u}_{F(e_i)}(x_j) - \overline{u}_{G(e_i)}(x_j)|), (|\underline{v}_{F(e_i)}(x_j) - \underline{v}_{G(e_i)}(x_j)| \vee |\overline{v}_{F(e_i)}(x_j) - \overline{v}_{G(e_i)}(x_j)|))]^2\};$$

$$\overline{v}_{\psi_1'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 + [min((|\underline{u}_{F(e_i)}(x_j) - \underline{u}_{G(e_i)}(x_j)| \vee |\overline{u}_{F(e_i)}(x_j) - \overline{u}_{G(e_i)}(x_j)|), (|\underline{v}_{F(e_i)}(x_j) - \underline{v}_{G(e_i)}(x_j)| \vee |\overline{v}_{F(e_i)}(x_j) - \overline{v}_{G(e_i)}(x_j)|))]\},$$

then $I((\psi_1'(F,G),P))$ is a similarity measure of (F,P) and (G,P).

Definition 18. Let (F,P) and (G,P) be two $IVIFSS(U)$ in universe $U=\{x_1,x_2,...x_n\}$, assume that: for any $e_i \in P$,

$F(e_i)=\{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\}=\{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$,

$G(e_i)=\{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\}=\{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}$.

Suppose that M_1, M_2 are aggregation operators which satisfy that

(1) M_1 is a top-aggregation operator,

(2) $M_1(x_1,x_2,x_3,x_4) \leqslant M_2(x_1,x_2,x_3,x_4)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(3) f is a strict fuzzy negation,

(4) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for any $\alpha \in [1,+\infty), \beta \in [1,+\infty)$, we can define a new interval-valued intuitionistic fuzzy set $(\psi_2(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i \in P$, $x_j \in U$,

$$\underline{u}_{\psi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M_1(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)), E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{1/\alpha}\};$$

$$\overline{u}_{\psi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M_1(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)), E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]\};$$

$$\underline{v}_{\psi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M_2(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)), E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\beta}\};$$

$$\overline{v}_{\psi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M_2(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)), E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]\}.$$

Theorem 8. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P) \in IVIFSS(U)$, then $I((\psi_2(F,G),P))$ is a similarity measure of (F,P) and (G,P).*

Example 8. Let I be an entropy measure of interval-valued intuitionistic fuzzy soft sets. For $(F,P),(G,P) \in IVIFSS(U)$, assume that: for any $e_i \in P$,

$F(e_i)=\{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\}=\{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$,

$G(e_i)=\{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\}=\{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}$.

let

(1) $M_1(x_1,x_2,x_3,x_4)=\frac{(x_1+x_2)\wedge(x_3+x_4)}{2}$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(2) $M_2(x_1,x_2,x_3,x_4)=\frac{(x_1+x_2)\vee(x_3+x_4)}{2}$ for any $x_1,x_2,x_3,x_4 \in [0,1]$;

(3) $E_1(x_1,x_2)=E_2(x_1,x_2)=E_3(x_1,x_2)=E_4(x_1,x_2)=\frac{2x_1x_2}{x_1^2+x_2^2}$ for any $x_1,x_2 \in [0,1]$.

(4) $\alpha=8$, $\beta=4$, $f(x)=1-x$,

we get an interval-valued intuitionistic fuzzy soft set $(\psi_2'(F,G),P)$ from (F,P) and (G,P) by Definition 18 as follows: for any $e_i \in P$, $x_j \in U$,

$$\underline{u}_{\psi_1'(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[1-\frac{1}{2}(\frac{2\underline{u}_{F(e_i)}(x_j)\underline{u}_{G(e_i)}(x_j)}{\underline{u}_{F(e_i)}(x_j)^2+\underline{u}_{G(e_i)}(x_j)^2} + \frac{2\overline{u}_{F(e_i)}(x_j)\overline{u}_{G(e_i)}(x_j)}{\overline{u}_{F(e_i)}(x_j)^2+\overline{u}_{G(e_i)}(x_j)^2}) \wedge (\frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2} + \frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2})]^{1/8}\};$$

$$\overline{u}_{\psi_2'(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[1-\frac{1}{2}(\frac{2\underline{u}_{F(e_i)}(x_j)\underline{u}_{G(e_i)}(x_j)}{\underline{u}_{F(e_i)}(x_j)^2+\underline{u}_{G(e_i)}(x_j)^2} + \frac{2\overline{u}_{F(e_i)}(x_j)\overline{u}_{G(e_i)}(x_j)}{\overline{u}_{F(e_i)}(x_j)^2+\overline{u}_{G(e_i)}(x_j)^2}) \wedge (\frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2} + \frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2})]\};$$

$$\underline{v}_{\psi_2'(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[1-\frac{1}{2}(\frac{2\underline{u}_{F(e_i)}(x_j)\underline{u}_{G(e_i)}(x_j)}{\underline{u}_{F(e_i)}(x_j)^2+\underline{u}_{G(e_i)}(x_j)^2}+\frac{2\overline{u}_{F(e_i)}(x_j)\overline{u}_{G(e_i)}(x_j)}{\overline{u}_{F(e_i)}(x_j)^2+\overline{u}_{G(e_i)}(x_j)^2})\vee(\frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2}+\frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2})]^4\};$$

$$\overline{v}_{\psi_2'(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[1-\frac{1}{2}(\frac{2\underline{u}_{F(e_i)}(x_j)\underline{u}_{G(e_i)}(x_j)}{\underline{u}_{F(e_i)}(x_j)^2+\underline{u}_{G(e_i)}(x_j)^2}+\frac{2\overline{u}_{F(e_i)}(x_j)\overline{u}_{G(e_i)}(x_j)}{\overline{u}_{F(e_i)}(x_j)^2+\overline{u}_{G(e_i)}(x_j)^2})\vee(\frac{2\underline{v}_{F(e_i)}(x_j)\underline{v}_{G(e_i)}(x_j)}{\underline{v}_{F(e_i)}(x_j)^2+\underline{v}_{G(e_i)}(x_j)^2}+\frac{2\overline{v}_{F(e_i)}(x_j)\overline{v}_{G(e_i)}(x_j)}{\overline{v}_{F(e_i)}(x_j)^2+\overline{v}_{G(e_i)}(x_j)^2})]\},$$

then $I((\psi_2'(F,G),P))$ is a similarity measure of (F,P) and (G,P).

