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Linguistic hesitant intuitionistic fuzzy cross-entropy measures

Wei Yang * **Yongfeng Pang** **Jiarong Shi**

Department of Mathematics, School of Science,
Xi'an University of Architecture and Technology, 710055 Xi'an, Shaanxi, P. R. China

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Abstract

In this paper, several cross-entropy measures for linguistic hesitant intuitionistic fuzzy information have been developed which integrating cross-entropy measures of intuitionistic fuzzy sets and hesitant fuzzy linguistic sets. Some desirable properties of new cross-entropy measures have been studied. Two new multiple attribute decision making methods have been presented based on the new cross-entropy measures in which attribute values are given in the form of linguistic hesitant intuitionistic fuzzy values to reflect human hesitantation and fuzzy thinking comprehensively. We consider different attribute weight situations including attribute weights are completely known, partly known and completely unknown. An optimization model is established to determine attribute weights if they are partly known and a formula is given if attribute weights are completely unknown. Finally, a numerical example is presented to illustrate practical advantages and effectiveness of the proposed approaches.

Keywords: Hesitant fuzzy set; intuitionisic fuzzy set; linguistic argument; aggregation operator; linguistic hesitant intuitionistic fuzzy cross-entropy.

1. Introduction

Fuzziness and uncertainty exists extensively in decision making process due to complicated decision problems, limited decision time and fuzzy nature of human thinking, etc. Many useful tools have been developed including fuzzy set, intuitionistic fuzzy set[1], hesitant fuzzy set[2–3], linguistic arguments[4–8], etc. In fuzzy set, the membership of each element is a real number between 0 and 1. Intuitionistic fuzzy set is the extension of fuzzy set, which is characterized by membership and non-membership summing less than 1. Hesitant fuzzy set is another extension of fuzzy set, in which several values are possible for the definition of a membership function of a fuzzy set. The envelope of hesitant fuzzy set is intuitionistic fuzzy set. Comparing with other tools to model fuzzy and uncertain information, hesitant fuzzy set is more powerful and accurate. The hesitant fuzzy set has been studied and applied extensively[9–14]. Some hesitant aggregation operators have been proposed[15–17]. Some classic multiple attribute decision making methods have been extended to hesitant fuzzy environment[18–20]. Some distance measures, entropy measures and correlation coefficients have been generalized to accommodate hesitant fuzzy information[21–25]. Hesitant fuzzy set has been extended to accommodate intuitionistic fuzzy values[26], interval-values[27], triangular fuzzy values[28], linguistic arguments[29–35], etc. Due to fuzzy nature of human thinking, complicated decision problems and limited decision time, decision makers would like to evaluate with linguistic terms rather than exact numerical values. Sev-

* Corresponding author. E-mail: yangweipyf@163.com.

eral types of linguistic models have been developed. Herrera and Martínez[4] proposed 2-tuple linguistic model to avoid information distortion and loss. Dong et al.[5] defined numerical scale and extended the 2-tuple fuzzy linguistic representation models under numerical scale. By using 2-tuple linguistic model, each evaluation value only has one linguistic evaluation value. Rodríguez et al.[32] developed hesitant linguistic fuzzy set in which each element has several linguistic terms. Wang[35] generalized hesitant fuzzy linguistic term sets by enabling any non-consecutive linguistic terms in them. Pang et al.[6] developed probabilistic linguistic term set in which possible linguistic values may have different importance degrees. The fuzzy linguistic modelling based on discrete fuzzy numbers is proposed by Riera et al.[7] to manage hesitant fuzzy linguistic information. Comparing with fuzzy numbers, intuitionistic fuzzy values are more accurately to model hesitation. The concept of a possibility distribution for hesitant fuzzy linguistic information has been defined[11]. But hesitation in decision making hasn't been modeled properly by existing linguistic models. In decision making process, experts would express some hesitation in evaluating with linguistic terms. Intuitionistic fuzzy values can be used to model hesitation accurately. By using intuitionistic fuzzy values to model hesitation in linguistic evaluating process, fuzzy nature of human thinking can be reflected accurately and hesitation can be modeled properly. If an expert uses linguistic term s_α in evaluating some alternative with respect to some attribute and he/she thinks the membership of alternative satisfying the attribute is μ and non-membership is ν, then linguistic intuitionistic fuzzy element (LIFE) $(s_\alpha,(\mu,\nu))$ can be got. If two experts use the same linguistic term and different intuitionistic fuzzy values, intuitionistic fuzzy values are merged together. For example, in evaluating performance of a candidate for a college dean, one expert thinks the degree of 'slightly good (S_6)' the candidate belonging to is (0.6,0.3) and degree of 'good (S_7)' is (0.7,0.2). Another expert thinks the degree of 'fair (S_5)' the candidate belonging to is (0.5,0.4), the degree of 'slightly good' is (0.7,0.1) and the degree of 'good' is (0.6,0.2). Then we can get linguistic hesitant intuitionistic fuzzy information as $\check{h} = \{(S_5,(0.5,0.4)),(S_6,(0.6,0.3),(0.7,0.1)),(S_7,(0.6,0.2),(0.7,0.2))\}$. The linguistic hesitant intuitionistic fuzzy set is developed by Yang et al.[36]. Comparing with other tools, linguistic hesitant intuitionisic fuzzy values can model fuzzy and uncertain information existing in decision making process more accurate, which is the prerequisite to get scientific and reasonable decision-making results.

A growing number of studies focus on entropy measures and cross-entropy measures due to advantages of measuring fuzziness and discrimination information[37–49]. Kullback and Leibler[39] defined a cross-entropy measure between two probability distribution. The fuzzy cross-entropy has been defined by Bhandari and Pal[40] by using its membership function. Zhang and Jiang[41] developed vague cross-entropy by analogy with the cross-entropy of probability distributions. Chen et al.[42] developed several cross-entropy measures for uncertain variables. Mao et al.[43] proposed a novel symmetric intuitionistic fuzzy cross-entropy formula taking into account intuitionistic fuzzy entropy and fuzzy entropy simultaneously. Xia and Xu[44] defined two cross-entropy measures for intuitionistic fuzzy values by normalizing the J-divergence intuitionistic fuzzy values introduced by Hung and Yang[45]. Wang and Li[46] proposed a cross-entropy measure of the membership degree from the non-membership degree for intuitionistic fuzzy values. Qi et al.[47] defined a new generalized interval-valued intuitionistic fuzzy cross-entropy measure and gave a new method to determine unknown attribute weights and expert weights based on the new cross-entropy measure. Xu and Xia[48] defined two cross-entropy measures for hesitant fuzzy information. Peng et al.[26] have developed some fuzzy cross-entropy measures for intuitionistic hesitant fuzzy information. The cross-entropy methods have been used extensively, such as traffic signal optimization, portfolio selection, clustering, energy management, etc.