Definition 19. Let $(F,P),(G,P)\in IVIFSS(U)$, assume that: for any $e_i\in P$,

$$F(e_i)=\{\langle x_j,u_{F(e_i)}(x_j),v_{F(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)],[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\rangle|x_j\in U\},$$

$$G(e_i)=\{\langle x_j,u_{G(e_i)}(x_j),v_{G(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\rangle|x_j\in U\}.$$

Suppose that,

(1) M is a top-aggregation operator,

(2) f is a strict fuzzy negation,

(3) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for $0<\alpha_1\leqslant\alpha_2\leqslant\alpha_4\leqslant\alpha_3$, we can define a new interval-valued intuitionistic fuzzy set $(\psi_3(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i\in P$, $x_j\in U$,

$$\underline{u}_{\psi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\alpha_1}\};$$

$$\overline{u}_{\psi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\alpha_2}\};$$

$$\underline{v}_{\psi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\alpha_3}\};$$

$$\overline{v}_{\psi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j))))]^{\alpha_4}\}.$$

Theorem 9. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P)\in IVIFSS(U)$, then $I((\psi_3(F,G),P))$ is a similarity measure of (F,P) and (G,P).*

Proof. We only need to prove that all the properties in Definition 10 hold.

(1)If (F,P) is a classical soft set, then for $\forall e_i\in P$, $x_j\in U$, we know
$u_{F(e_i)}(x_j)=[1,1]$, $v_{F(e_i)}(x_j)=[0,0]$, $u_{F^C(e_i)}(x_j)=[0,0]$, $v_{F^C(e_i)}(x_j)=[1,1]$ or
$u_{F(e_i)}(x_j)=[0,0]$, $v_{F(e_i)}(x_j)=[1,1]$, $u_{F^C(e_i)}(x_j)=[1,1]$, $v_{F^C(e_i)}(x_j)=[0,0]$,
so we get
$E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F^C(e_i)}(x_j))=E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F^C(e_i)}(x_j))=E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F^C(e_i)}(x_j))=E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F^C(e_i)}(x_j))=0.$
For $\forall\alpha_i\in\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ we have
$[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F^C(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F^C(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F^C(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F^C(e_i)}(x_j))))]^{\alpha_i}=1,\forall e_i\in P,x_j\in U.$
Hence, $\underline{u}_{\psi_3(F,F^C)(e_i)}(x_j)=\overline{u}_{\psi_3(F,F^C)(e_i)}(x_j)=0$,
$\underline{v}_{\psi_3(F,F^C)(e_i)}(x_j)=\overline{v}_{\psi_3(F,F^C)(e_i)}(x_j)=1$,
$\forall e_i\in P,x_j\in U$.
Thus, $(\psi_3(F,F^C),P)$ is classical soft set in U.
By Definition 12 of entropy for $IVIFSSs$, we have
$S((F,P),(F^C,P))=I((\psi_3(F,F^C),P))=0.$
(2)$S((F,P),(G,P))=I((\psi_3(F,G),P))=1$
$\Leftrightarrow u_{\psi_3(F,G)(e_i)}(x_j)=v_{\psi_3(F,G)(e_i)}(x_j)=[\frac{1}{2},\frac{1}{2}]$,
$\forall e_i\in P,x_j\in U$,
$\Leftrightarrow$ for $\forall\alpha_i\in\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$,
$[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)),$

$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_i} = 0, \forall e_i \in P, x_j \in U,$
$\Leftrightarrow M(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)),$
$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)))) = 1, \forall e_i \in P, x_j \in U,$
$\Leftrightarrow E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)) = E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)) = E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)) = E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)) = 1, \forall e_i \in P, x_j \in U,$
$\Leftrightarrow \underline{u}_{F(e_i)}(x_j) = \underline{u}_{G(e_i)}(x_j),\ \overline{u}_{F(e_i)}(x_j) = \overline{u}_{G(e_i)}(x_j),$
$\underline{v}_{F(e_i)}(x_j) = \underline{v}_{G(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j) = \overline{v}_{G(e_i)}(x_j),$
$\forall e_i \in P, x_j \in U,$
$\Leftrightarrow (F,P) = (G,P).$

(3)From the definition of $(\psi_3(F,G),P)$ we easily know that
$u_{\psi_3(F,G)(e_i)}(x_j) = u_{\psi_3(G,F)(e_i)}(x_j),$
$v_{\psi_3(F,G)(e_i)}(x_j) = v_{\psi_3(G,F)(e_i)}(x_j),$
i.e. $(\psi_3(F,G),P) = (\psi_3(G,F),P)$.
Thus, $I((\psi_3(F,G),P)) = I((\psi_3(G,F),P))$
$\Leftrightarrow S((F,P),(G,P)) = S((G,P),(F,P)).$

(4)If $(F,P) \subseteq (G,P) \subseteq (H,P)$, then we know
$\underline{u}_{F(e_i)}(x_j) \leqslant \underline{u}_{G(e_i)}(x_j) \leqslant \underline{u}_{H(e_i)}(x_j),$
$\overline{u}_{F(e_i)}(x_j) \leqslant \overline{u}_{G(e_i)}(x_j) \leqslant \overline{u}_{H(e_i)}(x_j),$
$\underline{v}_{F(e_i)}(x_j) \geqslant \underline{v}_{G(e_i)}(x_j) \geqslant \underline{v}_{H(e_i)}(x_j),$
$\overline{v}_{F(e_i)}(x_j) \geqslant \overline{v}_{G(e_i)}(x_j) \geqslant \overline{v}_{H(e_i)}(x_j),$
$\forall e_i \in P, x_j \in U,$
hence,
$E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j)) \leqslant E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)),$
$E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{H(e_i)}(x_j)) \leqslant E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)),$
$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j)) \leqslant E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)),$
$E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{H(e_i)}(x_j)) \leqslant E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)),$
$\forall e_i \in P, x_j \in U,$
then we have, for $\forall \alpha_i \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$,
$[f(M(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{H(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{H(e_i)}(x_j)),$
$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{H(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{H(e_i)}(x_j))))]^{\alpha_i} \geqslant$
$[f(M(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j)), E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)),$
$E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j)), E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_i},$
$\forall e_i \in P, x_j \in U,$
so we get,
$u_{\psi_3(F,H)(e_i)}(x_j) \leqslant u_{\psi_3(F,G)(e_i)}(x_j) \leqslant [\frac{1}{2}, \frac{1}{2}],$
$v_{\psi_3(F,H)(e_i)}(x_j) \geqslant v_{\psi_3(F,G)(e_i)}(x_j) \geqslant [\frac{1}{2}, \frac{1}{2}],$
$\forall e_i \in P, \forall x_j \in U.$
Let $(Q,P) \in IVIFSS(U)$ and $Q(e_i) = \{\langle x_j, [1/2, 1/2]), [1/2, 1/2] \rangle | x_j \in U\}$ for any $e_i \in P$, then we get
$(\psi_3(F,H),P) \subseteq (\psi_3(F,G),P) \subseteq (Q,P).$
Similarly, we get
$(\psi_3(F,H),P) \subseteq (\psi_3(G,H),P) \subseteq (Q,P).$
By Definition 9 of distance measure for $IVIFSSs$, we know
$D((\psi_3(F,G),P),(Q,P)) \leqslant D((\psi_3(F,H),P),(Q,P)),$
$D((\psi_3(G,H),P),(Q,P)) \leqslant D((\psi_3(F,H),P),(Q,P)).$
By Definition 12 of entropy for $IVIFSSs$, we conclude that
$I((\psi_3(F,H),P)) \leqslant I((\psi_3(F,G),P)),$
$I((\psi_3(F,H),P)) \leqslant I((\psi_3(G,H),P)).$
Hence,
$I((\psi_3(F,H),P)) \leqslant I((\psi_3(F,G),P)) \wedge I((\psi_3(G,H),P)),$
that is,
$S((F,P),(H,P)) \leqslant S((F,P),(G,P)) \wedge S((G,P),(H,P)).$
□

Definition 20. Let $(F,P),(G,P) \in IVIFSS(U)$, assume that: for any $e_i \in P$,

$$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\},$$

$$G(e_i) = \{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}.$$

Suppose that,

(1) M is a bottom-aggregation operator,

(2) f is a strict fuzzy negation,

(3) E_l $(l = 1,2,3,4)$ are fuzzy equivalence operators,

then for $0 < \alpha_1 \leqslant \alpha_2 \leqslant \alpha_4 \leqslant \alpha_3$, we can define a new interval-valued intuitionistic fuzzy set $(\psi_4(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i \in P$, $x_j \in U$,

$$\underline{u}_{\psi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_1}\};$$

$$\overline{u}_{\psi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_2}\};$$

$\underline{v}_{\psi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_3}\}$;

$\overline{v}_{\psi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j))))]^{\alpha_4}\}$.

Theorem 10. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P) \in IVIFSS(U)$, then $I((\psi_4(F,G),P))$ is a similarity measure of (F,P) and (G,P).*

Example 9. Let I be an entropy measure of interval-valued intuitionistic fuzzy soft sets. For $(F,P),(G,P) \in IVIFSS(U)$, for any $e_i \in P$,

$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$,

$G(e_i) = \{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}$.

Let

(1) $M(x_1,x_2,x_3,x_4) = \frac{x_1+x_2+x_3+x_4}{4}$ for any $x_1,x_2,x_3,x_4 \in [0,1]$,

(2) $E_1(x_1,x_2) = E_2(x_1,x_2) = E_3(x_1,x_2) = E_4(x_1,x_2) = 1-|x_1-x_2|$ for any $x_1,x_2 \in [0,1]$,

(3) $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 5$, $\alpha_4 = 4$,

(4) $f(x) = 1-x$,

we get an interval-valued intuitionistic fuzzy soft set $(\psi_4'(F,G),P)$ from (F,P) and (G,P) by Definition 20 as follows: for any $e_i \in P$, $x_j \in U$,

$\underline{u}_{\psi_4'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[\frac{1}{4}(|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)| + |\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)| + |\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j)| + |\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|)]^2\}$;

$\overline{u}_{\psi_4'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[\frac{1}{4}(|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)| + |\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)| + |\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j)| + |\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|)]^3\}$;

$\underline{v}_{\psi_4'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[\frac{1}{4}(|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)| + |\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)| + |\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j)| + |\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|)]^5\}$;

$\overline{v}_{\psi_4'(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[\frac{1}{4}(|\underline{u}_{F(e_i)}(x_j)-\underline{u}_{G(e_i)}(x_j)| + |\overline{u}_{F(e_i)}(x_j)-\overline{u}_{G(e_i)}(x_j)| + |\underline{v}_{F(e_i)}(x_j)-\underline{v}_{G(e_i)}(x_j)| + |\overline{v}_{F(e_i)}(x_j)-\overline{v}_{G(e_i)}(x_j)|)]^4\}$,

then $I((\psi_4'(F,G),P))$ is a similarity measure of (F,P) and (G,P).

Theorem 11. *If I is an entropy measure of $IVIFSSs$ and $(\psi_h(F,G),P)(h = 1,2,3,4)$ is given by Definition 17-20, then $I((\psi_h(F,G)^C,P))(h = 1,2,3,4)$ is also a similarity measure between (F,P) and (G,P).*

Remark 5. Based on Definition 17-20, by selecting different aggregation operators and fuzzy equivalences, we can obtain a large number of $IVIFSSs$, which can be used to transform an entropy measure into a similarity measure for $IVIFSSs$.