From the above analysis we can find that all the existing cross-entropy measures are based on exact numerical values, fuzzy values, intuitionistic fuzzy values, hesitant fuzzy values. Linguistic hesitant intuitionistic fuzzy values are more accurate and flexible in modeling fuzzy and uncertain information. However, the study on cross-entropy measures for linguistic hesitant intuitionistic fuzzy information hasn't been found yet. Due to the fact hesitation is common existing in actual decision making process, it is necessary to develop some cross entropy measures for linguistic hesitant intuitionistic fuzzy information. The aim of this paper is to propose several linguistic hesitant intuitionistic fuzzy cross-entropy measures by extending intuitionistic fuzzy cross-entropy measures and hesitant fuzzy cross-entropy measures to linguistic hesitant intuitionistic fuzzy environment, which can comprehensively accommodate membership, nonmembership and hesitation degree in linguistic evaluation process. Based on the new cross-entropy measures, a programming model is set up to derive unknown attribute weights by considering deviation between attribute assessment values and a formula is given to derive attribute weights if they are completely unknown. Then we develop two algorithms based on the new cross-entropy measures integrating the afore presented methods considering different situations of attribute weights. A numerical example of supplier selection problem is presented to illustrate the new algorithms. Additionally, it is important to note that the decision making methods proposed in this paper can also be used to solve other decision making problems with high uncertainty and hesitation degrees.

In order to do so, the rest of the paper is organized as follows. In section 2, we first review some basic concepts on linguistic hesitant intuitionistic fuzzy set. Then we define several linguistic hesitant intuitionistic fuzzy aggregation operators. In section 3, several cross-entropy measures for linguistic hesitant intuitionistic fuzzy information have been developed and some desirable properties have been studied. In section 4, we propose two new multiple attribute decision making methods based on the new cross-entropy measures. In section 5, an example of supplier selection is given to illustrate feasibility and practical advantages of new methods. The conclusions are given in the last section.

2. Linguistic hesitant intuitionistic fuzzy term set

An HFS is defined in terms of a function that returns a set of membership values of each element in the domain.

Definition 2.1[2]. Let X be a reference set, an HFS A on X is a function h that returns a subset of values in $[0,1]$ when it is applied to X:

$$A = \{< x, h_A(x) >| \, x \in X\},$$

where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of $x \in X$ to A. $h_A(x)$ is called a hesitant fuzzy element (HFE)[14].

Suppose that $S = \{s_i | \, i = 1,...,g\}$ is a finite and totally ordered discrete term set, where s_i represents a possible value for a linguistic variable. A set of nine terms S^{50} can be expressed as follows $S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\}$. In order to preserve all information, the discrete linguistic term set S can be extended to a continuous one $\overline{S} = \{s_\alpha \mid s_0 \leqslant s_\alpha \leqslant s_g, \alpha \in [0,g]\}$.

By extending hesitant fuzzy set, Zhang and Wu[31] develop hesitant fuzzy linguistic term set (HFLS), in which a linguistic variable has several linguistic terms.

Definition 2.2. Let X be a reference set and $\overline{S} = \{s_\alpha \mid s_0 \leqslant s_\alpha \leqslant s_g\}$ be a linguistic term set. A hesitant fuzzy linguistic term set $\overline{A}$ on X is an ordered finite subset of the consecutive linguistic term set $\overline{S}$

$$\overline{A} = \{< x_i, \overline{h}_{\overline{A}}(x_i) >| \, x_i \in X, i = 1,2,...,n\},$$

where $\overline{h}_{\overline{A}}(x_i) : X \to \overline{S}$ denotes all the possible linguistic evaluation values of element $x_i \in X$. For convenience, we call $\overline{h}_{\overline{A}}(x_i)$ a hesitant fuzzy linguistic element (HFLE), which can be represented as

$$\overline{h}_{\overline{A}}(x_i) = \{s_i \mid s_i \in \overline{h}_{\overline{A}}(x_i)\},$$

here s_i is a linguistic argument.

Since experts would express some hesitation in evaluation using linguistic terms, we use intuitionistic fuzzy value to model hesitation. If multiple experts evaluate alternatives with respect to attributes using different linguistic terms and different intuitionistic fuzzy values, we get linguistic hesitant intuitionistic fuzzy set, which can be defined as follows.

Definition 2.3[36]. Let $X = \{x_1, x_2, ..., x_n\}$ be a reference set and $\overline{S} = \{s_\alpha \mid s_0 \leqslant s_\alpha \leqslant s_g\}$ be a linguistic term set. A linguistic hesitant intuitionistic fuzzy set (LHIFS) $\check{A}$ on X is defined as

$$\check{A} = \{< x_i, \check{h}_{\check{A}}(x_i) > \mid x_i \in X, i = 1, 2, ..., n\},$$

where $\check{h}_{\check{A}}(x_i) : X \to H$ denotes all possible linguistic intuitionistic fuzzy evaluation values of element $x_i \in X$. For convenience, we call $\check{h}_{\check{A}}(x_i)$ a linguistic hesitant intuitionistic fuzzy element (LHIFE), which can be represented as

$$\check{h}_{\check{A}}(x_i) = \left\{ \left(s_{\theta_i}, lh(s_{\theta_i})\right) \mid x_i \in X \right\},$$

s_{θ_i} is a linguistic argument and $lh(s_{\theta_i}) = \{(\mu_i^{(k)}, \nu_i^{(k)})\}$ is the set of intuitionistic fuzzy membership values that s_{θ_i} satisfies x_i. $\left(s_{\theta_i}, lh(s_{\theta_i})\right)$ is the linguistic intuitionistic fuzzy element (LIFE). Let H be the set of all LHIFEs.