Remark 6. If $(F,P),(G,P) \in IVIFSS(U)$ degenerate to $F,G \in IVIFS(U)$, the specific interval-valued intuitionistic fuzzy soft set $(\psi_1'(F,G),P)$ in Example 7 degenerates to $\psi_1'(F,G) \in IVIFS(U)$. The entropy of $\psi_1'(F,G)$ has been proven a similarity measure between F and G in Ref. [17]. Our research in this subsection can be regarded as a generalization and extension of the research in Ref. [17] based on fuzzy equivalences and aggregation operators. However, even if it degenerates to the $IVIFSs$ situation, all the formulae given by Definition 18-20 in this work are new.

4.4. Transformation of entropies into inclusion measures for $IVIFSSs$

Definition 21. Let $(F,P),(G,P) \in IVIFSS(U)$, assume that: for any $e_i \in P$,

$F(e_i) = \{\langle x_j, u_{F(e_i)}(x_j), v_{F(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j)], [\underline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)]\rangle | x_j \in U\}$,

$G(e_i) = \{\langle x_j, u_{G(e_i)}(x_j), v_{G(e_i)}(x_j)\rangle | x_j \in U\} = \{\langle x_j, [\underline{u}_{G(e_i)}(x_j), \overline{u}_{G(e_i)}(x_j)], [\underline{v}_{G(e_i)}(x_j), \overline{v}_{G(e_i)}(x_j)]\rangle | x_j \in U\}$.

Suppose that,

(1) M_1 is a bottom-aggregation operator,

(2) $M_1(x_1,x_2,x_3,x_4) \geqslant M_2(x_1,x_2,x_3,x_4)$ for any $x_1,x_2,x_3,x_4 \in [0,1]$,

(3) f is a strict fuzzy negation,

(4) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for any $\alpha \in [1,+\infty), \beta \in [1,+\infty)$, we can define a new interval-valued intuitionistic fuzzy set $(\phi_1(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i \in P$, $x_j \in U$,

$$\underline{u}_{\phi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j))))]^{1/\alpha}\};$$

$$\overline{u}_{\phi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1-[M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j))))]\};$$

$$\underline{v}_{\phi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j))))]^{\beta}\};$$

$$\overline{v}_{\phi_1(F,G)(e_i)}(x_j) = \frac{1}{2}\{1+[M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j))))]\}.$$

Theorem 12. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P) \in IVIFSS(U)$, then $I((\phi_1(F,G),P))$ is an inclusion measure between (F,P) and (G,P).*

Proof. We only need to prove that all the properties in Definition 11 hold.
(1) If $(F,P)=(U,P)$, $(G,P)=(\emptyset,P)$,
we get $F(e_i)=\{\langle x_j,[1,1],[0,0]\rangle | x_j \in U\}$, $G(e_i)=\{\langle x_j,[0,0],[1,1]\rangle | x_j \in U\}$ for $\forall e_i \in P$,
then we have for any $x_j \in U, e_i \in P$
$E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j)) = E_1(1,1\wedge 0) = 0$,
$E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j)) = E_2(1,1\wedge 0) = 0$,
$E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j)) = E_3(0,0\vee 1) = 0$,
$E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j)) = E_4(0,0\vee 1) = 0$,
so we get
$M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j)))) = M_1(1,1,1,1) = 1$ and
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j)))) = M_2(1,1,1,1) = 1$.
Thus, it is easy to get that
$[\underline{u}_{\phi_1(F,G)(e_i)}(x_j),\overline{u}_{\phi_1(F,G)(e_i)}(x_j)] = [0,0]$,
$[\underline{v}_{\phi_1(F,G)(e_i)}(x_j),\overline{v}_{\phi_1(F,G)(e_i)}(x_j)] = [1,1]$,
$\forall x_j \in U, e_i \in P$.
By Definition 12 of entropy for $IVIFSSs$, we know $I((\phi_1(F,G),P)) = 0 \Leftrightarrow J((F,P),(G,P)) = 0$.
(2) $I((\phi_1(F,G),P)) = J((F,P),(G,P)) = 1$,
$\Leftrightarrow [\underline{u}_{\phi_1(F,G)(e_i)}(x_j),\overline{u}_{\phi_1(F,G)(e_i)}(x_j)] = [\frac{1}{2},\frac{1}{2}]$,
$[\underline{v}_{\phi_1(F,G)(e_i)}(x_j),\overline{v}_{\phi_1(F,G)(e_i)}(x_j)] = [\frac{1}{2},\frac{1}{2}]$,
$\forall x_j \in U, e_i \in P$.
$\Leftrightarrow M_1(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j)))) = 0$,
and
$M_2(f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee \overline{v}_{G(e_i)}(x_j)))) = 0$,
$\forall x_j \in U, e_i \in P$.
$\Leftrightarrow f(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge \underline{u}_{G(e_i)}(x_j))) = 0$,