Definition 2.4[36]. Let $\check{h}$, $\check{h}_1$ and $\check{h}_2$ be LHIFEs, $\lambda > 0$. $a_k = (s_{\theta_k}, lh(s_{\theta_k})) \in \check{h}$, $a_i = (s_{\theta_i}, lh(s_{\theta_i})) \in \check{h}_1$, $a_j = (s_{\theta_j}, lh(s_{\theta_j})) \in \check{h}_2$. Some operations on these LHIFEs can be defined as follows

(1) $\check{h}_1 \oplus \check{h}_2 = \bigcup_{a_i \in \check{h}_1, a_j \in \check{h}_2} \left\{ \left(s_{\theta_i + \theta_j}, \bigcup_{(\mu_i^{(l)}, \nu_i^{(l)}) \in lh(s_{\theta_i}), (\mu_j^{(m)}, \nu_j^{(m)}) \in lh(s_{\theta_j})} \{(\mu_i^{(l)} + \mu_j^{(m)} - \mu_i^{(l)} \mu_j^{(m)}, \nu_i^{(l)} \nu_j^{(m)})\} \right) \right\}$,

(2) $\check{h}_1 \otimes \check{h}_2 = \bigcup_{a_i \in \check{h}_1, a_j \in \check{h}_2} \left\{ \left(s_{\theta_i \theta_j}, \bigcup_{(\mu_i^{(l)}, \nu_i^{(l)}) \in lh(s_{\theta_i}), (\mu_j^{(m)}, \nu_j^{(m)}) \in lh(s_{\theta_j})} \{(\mu_i^{(l)} \mu_j^{(m)}, \nu_i^{(l)} + \nu_j^{(m)} - \nu_i^{(l)} \nu_j^{(m)})\} \right) \right\}$,

(3) $\lambda \check{h} = \bigcup_{a_k \in \check{h}} \left\{ \left(s_{\lambda \theta_k}, \bigcup_{(\mu_k^{(n)}, \nu_k^{(n)}) \in lh(s_{\theta_k})} \{(1 - (1 - \mu_k^{(n)})^\lambda, (\nu_k^{(n)})^\lambda)\} \right) \right\}$,

(4) $(\check{h})^\lambda = \bigcup_{a_k \in \check{h}} \left\{ \left(s_{\theta_k^\lambda}, \bigcup_{(\mu_k^{(n)}, \nu_k^{(n)}) \in lh(s_{\theta_k})} \{((\mu_k^{(n)})^\lambda, 1 - (1 - \nu_k^{(n)})^\lambda)\} \right) \right\}$.

Theorem 1[36]. Let $\check{h}$, $\check{h}_1$ and $\check{h}_2$ be LHIFEs and $\lambda, \lambda_1, \lambda_2 > 0$, then

(1) $\check{h}_1 \oplus \check{h}_2 = \check{h}_2 \oplus \check{h}_1$,
(2) $\check{h}_1 \otimes \check{h}_2 = \check{h}_2 \otimes \check{h}_1$,
(3) $\lambda(\check{h}_1 \oplus \check{h}_2) = \lambda \check{h}_1 \oplus \lambda \check{h}_2$,
(4) $(\check{h}_1 \otimes \check{h}_2)^\lambda = (\check{h}_1)^\lambda \otimes (\check{h}_2)^\lambda$,
(5) $(\lambda_1 + \lambda_2)\check{h} = \lambda_1 \check{h} \oplus \lambda_2 \check{h}$,
(6) $\check{h}^{\lambda_1 + \lambda_2} = \check{h}^{\lambda_1} \otimes \check{h}^{\lambda_2}$.

Definition 2.5[36]. Let $a_i = (s_{\theta_i}, lh(s_{\theta_i}))$ be a LIFE, then the score function $s(a_i)$ of a_i can be defined as

$$s(a_i) = \frac{\theta_i}{g \mid lh(s_{\theta_i}) \mid} \sum_{(\mu_i^{(k)}, \nu_i^{(k)}) \in lh(s_{\theta_i})} (\mu_i^{(k)} - \nu_i^{(k)}),$$

and the accuracy function $h(a_i)$ of a_i can be defined as

$$h(a_i) = \frac{\theta_i}{g \mid lh(s_{\theta_i}) \mid} \sum_{(\mu_i^{(k)}, \nu_i^{(k)}) \in lh(s_{\theta_i})} (\mu_i^{(k)} + \nu_i^{(k)}),$$

where g is the number of linguistic arguments in linguistic term set S and $\mid lh(s_{\theta_i}) \mid$ is the number of intuitionistic fuzzy memberships in $lh(s_{\theta_i})$.

Based on the score function $s(a_i)$ and the accuracy function $h(a_i)$, we can rank LIFEs as follows. Let $a_i = (s_{\theta_i}, lh(s_{\theta_i}))$ and $a_j = (s_{\theta_j}, lh(s_{\theta_j}))$ be two LIFEs, then

(1) If $s(a_i) < s(a_j)$, then $a_i < a_j$,

(2) If $s(a_i) = s(a_j)$ and $h(a_i) < h(a_j)$, then $a_i < a_j$, else if $s(a_i) = s(a_j)$ and $h(a_i) = h(a_j)$, then $a_i \sim a_j$.

Definition 2.6[36]. Let $\check{h} = \left\{ \left(s_{\theta_i}, lh(s_{\theta_i})\right) \right\}$ be a LHIFE, the score function $S(\check{h})$ can be defined as

$$S(\check{h}) = \frac{1}{\mid \check{h} \mid} \Big(\sum \frac{\theta_i}{g \mid lh(s_{\theta_i}) \mid} \sum_{(\mu_i^{(k)}, \nu_i^{(k)}) \in lh(s_{\theta_i})} (\mu_i^{(k)} - \nu_i^{(k)}) \Big),$$

and the accuracy function $A(\check{h})$ can be defined as

$$A(\check{h}) = \frac{1}{\mid \check{h} \mid} \Big(\sum \frac{\theta_i}{g \mid lh(s_{\theta_i}) \mid} \sum_{(\mu_i^{(k)}, \nu_i^{(k)}) \in lh(s_{\theta_i})} (\mu_i^{(k)} + \nu_i^{(k)}) \Big),$$

where $\mid \check{h} \mid$ is the number of LIFEs in $\check{h}$ and g is the number of linguistic terms in linguistic term set S, $\mid lh(s_{\theta_i}) \mid$ is the number of intuitionistic fuzzy memberships in $lh(s_{\theta_i})$.