$f(E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)))=0,$
$f(E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)))=0,$
$f(E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))=0,$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j))=1,$
$E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j))=1,$
$E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j))=1,$
$E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j))=1,$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow \underline{u}_{F(e_i)}(x_j)=\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j),$
$\overline{u}_{F(e_i)}(x_j)=\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j),$
$\underline{v}_{F(e_i)}(x_j)=\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j),$
$\overline{v}_{F(e_i)}(x_j)=\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j),$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)]\leqslant[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],$
$[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\geqslant[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)],$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow(F,P)\subseteq(G,P)$
(3)If $(F,P)\subseteq(G,P)\subseteq(H,P)$, then for any $x_j\in U$, $e_i\in P$,
$[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)]\leqslant[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)]\leqslant[\underline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)]$ and $[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\geqslant[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\geqslant[\underline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)]$,
so we have
$E_1(\underline{u}_{H(e_i)}(x_j),\underline{u}_{H(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j))=E_1(\underline{u}_{H(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j))\leqslant E_1(\underline{u}_{G(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j))=E_1(\underline{u}_{G(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j)),$
$E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j))=E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j))\leqslant E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j))=E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j)),$
$E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{H(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j))=E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j))\leqslant E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j))=E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j)),$
$E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j))=E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j))\leqslant E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j))=E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j)),$
then we get
$f(E_l(\underline{u}_{H(e_i)}(x_j),\underline{u}_{H(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j)))\geqslant f(E_l(\underline{u}_{G(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j))),$
$f(E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j)))\geqslant f(E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j))),$
$f(E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{H(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j)))\geqslant f(E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j))),$
$f(E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j)))\geqslant f(E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j))).$
From the property of aggregation operators, we get
$[\underline{u}_{\phi_1(H,F)(e_i)}(x_j),\overline{u}_{\phi_1(H,F)(e_i)}(x_j)]\leqslant[\underline{u}_{\phi_1(G,F)(e_i)}(x_j),\overline{u}_{\phi_1(G,F)(e_i)}(x_j)]\leqslant[\frac{1}{2},\frac{1}{2}],$
$[\underline{v}_{\phi_1(H,F)(e_i)}(x_j),\overline{v}_{\phi_1(H,F)(e_i)}(x_j)]\geqslant[\underline{v}_{\phi_1(G,F)(e_i)}(x_j),\overline{v}_{\phi_1(G,F)(e_i)}(x_j)]\geqslant[\frac{1}{2},\frac{1}{2}].$
Let $(Q,P)\in IVIFSS(U)$ and $Q(e_i)=\{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle|x_j\in U\}$ for any $e_i\in P$, then we get
$(\phi_1(H,F),P)\subseteq(\phi_1(G,F),P)\subseteq(Q,P),$
thus,
$D((\phi_1(H,F),P),(Q,P))\geqslant D((\phi_1(G,F),P),(Q,P)).$
By Definition 12 of entropy for $IVIFSSs$, we get
$I((\phi_1(H,F),P))\leqslant I((\phi_1(G,F),P))$
$\Leftrightarrow J((H,P),(F,P))\leqslant J((G,P),(F,P)).$
By the similar way, we get
$I((\phi_1(H,F),P))\leqslant I((\phi_1(H,G),P))$
$\Leftrightarrow J((H,P),(F,P))\leqslant J((H,P),(G,P)).$ □

Definition 22. Let $(F,P),(G,P)\in IVIFSS(U)$, assume that: for any $e_i\in P$,

$$F(e_i)=\{\langle x_j,u_{F(e_i)}(x_j),v_{F(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)],[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\rangle|x_j\in U\},$$

$$G(e_i)=\{\langle x_j,u_{G(e_i)}(x_j),v_{G(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\rangle|x_j\in U\}.$$

Suppose that,

(1) M_1 is a top-aggregation operator,

(2) $M_1(x_1,x_2,x_3,x_4)\leqslant M_2(x_1,x_2,x_3,x_4)$ for any $x_1,x_2,x_3,x_4\in[0,1]$,

(3) f is a strict fuzzy negation,

(4) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for any $\alpha\in[1,+\infty),\beta\in[1,+\infty)$, we can define a new interval-valued intuitionistic fuzzy set $(\phi_2(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i\in P$, $x_j\in U$,

$$\underline{u}_{\phi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M_1(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j))))]^{1/\alpha}\};$$

$$\overline{u}_{\phi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M_1(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]\};$$

$$\underline{v}_{\phi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M_2(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]^{\beta}\};$$

$$\overline{v}_{\phi_2(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M_2(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]\}.$$

Theorem 13. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P)\in IVIFSS(U)$, then $I((\phi_2(F,G),P))$ is an inclusion measure between (F,P) and (G,P).*

Definition 23. Let $(F,P),(G,P)\in IVIFSS(U)$, assume that: for any $e_i\in P$,

$$F(e_i)=\{\langle x_j,u_{F(e_i)}(x_j),v_{F(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)],[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\rangle|x_j\in U\},$$

$$G(e_i)=\{\langle x_j,u_{G(e_i)}(x_j),v_{G(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\rangle|x_j\in U\}.$$

Suppose that,

(1) M is a top-aggregation operator,

(2) f is a strict fuzzy negation,

(3) E_l $(l=1,2,3,4)$ are fuzzy equivalence operators,

then for any $0<\alpha_1\leqslant\alpha_2\leqslant\alpha_4\leqslant\alpha_3$, we can define a new interval-valued intuitionistic fuzzy set $(\phi_3(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i\in P$, $x_j\in U$,

$$\underline{u}_{\phi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]^{\alpha_1}\};$$

$$\overline{u}_{\phi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1-[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]^{\alpha_2}\};$$

$$\underline{v}_{\phi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]^{\alpha_3}\};$$

$$\overline{v}_{\phi_3(F,G)(e_i)}(x_j)=\frac{1}{2}\{1+[f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))))]^{\alpha_4}\}.$$

Theorem 14. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P)\in IVIFSS(U)$, then $I((\phi_3(F,G),P))$ is an inclusion measure between (F,P) and (G,P).*

Proof. We only need to prove that all the properties in Definition 11 hold.

(1) If $(F,P)=(U,P)$, $(G,A)=(\emptyset,P)$, we get $F(e_i)=\{\langle x_j,[1,1],[0,0]\rangle|x_j\in U\}$, $G(e_i)=\{\langle x_j,[0,0],[1,1]\rangle|x_j\in U\}$ for $\forall e_i\in P$, then we have for any $x_j\in U,e_i\in P$,