Based on the score function and accuracy function, we present the following method to compare LHIFEs. Let $\check{h}_1$ and $\check{h}_2$ be two LHIFEs,

(1) If $S(\check{h}_1) < S(\check{h}_2)$, then $\check{h}_1 < \check{h}_2$;

(2) If $S(\check{h}_1) = S(\check{h}_2)$ and

(I) $A(\check{h}_1) < A(\check{h}_2)$, then $\check{h}_1 < \check{h}_2$,

(II) $A(\check{h}_1)=A(\check{h}_2)$, then $\check{h}_1 \sim \check{h}_2$.

Definition 2.7. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. The linguistic hesitant intuitionistic fuzzy weighted averaging (LHIFWA) operator is a mapping LHIFWA: $H^n \to H$, which can be defined as follows:

$$\begin{aligned} & \mathrm{LHIFWA}_w(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & w_1\check{h}_1 \oplus w_2\check{h}_2 \oplus ... \oplus w_n\check{h}_n. \end{aligned} \tag{1}$$

Theorem 2. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. The aggregated result of the LHIFWA operator is also a LHIFE, and

$$\begin{aligned} & \mathrm{LHIFWA}_w(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & \bigcup_{a_i\in\check{h}_i}\Big\{\big(s_{(\sum_{j=1}^{n} w_j\theta_j)}, \bigcup_{(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})} \{(1- \\ & \textstyle\prod_{j=1}^{n}(1-\mu_j^{(k)})^{w_j}, \prod_{j=1}^{n}(\nu_j^{(k)})^{w_j})\}\big)\Big\}, \end{aligned} \tag{2}$$

where $a_i=\{(s_{\theta_i},lh(s_{\theta_i}))\}$, $(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})$, $k=1,2,...,l_j$, l_j is the number of intuitionistic fuzzy memberships in $lh(s_{\theta_j})$.

Definition 2.8. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. The linguistic hesitant intuitionistic fuzzy weighted geometric (LHIFWG) operator is a mapping LHIFWG: $H^n \to H$, which can be defined as follows:

$$\begin{aligned} & \mathrm{LHIFWG}_w(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & \check{h}_1^{w_1} \otimes \check{h}_2^{w_2} \otimes ... \otimes \check{h}_n^{w_n}. \end{aligned} \tag{3}$$

Theorem 3. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. The aggregated result of the LHIFWG operator is also a LHIFE, and

$$\begin{aligned} & \mathrm{LHIFWG}_w(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & \bigcup_{a_i\in\check{h}_i}\Big\{\big(s_{(\prod_{j=1}^{n}(\theta_j)^{w_j})}, \bigcup_{(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})} \\ & \textstyle\{(\prod_{j=1}^{n}(\mu_j^{(k)})^{w_j}, 1-\prod_{j=1}^{n}(1-\nu_j^{(k)})^{w_j})\}\big)\Big\}, \end{aligned} \tag{4}$$

where $a_i=\{(s_{\theta_i},lh(s_{\theta_i}))\}$, $(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})$, $k=1,2,...,l_j$, l_j is the number of intuitionistic fuzzy memberships in $lh(s_{\theta_j})$.

Definition 2.9. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$, $\lambda>0$. The generalized linguistic hesitant intuitionistic fuzzy weighted averaging (GLHIFWA) operator is a mapping GLHIFWA: $H^n \to H$, which can be defined as follows:

$$\begin{aligned} & \mathrm{GLHIFWA}_{w,\lambda}(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & (w_1\check{h}_1^{\lambda} \oplus w_2\check{h}_2^{\lambda} \oplus ... \oplus w_n\check{h}_n^{\lambda})^{1/\lambda}. \end{aligned} \tag{5}$$

Theorem 4. Let $\check{h}_j$ $(j=1,2,...,n)$ be a collection of LHIFEs, $w=(w_1,w_2,...,w_n)$ be the weight vector of $\check{h}_j$ $(j=1,2,...,n)$ with $w_j \geqslant 0$ $(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. The aggregated result of the GLHIFWA operator is also a LHIFE, and

$$\begin{aligned} & \mathrm{GLHIFWA}_{w,\lambda}(\check{h}_1,\check{h}_2,...,\check{h}_n) \\ = \; & \bigcup_{a_i\in\check{h}_i}\Big\{\big(s_{(\sum_{j=1}^{n} w_j(\theta_j)^{\lambda})^{1/\lambda}}, \bigcup_{(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})} \{((1- \\ & \textstyle\prod_{j=1}^{n}(1-(\mu_j^{(k)})^{\lambda})^{w_j})^{1/\lambda}, 1-(1-\prod_{j=1}^{n}(1- \\ & (1-\nu_j^{(k)})^{\lambda})^{w_j})^{1/\lambda})\}\big)\Big\}. \end{aligned} \tag{6}$$

where $a_i=\{(s_{\theta_i},lh(s_{\theta_i}))\in\check{h}_i\}$, $(\mu_j^{(k)},\nu_j^{(k)})\in lh(s_{\theta_j})$, $k=1,2,...,l_j$ and l_j is the number of intuitionistic fuzzy memberships in $lh(s_{\theta_j})$, $\lambda>0$.

3. Cross-entropy measures for LHIFSs

Let $P=\{p_1,p_2,...,p_n\}$ and $Q=\{q_1,q_2,...,q_n\}$ be two probability distribution. In order to measure the divergence between P and Q, Kullback and Leibler[39] defined the cross-entropy measure as

$$CE_1(P,Q)=\sum_{i=1}^{n} p_i \ln\frac{p_i}{q_i}. \tag{7}$$

If $n=2$, $P=\{p,1-p\}$, $Q=\{q,1-q\}$, then $CE_1(P,Q)=p\ln\frac{p}{q}+(1-p)\ln\frac{1-p}{1-q}$.