$E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)=E_1(1,1\wedge 0)=0$,

$E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j))=E_2(1,1\wedge 0)=0$,

$E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j))=E_3(0,0\vee 1)=0$,

$E_4(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j))=E_4(0,0\vee 1)=$

0,
so we get,
$M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))$
$=M(0,0,0,0)=0.$
Thus, it is easy to get that
$[\underline{u}_{\phi_3(F,G)(e_i)}(x_j),\overline{u}_{\phi_3(F,G)(e_i)}(x_j)]=[0,0],$
$[\underline{v}_{\phi_3(F,G)(e_i)}(x_j),\overline{v}_{\phi_3(F,G)(e_i)}(x_j)]=[1,1],$
$\forall x_j\in U,e_i\in P.$
From Definition 12 of entropy for $IVIFSSs$, we know
$I((\phi_3(F,G),P))=0\Leftrightarrow J((F,P),(G,P))=0.$
(2)$I((\phi_3(F,G),P))=J((F,P),(G,P))=1,$
$\Leftrightarrow[\underline{u}_{\phi_3(F,G)(e_i)}(x_j),\overline{u}_{\phi_3(F,G)(e_i)}(x_j)]=[\frac{1}{2},\frac{1}{2}],$
$[\underline{v}_{\phi_3(F,G)(e_i)}(x_j),\overline{v}_{\phi_3(F,G)(e_i)}(x_j)]=[\frac{1}{2},\frac{1}{2}],$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow f(M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j))))=0,\ \forall x_j\in U,e_i\in P.$
$\Leftrightarrow M(E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j)),E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j)),E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j)),E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j)))=1,\ \forall x_j\in U,e_i\in P.$
$\Leftrightarrow E_1(\underline{u}_{F(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j))=E_2(\overline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j))=E_3(\underline{v}_{F(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j))=E_4(\overline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j))=1,$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow \underline{u}_{F(e_i)}(x_j)=\underline{u}_{F(e_i)}(x_j)\wedge\underline{u}_{G(e_i)}(x_j),$
$\overline{u}_{F(e_i)}(x_j)=\overline{u}_{F(e_i)}(x_j)\wedge\overline{u}_{G(e_i)}(x_j),$
$\underline{v}_{F(e_i)}(x_j)=\underline{v}_{F(e_i)}(x_j)\vee\underline{v}_{G(e_i)}(x_j),$
$\overline{v}_{F(e_i)}(x_j)=\overline{v}_{F(e_i)}(x_j)\vee\overline{v}_{G(e_i)}(x_j),$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)]\leqslant[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)]$ and
$[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\geqslant[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)],$
$\forall x_j\in U,e_i\in P.$
$\Leftrightarrow(F,P)\subseteq(G,P).$
(3)If $(F,E)\subseteq(G,P)\subseteq(H,P)$, then
$[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)]\leqslant[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)]\leqslant[\underline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)]$ and
$[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\geqslant[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\geqslant[\underline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)],$
$\forall x_j\in U,e_i\in P.$
So we have for $\forall x_j\in U,e_i\in P,$
$E_1(\underline{u}_{H(e_i)}(x_j),\underline{u}_{H(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j))=E_1(\underline{u}_{H(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j))\leqslant E_1(\underline{u}_{G(e_i)}(x_j),\underline{u}_{F(e_i)}(x_j))=E_1(\underline{u}_{G(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j)),$
$E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j))=E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j))\leqslant E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j))=E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j)),$
$E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{H(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j))=E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j))\leqslant E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{F(e_i)}(x_j))=E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j)),$
$E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j))=E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j))\leqslant E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j))=E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j)),$
so we get
$f(M(E_1(\underline{u}_{H(e_i)}(x_j),\underline{u}_{H(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j)),E_2(\overline{u}_{H(e_i)}(x_j),\overline{u}_{H(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j)),E_3(\underline{v}_{H(e_i)}(x_j),\underline{v}_{H(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j)),E_4(\overline{v}_{H(e_i)}(x_j),\overline{v}_{H(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j))))\geqslant f(M(E_1(\underline{u}_{G(e_i)}(x_j),\underline{u}_{G(e_i)}(x_j)\wedge\underline{u}_{F(e_i)}(x_j)),E_2(\overline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)\wedge\overline{u}_{F(e_i)}(x_j)),E_3(\underline{v}_{G(e_i)}(x_j),\underline{v}_{G(e_i)}(x_j)\vee\underline{v}_{F(e_i)}(x_j)),E_4(\overline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)\vee\overline{v}_{F(e_i)}(x_j)))).$
Hence,
$[\underline{u}_{\phi_3(H,F)(e_i)}(x_j),\overline{u}_{\phi_3(H,F)(e_i)}(x_j)]\leqslant[\underline{u}_{\phi_3(G,F)(e_i)}(x_j),\overline{u}_{\phi_3(G,F)(e_i)}(x_j)]\leqslant[\frac{1}{2},\frac{1}{2}]$ and
$[\underline{v}_{\phi_3(H,F)(e_i)}(x_j),\overline{v}_{\phi_3(H,F)(e_i)}(x_j)]\geqslant[\underline{v}_{\phi_3(G,F)(e_i)}(x_j),\overline{v}_{\phi_3(G,F)(e_i)}(x_j)]\geqslant[\frac{1}{2},\frac{1}{2}]$, $\forall x_j\in U,e_i\in P.$
Let $(Q,P)\in IVIFSS(U)$ and $Q(e_i)=\{\langle x_j,[1/2,1/2]),[1/2,1/2]\rangle|x_j\in U\}$ for any $e_i\in P$, then we get
$(\phi_3(H,F),P)\subseteq(\phi_3(G,F),P)\subseteq(Q,P),$
by Definition 9 of distance measure for $IVIFSSs$, we get
$D((\phi_3(H,F),P),(Q,P))\geqslant D((\phi_3(G,F),P),(Q,P)).$
Therefore, by Definition 12 of entropy for $IVIFSSs$, we get
$I((\phi_3(H,F),P))\leqslant I((\phi_3(G,F),P))$
$\Leftrightarrow J((H,P),(F,P))\leqslant J((G,P),(F,P)).$
By the similar way, we get
$I((\phi_3(H,F),P))\leqslant I((\phi_3(H,G),P))$
$\Leftrightarrow J((H,P),(F,P))\leqslant J((H,P),(G,P)).$ □

Definition 24. Let $(F,P),(G,P)\in IVIFSS(U)$, assume that: for any $e_i\in P$,

$$F(e_i)=\{\langle x_j,u_{F(e_i)}(x_j),v_{F(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{F(e_i)}(x_j),\overline{u}_{F(e_i)}(x_j)],[\underline{v}_{F(e_i)}(x_j),\overline{v}_{F(e_i)}(x_j)]\rangle|x_j\in U\},$$

$$G(e_i)=\{\langle x_j,u_{G(e_i)}(x_j),v_{G(e_i)}(x_j)\rangle|x_j\in U\}=\{\langle x_j,[\underline{u}_{G(e_i)}(x_j),\overline{u}_{G(e_i)}(x_j)],[\underline{v}_{G(e_i)}(x_j),\overline{v}_{G(e_i)}(x_j)]\rangle|x_j\in U\}.$$

Suppose that,

(1) M is a bottom-aggregation operator,

(2) f is a strict fuzzy negation,

(3) E_l $(l = 1,2,3,4)$ are fuzzy equivalence operators,

then for any $0 < \alpha_1 \leqslant \alpha_2 \leqslant \alpha_4 \leqslant \alpha_3$, we can define a new interval-valued intuitionistic fuzzy set $(\phi_4(F,G),P)$ from (F,P) and (G,P) as follows: for any $e_i \in P$, $x_j \in U$,

$$\underline{u}_{\phi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j) \wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{F(e_i)}(x_j) \vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j) \vee \overline{v}_{G(e_i)}(x_j))))]^{\alpha_1}\};$$

$$\overline{u}_{\phi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 - [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j) \wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{F(e_i)}(x_j) \vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j) \vee \overline{v}_{G(e_i)}(x_j))))]^{\alpha_2}\};$$

$$\underline{v}_{\phi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 + [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j) \wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{F(e_i)}(x_j) \vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j) \vee \overline{v}_{G(e_i)}(x_j))))]^{\alpha_3}\};$$

$$\overline{v}_{\phi_4(F,G)(e_i)}(x_j) = \frac{1}{2}\{1 + [M(f(E_1(\underline{u}_{F(e_i)}(x_j), \underline{u}_{F(e_i)}(x_j) \wedge \underline{u}_{G(e_i)}(x_j))), f(E_2(\overline{u}_{F(e_i)}(x_j), \overline{u}_{F(e_i)}(x_j) \wedge \overline{u}_{G(e_i)}(x_j))), f(E_3(\underline{v}_{F(e_i)}(x_j), \underline{v}_{F(e_i)}(x_j) \vee \underline{v}_{G(e_i)}(x_j))), f(E_4(\overline{v}_{F(e_i)}(x_j), \overline{v}_{F(e_i)}(x_j) \vee \overline{v}_{G(e_i)}(x_j))))]^{\alpha_4}\}.$$

Theorem 15. *Let I be an entropy measure of interval-valued intuitionistic fuzzy soft set. For $(F,P),(G,P) \in IVIFSS(U)$, then $I((\phi_4(F,G),P))$ is an inclusion measure between (F,P) and (G,P).*

Theorem 16. *If I is an entropy measure of $IVIFSSs$ and $(\phi_h(F,G),P)(h = 1,2,3,4)$ is given by Definition 21-24, then $I((\phi_h(F,G)^C,P))$ $(h = 1,2,3,4)$ is also an inclusion measure between (F,P) and (G,P).*

Remark 7. Based on Definition 21-24, by selecting different aggregation operators and fuzzy equivalences, we can obtain a large number of $IVIFSSs$, which can be used to transform an entropy measure into an inclusion measure for $IVIFSSs$.

Remark 8. In Ref. [17], the authors provided a specific interval-valued intuitionistic fuzzy set, the entropy of which have been proved the inclusion measure for $IVIFSs$. If we extend this interval-valued intuitionistic fuzzy set into $IVIFSSs$, the corresponding interval-valued intuitionistic fuzzy soft set can be constructed by Definition 24, Theorem 15 and 16 in this work, by selecting a specific aggregation operator, a specific equivalence operator, a specific fuzzy negation operator and several specific power exponents. To a certain degree, our research is the extension of the research in Ref. [17] based on fuzzy equivalence and aggregation operators. However, even if it degenerates to the $IVIFSs$ situation, all the formulae given by Definition 21-23 in this work are new.

4.5. Transformation of similarity measures into inclusion measures for $IVIFSSs$

Theorem 17. *Let S be a similarity measure of interval-valued intuitionistic fuzzy soft sets and $(F,P),(G,P) \in IVIFSS(U)$, then $J((F,P),(G,P)) = S((G,P),(F,P) \cup (G,P))$ is an inclusion measure between (F,P) and (G,P).*

Proof. We only need to verify that the following three properties of inclusion measure hold.

(1) $J((U,P),(\emptyset,P)) = S((\emptyset,P),(U,P)) = 0$;

(2) $J((F,P),(G,P)) = 1 \Leftrightarrow S((G,P),(F,P) \cup (G,P)) = 1 \Leftrightarrow (G,P) = (F,P) \cup (G,P) \Leftrightarrow (F,P) \subseteq (G,P)$.

(3) If $(F,P) \subseteq (G,P) \subseteq (H,P)$, we easily get that

$J((H,P),(F,P)) = S((F,P),(H,P) \cup (F,P)) = S((F,P),(H,P)) \leqslant S((F,P),(G,P))$
$= S((F,P),(G,P) \cup (F,P)) = J((G,P),(F,P))$,

and

$J((H,P),(F,P)) = S((F,P),(H,P) \cup (F,P)) = S((F,P),(H,P)) \leqslant S((G,P),(H,P))$

$= S((G,P),(H,P) \cup (G,P)) = J((H,P),(G,P))$.
So, we have $J((H,P),(F,P)) \leqslant J((G,P),(F,P))$ and $J((H,P),(F,P)) \leqslant J((H,P),(G,P))$.
Thus, J is an inclusion measure of $IVIFSSs$. □

5. Disease diagnosis based on entropy and distance measure of $IVIFSSs$

An application of similarity measure of intuitionistic fuzzy soft set in disease diagnosis can be found in [23]. Benefiting from their idea, an application of the entropy and the distance measure of $IVIFSSs$ in disease diagnosis is given. In oder to estimate if an ill person is suffering from a certain disease or not, with the help of experts, we will construct an interval-valued intuitionistic fuzzy soft set for the disease and an interval-valued intuitionistic fuzzy soft set for the ill person, respectively. The algorithm is stated as follows:

Algorithm 1

Step 1. Select the threshold $\alpha \in [0,1]$ for judging the sample set of a disease and the threshold $\beta \in [0,1]$ for assessing if a patient is suffering from a disease or not;

Step 2. Constructs an interval-valued intuitionistic fuzzy soft set (F,P) over U for the disease.