Bhandari and Pal[40] generalized the cross-entropy measure based on probability distribution to accommodate fuzzy information. Let A and B be two fuzzy sets in the finite universe $X=\{x_1,x_2,...,x_n\}$, then the cross-entropy measure for fuzzy values can be defined as

$$CE_2(A,B) = \sum_{i=1}^{n}\Big(\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\mu_B(x_i)}+ (1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\Big). \tag{8}$$

Vlachos and Sergiadis[49] developed a cross-entropy measure for intuitionistic fuzzy information by extending fuzzy cross-entropy measure to intuitionistic fuzzy environment. Let A' and B' be two intuitionistic fuzzy sets in the finite universe $X=\{x_1,x_2,...,x_n\}$, then the cross-entropy measure for intuitionsitic fuzzy values can be defined as

$$CE_3(A',B')=\sum_{i=1}^{n}\Big(\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\mu_B(x_i)}+\nu_A(x_i)\ln\frac{\nu_A(x_i)}{\nu_B(x_i)}\Big). \tag{9}$$

If $\mu_B(x_i)=0$, $\mu_A(x_i)\neq 0$ or $\nu_B(x_i)=0, \nu_A(x_i)\neq 0$, $CE_3(A',B')$ is undefined. Vlachos and Sergiadis further gave the following cross-entropy

$$CE_4(A',B') = \sum_{i=1}^{n}\Big(\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i))}+ \nu_A(x_i)\ln\frac{\nu_A(x_i)}{\frac{1}{2}(\nu_A(x_i)+\nu_B(x_i))}\Big). \tag{10}$$

Uncertainty of intuitionistic fuzzy values is decomposed into intuitionism and fuzziness. Intuitionism is determined by hesitancy degree $\pi_A(x_i)=1-\mu_A(x_i)-\nu_A(x_i)$ and fuzziness is determined by the closeness of membership $\mu_A(x_i)$ and non-membership $\nu_A(x_i)$ as $\triangle_A(x_i)=|\mu_A(x_i)-\nu_A(x_i)|$. Mao et al.[43] presented the cross-entropy measure for intuitionistic fuzzy information by considering intuitionism and fuzziness simultaneously as follows

$$CE_5(A',B') = \sum_{i=1}^{n}\Big(\pi_A(x_i)\ln\frac{\pi_A(x_i)}{\frac{1}{2}(\pi_A(x_i)+\pi_B(x_i))}+ \triangle_A(x_i)\ln\frac{\triangle_A(x_i)}{\frac{1}{2}(\triangle_A(x_i)+\triangle_B(x_i))}\Big). \tag{11}$$

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Xu and Xia[48] proposed some cross-entropy formulas for hesitant fuzzy information by using two concave-up functions $f(x)=(1+qx)ln(1+qx)$ and $g(x)=x^p$. Let $\widetilde{A}=\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,...,\widetilde{\alpha}_n\}$ and $\widetilde{B}=\{\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\}$ be two hesitant fuzzy sets in the finite universe $X=\{x_1,x_2,...,x_n\}$. The number of elements in all $\widetilde{\alpha}_i$ $(i=1,2,...,n)$ and $\widetilde{\beta}_i$ $(i=1,2,...,n)$ is the same. $|\widetilde{\alpha}_i|$ is the number of elements in $\widetilde{\alpha}_i$. $\widetilde{\alpha}_{\sigma(i)}$ is the ith largest values in $\widetilde{\alpha}_i$. Then the cross-entropy measures can be defined as follows in Eq. (12) and Eq.(13). Here $l=|\widetilde{\alpha}_i|$, $T=(1+q)\ln(1+q)-(2+q)(\ln(2+q)-\ln 2)$, $q>0$.

$$\begin{aligned} CE_5(\widetilde{A},\widetilde{B}) = {} & \tfrac{1}{lT}\sum_{i=1}^{n}\sum_{j=1}^{l}\Big(\tfrac{(1+q\widetilde{\alpha}_{\sigma(j)}(x_i))\ln(1+q\widetilde{\alpha}_{\sigma(j)}(x_i))+(1+q\widetilde{\beta}_{\sigma(j)}(x_i))\ln(1+q\widetilde{\beta}_{\sigma(j)}(x_i))}{2}- \\ & \tfrac{(2+q\widetilde{\alpha}_{\sigma(j)}(x_i)+q\widetilde{\beta}_{\sigma(j)}(x_i))}{2}\ln\tfrac{(2+q\widetilde{\alpha}_{\sigma(j)}(x_i)+q\widetilde{\beta}_{\sigma(j)}(x_i))}{2}+ \\ & \tfrac{(1+q(1-\widetilde{\alpha}_{\sigma(l-i+1)}(x_i))\ln(1+q(1-\widetilde{\alpha}_{\sigma(l-j+1)}(x_i))+(1+q(1-\widetilde{\beta}_{2\sigma(l-j+1)}(x_i))\ln(1+q(1-\widetilde{\beta}_{\sigma(l-j+1)}(x_i))}{2} \\ & -\tfrac{2+q(1-\widetilde{\alpha}_{\sigma(l-i+1)}(x_i))+1-\widetilde{\beta}_{\sigma(l-i+1)}(x_i))}{2}\ln\tfrac{2+q(1-\widetilde{\alpha}_{\sigma(l-i+1)}(x_i))+1-\widetilde{\beta}_{\sigma(l-i+1)}(x_i))}{2}\Big). \end{aligned} \tag{12}$$

$$\begin{aligned} CE_6(\widetilde{A},\widetilde{B}) = {} & \tfrac{1}{l(1-2^{1-p})}\sum_{i=1}^{n}\sum_{j=1}^{l}\Big(\tfrac{\widetilde{\alpha}^p_{\sigma(j)}(x_i)+\widetilde{\beta}^p_{\sigma(j)}(x_i)}{2}+\tfrac{(1-\widetilde{\alpha}_{\sigma(l-j+1)}(x_i))^p+(1-\widetilde{\beta}_{\sigma(l-j+1)}(x_i))^p}{2}- \\ & \big(\tfrac{\widetilde{\alpha}_{\sigma(j)}(x_i)+\widetilde{\beta}_{\sigma(j)}(x_i)}{2}\big)^p+\tfrac{(1-\widetilde{\alpha}_{\sigma(l-j+1)}(x_i)+1-\widetilde{\beta}_{\sigma(l-j+1)}(x_i))^p}{2}\Big), l=|\widetilde{\alpha}_i|. \end{aligned} \tag{13}$$