Step 3. Calculate the entropy of (F,P). If $I((F,P)) < \alpha$, (F,P) can be regarded as a sample set for the disease; if else, collect more relevant information and reconstruct the interval-valued intuitionistic fuzzy soft set for the disease;

Step 4. Constructs an interval-valued intuitionistic fuzzy soft set (G,P) over U for the patient;

Step 5. Calculate the distance measure between (F,P) and (G,P), i.e., $D((F,P),(G,P))$;

Step 6. We say the patient is suffering from the disease if $D((F,P),(G,P)) < \beta$; if else, we say the patient is not suffering from the disease.

The thresholds α and β in Step 1 can be selected according to the actual situation with the help of experts. Step 2 is based on the consideration that if the uncertain degree of an interval-valued intuitionistic fuzzy soft set for the disease is too large, it maybe not suitable to be a reference sample.

Example 10. Assume that our universal set contain three elements $U = \{x_1,x_2,x_3\}$, where $x_1 =$ on the first day of illness, $x_2 =$ on the second day of illness, $x_3 =$ on the third day of illness. Here the set of parameters P is the set of certain visible symptoms, assume that $P = \{e_1,e_2,e_3,e_4,e_5\}$ where $e_1 =$ fever, $e_2 =$ cough, $e_3 =$ vomit, $e_4 =$ twitch, $e_5 =$ trouble breathing. We will try to estimate if a patient is suffering from a certain disease or not.

Step 1. Let $\alpha = 0.5$ and $\beta = 0.1$.

Step 2. Constructs an interval-valued intuitionistic fuzzy soft set (F,P) over U for the disease which can be prepared with the help of experienced doctors:
$F(e_1) = \{(x_1,[0.7,0.8],[0.15,0.2]),(x_2,[0.6,0.7],[0.15,0.21]),(x_3,[0.55,0.65],[0.15,0.25])\}$,
$F(e_2) = \{(x_1,[0.7,0.8],[0.1,0.2]),(x_2,[0.55,0.65],[0.2,0.25]),(x_3,[0.60,0.70],[0.05,0.1])\}$,
$F(e_3) = \{(x_1,[0.7,0.8],[0.1,0.2]),(x_2,[0.65,0.75],[0.2,0.25]),(x_3,[0.77,0.88],[0.1,0.1])\}$,
$F(e_4) = \{(x_1,[0.6,0.7],[0.1,0.2]),(x_2,[0.55,0.65],[0.2,0.25]),(x_3,[0.66,0.7],[0.05,0.1])\}$, $F(e_5) = \{(x_1,[0.6,0.6],[0.2,0.3]),(x_2,[0.55,0.60],[0.2,0.25]),(x_3,[0.7,0.8],[0.05,0.1])\}$.

Step 3. Calculate the entropy of (F,P). Here we use the entropy measure of $IVIFSSs$ constructed by Theorem 3. Let $D_2((F,P),(Q,P))$ be the Normalized hamming distance between (F,P) and (Q,P) and $f'(x) = 1-x$ for all $x \in [0,1]$. Then we get $I_2((F,P)) = f'(2D_2((F,P),(Q,P))) = 0.49 < 0.5$, that is to say, (F,P) can be regarded as a sample set for the disease.

Step 4. Constructs an interval-valued intuitionistic fuzzy soft set (G,P) over U based on the data of a patient:
$G(e_1) = \{(x_1,[0.7,0.8],[0.15,0.2]),(x_2,[0.6,0.7],[0.15,0.21]),(x_3,[0.55,0.75],[0.15,0.25])\}$,
$G(e_2) = \{(x_1,[0.6,0.7],[0.2,0.3]),(x_2,[0.55,0.65],[0.2,0.25]),(x_3,[0.7,0.88],[0.05,0.1])\}$,
$G(e_3) = \{(x_1,[0.5,0.6],[0.2,0.3]),(x_2,[0.45,0.55],[0.2,0.25]),(x_3,[0.7,0.78],[0.05,0.1])\}$,
$G(e_4) = \{(x_1,[0.3,0.4],[0.3,0.4]),(x_2,[0.55,0.65],[0.2,0.25]),(x_3,[0.7,0.88],[0.05,0.1])\}$,
$G(e_5) = \{(x_1,[0.4,0.5],[0.2,0.3]),(x_2,[0.35,0.40],[0.2,0.25]),(x_3,[0.7,0.88],[0.05,0.1])\}$.

Step 5. Here we use the Normalized hamming distance between (F,P) and (G,P), which is denoted by $D_2((F,P),(G,P))$. It is easy to get that

$D_2((F,P),(G,P)) \approx 0.067$.

Step 6. We conclude that the patient is suffering from the disease since

$D_2((F,P),(G,P)) < 0.1$.

6. Conclusions and Discussion

In this paper, we give eight general formulae to calculate the distance measures of $IVIFSSs$ by aggregating fuzzy equivalencies. Consistently with a new axiomatic definition of entropy for $IVIFSSs$, we prove some theorems which demonstrate that distance measures can be transformed into entropies for $IVFSSs$. Besides, we prove some theorems which demonstrate that entropies can be transformed into the inclusion measure and the similarity measure for $IVIFSSs$ based on fuzzy equivalencies.

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