Peng et al.[26] developed several cross-entropy measures for intuitionistic hesitant fuzzy numbers (IHFNs) by extending cross-entropy measures for hesitant fuzzy elements introduced by Xu and Xia[48]. Let $\widetilde{\alpha}_j=<\Gamma_{\widetilde{\alpha}_j},\Psi_{\widetilde{\alpha}_j}>$ $(j=1,2)$ be IHFNs, $\Gamma_{\widetilde{\alpha}_j}$ and $\Psi_{\widetilde{\alpha}_j}$ denote the possible degrees of membership and non-membership, respectively. $\Pi_{\widetilde{\alpha}_j}$ denotes the possible degrees of hesitation. Here $T=(1+q)\ln(1+q)-(2+q)(\ln(2+q)-\ln 2)$, $q>0$.

$$
\begin{aligned}
CE_7(\widetilde{\alpha}_1,\widetilde{\alpha}_2) \;=\; & \tfrac{1}{T}\Big(\max_{\mu_{\widetilde{\alpha}_1}\in\Gamma_{\widetilde{\alpha}_1}}\Big\{\min_{\mu_{\widetilde{\alpha}_2}\in\Gamma_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\mu_{\widetilde{\alpha}_1})\ln(1+q\mu_{\widetilde{\alpha}_1})+(1+q\mu_{\widetilde{\alpha}_2})\ln(1+q\mu_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\mu_{\widetilde{\alpha}_1}+q\mu_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\mu_{\widetilde{\alpha}_1}+q\mu_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}+\\
& \max_{\nu_{\widetilde{\alpha}_1}\in\Psi_{\widetilde{\alpha}_1}}\Big\{\min_{\nu_{\widetilde{\alpha}_2}\in\Psi_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\nu_{\widetilde{\alpha}_1})\ln(1+q\nu_{\widetilde{\alpha}_1})+(1+q\nu_{\widetilde{\alpha}_2})\ln(1+q\nu_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\nu_{\widetilde{\alpha}_1}+q\nu_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\nu_{\widetilde{\alpha}_1}+q\nu_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}+\\
& \max_{\pi_{\widetilde{\alpha}_1}\in\Pi_{\widetilde{\alpha}_1}}\Big\{\min_{\pi_{\widetilde{\alpha}_2}\in\Pi_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\pi_{\widetilde{\alpha}_1})\ln(1+q\pi_{\widetilde{\alpha}_1})+(1+q\pi_{\widetilde{\alpha}_2})\ln(1+q\pi_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\pi_{\widetilde{\alpha}_1}+q\pi_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\pi_{\widetilde{\alpha}_1}+q\pi_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}\Big), p\geqslant 1.
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
CE_8(\widetilde{\alpha}_1,\widetilde{\alpha}_2) \;=\; & \Big(\tfrac{1}{|\Gamma_{\widetilde{\alpha}_1}|}\sum_{\mu_{\widetilde{\alpha}_1}\in\Gamma_{\widetilde{\alpha}_1}}\tfrac{1}{T}\Big(\Big\{\min_{\mu_{\widetilde{\alpha}_2}\in\Gamma_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\mu_{\widetilde{\alpha}_1})\ln(1+q\mu_{\widetilde{\alpha}_1})+(1+q\mu_{\widetilde{\alpha}_2})\ln(1+q\mu_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\mu_{\widetilde{\alpha}_1}+q\mu_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\mu_{\widetilde{\alpha}_1}+q\mu_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}\Big)^p\Big)^{1/p}+\\
& \Big(\tfrac{1}{|\Psi_{\widetilde{\alpha}_1}|}\sum_{\nu_{\widetilde{\alpha}_1}\in\Psi_{\widetilde{\alpha}_1}}\tfrac{1}{T}\Big(\Big\{\min_{\nu_{\widetilde{\alpha}_2}\in\Psi_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\nu_{\widetilde{\alpha}_1})\ln(1+q\nu_{\widetilde{\alpha}_1})+(1+q\nu_{\widetilde{\alpha}_2})\ln(1+q\nu_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\nu_{\widetilde{\alpha}_1}+q\nu_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\nu_{\widetilde{\alpha}_1}+q\nu_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}\Big)^p\Big)^{1/p}\\
& \Big(\tfrac{1}{|\Pi_{\widetilde{\alpha}_1}|}\sum_{\pi_{\widetilde{\alpha}_1}\in\Pi_{\widetilde{\alpha}_1}}\tfrac{1}{T}\Big(\Big\{\min_{\pi_{\widetilde{\alpha}_2}\in\Pi_{\widetilde{\alpha}_2}}\Big\{\tfrac{(1+q\pi_{\widetilde{\alpha}_1})\ln(1+q\pi_{\widetilde{\alpha}_1})+(1+q\pi_{\widetilde{\alpha}_2})\ln(1+q\pi_{\widetilde{\alpha}_2})}{2}-\\
& \tfrac{2+q\pi_{\widetilde{\alpha}_1}+q\pi_{\widetilde{\alpha}_2}}{2}\ln\Big(\tfrac{2+q\pi_{\widetilde{\alpha}_1}+q\pi_{\widetilde{\alpha}_2}}{2}\Big)\Big\}\Big\}\Big)^p\Big)^{1/p}, p\geqslant 1.
\end{aligned} \tag{15}
$$

In the following, we propose the axiomatic definition of cross-entropy measure for linguistic hesitant intuitionistic fuzzy information as follows, motivated by Xu and Xia [48], Peng et al. [26], etc.

Definition 3.1. Let $\check{h}_1,\check{h}_2\in H$, $CE':H\times H\to R^+$, then cross-entropy $CE'(\check{h}_1,\check{h}_2)$ of $\check{h}_1$ and $\check{h}_2$ should satisfy the following conditions:

(1) $CE'(\check{h}_1,\check{h}_2)\geqslant 0,\forall\,\check{h}_1,\check{h}_2\in H$,

(2) $CE'(\check{h}_1,\check{h}_2)=0$, if $\check{h}_1=\check{h}_2$,

(3) $CE'(\check{h}_1^c,\check{h}_2^c)=CE'(\check{h}_1,\check{h}_2),\forall\,\check{h}_1,\check{h}_2\in H$.

Here $\check{h}_i^c=\big\{\big(s_{(g-\theta_i)},\ \ lh(s_{\theta_i})^c\big)\big\}, lh(s_{\theta_i})^c=\{(\nu_i^{(k)},\mu_i^{(k)})\},(\mu_i^{(k)},\nu_i^{(k)})\in lh(s_{\theta_i}),i=1,2.$

We develop several cross-entropy measures for linguistic hesitant intuitionistic fuzzy information, motivated by Vlachos and Sergiadis[49], Mao et al.[43], Xu and Xia[48], Peng et al.[26], etc.

Let $\check{h}_1=\{a_{1i}\}=\{(s_{\theta_{1i}},lh(s_{\theta_{1i}}))\},(\mu_{1i}^{(k)},\nu_{1i}^{(k)})\in lh(s_{\theta_{1i}})$, $\check{h}_2=\{a_{2j}\}=\{(s_{\theta_{2j}},lh(s_{\theta_{2j}}))\}$, $(\mu_{2j}^{(k)},\nu_{2j}^{(k)})\in lh(s_{\theta_{2j}})$. Then two cross entropy measures can be defined as follows by considering linguistic variables, intuitionistic fuzzy memberships. Since each LHIFE has several linguistic terms and intuitionistic fuzzy memberships, we can define cross-entropy measures by considering the maximum value and the average value.

$$
\begin{aligned}
& CE'_1(\check{h}_1,\check{h}_2)\\
=\; & \max_{a_{ki}\in\check{h}_k}\Big\{\tfrac{\theta_{1\sigma(i)}}{g}\log_2\tfrac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}+\\
& (1-\tfrac{\theta_{1\sigma(i)}}{g})\log_2\tfrac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\Big\}+\\
& \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big\{\mu_{1\sigma(j)}^{(k)}\log_2\tfrac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}+\\
& \nu_{1\sigma(j)}^{(k)}\log_2\tfrac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\Big\}.
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
& CE'_2(\check{h}_1,\check{h}_2)\\
=\; & \tfrac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\Big(\Big(\tfrac{\theta_{1\sigma(i)}}{g}\log_2\tfrac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}+\\
& (1-\tfrac{\theta_{1\sigma(i)}}{g})\log_2\tfrac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\Big)\\
& \tfrac{1}{|lh(s_{(\theta_{ki})})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big(\mu_{1\sigma(j)}^{(k)}\log_2\tfrac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}\\
& +\nu_{1\sigma(j)}^{(k)}\log_2\tfrac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\Big)\Big).
\end{aligned} \tag{17}
$$

Similarly, other cross-entropy measures for linguistic hesitant intuitionistic fuzzy elements can be defined as follows. By using the generalized mean operator, we can get the cross-entropy measures $CE_3'(\check{h}_1,\check{h}_2)$ and $CE_4'(\check{h}_1,\check{h}_2)$.

$$\begin{aligned} & CE_3'(\check{h}_1,\check{h}_2) \\ = & \Bigg(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\Bigg(\left(\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}\right)^p+ \\ & \left((1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\right)^p \\ & +\frac{1}{|lh(s_{\theta_{ki}})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Bigg(\left(\mu_{1\sigma(j)}^{(k)}\log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}\right)^p \\ & +\left(\nu_{1\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\right)^p\Bigg)\Bigg)^{1/p},\quad p\geqslant 1. \end{aligned} \tag{18}$$

$$\begin{aligned} & CE_4'(\check{h}_1,\check{h}_2) \\ = & \left(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\left(\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}\right)^p\right)^{1/p}+ \\ & \left(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\left((1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\right)^p\right)^{1/p} \\ & +\Bigg(\frac{1}{|a_{1i}||lh(s_{(\theta_{ki})})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Bigg(\mu_{i\sigma(j)}^{(k)}* \\ & \log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}\Bigg)^p\Bigg)^{1/p}+\Bigg(\frac{1}{|a_{1i}||lh(s_{\theta_{ki}})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})} \\ & \left(\nu_{i\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\right)^p\Bigg)^{1/p},\quad p\geqslant 1. \end{aligned} \tag{19}$$

Ye[38] developed the cross-entropy measure for intuitionistic fuzzy value by considering the complementary set of the intuitionistic fuzzy set to get

$$\begin{aligned} & CE(A',B') \\ = & \sum_{i=1}^n\Bigg(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}* \\ & \log_2\frac{\mu_A(x_i)+1-\nu_A(x_i)}{\frac{1}{2}[(\mu_A(x_i)+1-\nu_A(x_i))+\mu_B(x_i)+1-\nu_B(x_i)]}+ \\ & \frac{1-\mu_A(x_i)+\nu_A(x_i)}{2}* \\ & \log_2\frac{1-\mu_A(x_i)+\nu_A(x_i)}{\frac{1}{2}[(1-\mu_A(x_i)+\nu_A(x_i))+1-\mu_B(x_i)+\nu_B(x_i)]}\Bigg). \end{aligned}$$

We extend the cross-entropy measure $CE(A',B')$ in Ye[38] to accommodate linguistic hesitant intuitionitic fuzzy values to get cross-entropy measures $CE_5'(\check{h}_1,\check{h}_2)$ and $CE_6'(\check{h}_1,\check{h}_2)$

$$\begin{aligned} & CE_5'(\check{h}_1,\check{h}_2) \\ = & \max_{a_{ki}\in\check{h}_k}\Bigg\{\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}+ \\ & (1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\Bigg\}+ \\ & \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Bigg\{\frac{1+\mu_{1\sigma(j)}^{(k)}-\nu_{1\sigma(j)}^{(k)}}{2}* \\ & \log_2\frac{2(1+\mu_{1\sigma(j)}^{(k)}-\nu_{1\sigma(j)}^{(k)})}{2+\mu_{1\sigma(j)}^{(k)}-\nu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}-\nu_{2\sigma(j)}^{(k)}}+ \\ & \frac{1-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}}{2}\log_2\frac{2(1-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)})}{2-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}-\mu_{2\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\Bigg\}. \end{aligned} \tag{20}$$

$$\begin{aligned} CE_6'(\check{h}_1,\check{h}_2) = & \left(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\left(\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}\right)^p\right)^{1/p}+\left(\frac{1}{|a_{1i}|}\sum_{a_{1i}\in\check{h}_1}\left((1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\right)^p\right)^{1/p} \\ & \left(\frac{1}{|a_{1i}||lh(s_{(\theta_{ki})})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\left(\frac{\mu_{1\sigma(j)}^{(k)}+1-\nu_{1\sigma(j)}^{(k)}}{2}\log_2\frac{2(\mu_{1\sigma(j)}^{(k)}+1-\nu_{1\sigma(j)}^{(k)})}{2+\mu_{1\sigma(j)}^{(k)}-\nu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}-\nu_{2\sigma(j)}^{(k)}}\right)^p\right)^{1/p}+ \\ & \left(\frac{1}{|a_{1i}||lh(s_{\theta_{ki}})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\left(\frac{1-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}}{2}\log_2\frac{2(1-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)})}{2-\mu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}-\mu_{2\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\right)^p\right)^{1/p},\ p\geqslant 1 \end{aligned} \tag{21}$$

By using the concave-up function $f(x)=x^p$, we extend the cross-entropy $CE_6(\widetilde{A},\widetilde{B})$ developed by Xu and Xia[48] to linguistic hesitant intuitionistic fuzzy environment to get $CE_7'(\check{h}_1,\check{h}_2)$. By using the concave-up function $f(x)=x^p$ and the generalized mean operator, we can further get cross-entropy measure $CE_8'(\check{h}_1,\check{h}_2)$.

$$
\begin{aligned}
CE_7'(\check{h}_1,\check{h}_2) \;=\;& \frac{1}{1-2^{1-p}}\Bigg(\max_{a_{ki}\in\check{h}_k}\left\{ \frac{(\theta_{1\sigma(i)}/g)^p+(\theta_{2\sigma(i)}/g)^p}{2} - \left(\frac{\theta_{1\sigma(i)}/g+\theta_{2\sigma(i)}/g}{2}\right)^p \right\} + \\
& \max_{a_{ki}\in\check{h}_k}\left\{ \frac{(1-\theta_{1\sigma(i)}/g)^p+(1-\theta_{2\sigma(i)}/g)^p}{2} - \left(1-\frac{\theta_{1\sigma(i)}/g+\theta_{2\sigma(i)}/g}{2}\right)^p \right\} \\
& + \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})} \left\{ \frac{(\mu_{1\sigma(j)}^{(k)})^p+(\mu_{2\sigma(j)}^{(k)})^p}{2} - \left(\frac{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}{2}\right)^p \right\} + \\
& \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})} \left\{ \frac{(\nu_{1\sigma(j)}^{(k)})^p+(\nu_{2\sigma(j)}^{(k)})^p}{2} - \left(\frac{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}{2}\right)^p \right\}\Bigg),\; 2\geqslant p>1;
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
& CE_8'(\check{h}_1,\check{h}_2) \\
=\;& \left(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\frac{1}{1-2^{1-q}}\left(\frac{(\theta_{1\sigma(j)}/g)^q+(\theta_{2\sigma(j)}/g)^q}{2}-\left(\frac{\theta_{1\sigma(j)}/g+\theta_{2\sigma(j)}/g}{2}\right)^q\right)^p\right)^{1/p}+ \\
& \left(\frac{1}{|a_{1i}|}\sum_{a_{ki}\in\check{h}_k}\frac{1}{1-2^{1-q}}\left(\frac{(1-\theta_{1\sigma(j)}/g)^q+(1-\theta_{2\sigma(j)}/g)^q}{2}-\left(1-\frac{\theta_{1\sigma(j)}/g+\theta_{2\sigma(j)}/g}{2}\right)^q\right)^p\right)^{1/p} \\
& +\left(\frac{1}{|a_{1i}||lh(s_{(\theta_{ki})})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\frac{1}{1-2^{1-q}}\left(\frac{(\mu_{1\sigma(j)}^{(k)})^q+(\mu_{2\sigma(j)}^{(k)})^q}{2}-\left(\frac{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}{2}\right)^q\right)^p\right)^{1/p} \\
& +\left(\frac{1}{|a_{1i}||lh(s_{(\theta_{ki})})|}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\frac{1}{1-2^{1-q}}\left(\frac{(\nu_{1\sigma(j)}^{(k)})^q+(\nu_{2\sigma(j)}^{(k)})^q}{2}-\left(\frac{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}{2}\right)^q\right)^p\right)^{1/p}, \\
& 2\geqslant q>1, p\geqslant 1.
\end{aligned} \tag{23}
$$

By using the concave-up function $f(x)=(1+qx)ln(1+qx)$, we can extend the cross-entropy measures $CE_7(\widetilde{\alpha}_1,\widetilde{\alpha}_2)$ and $CE_8(\widetilde{\alpha}_1,\widetilde{\alpha}_2)$ for intuitionistic hesitant fuzzy values to linguistic hesitant intuitionistic fuzzy environment to get cross-entropy measures $CE_9'(\check{h}_1,\check{h}_2)$ and $CE_{10}'(\check{h}_1,\check{h}_2)$.

$$
\begin{aligned}
& CE_9'(\check{h}_1,\check{h}_2) \\
=\;& \frac{1}{T}\Bigg(\max_{a_{ki}\in\check{h}_k}\Bigg\{\frac{(1+q(\theta_{1\sigma(i)}/g))\ln(1+q(\theta_{1\sigma(i)}/g))+(1+q(\theta_{2\sigma(i)}/g))\ln(1+q(\theta_{2\sigma(i)}/g))}{2}- \\
& \frac{2+q(\theta_{1\sigma(i)}/g)+q(\theta_{2\sigma(i)}/g)}{2}\ln\frac{2+q(\theta_{1\sigma(i)}/g)+q(\theta_{2\sigma(i)}/g)}{2}\Bigg\}+ \\
& \max_{a_{ki}\in\check{h}_k}\Bigg\{\frac{(1+q(1-\theta_{1\sigma(i)}/g))\ln(1+q(1-\theta_{1\sigma(i)}/g))+(1+q(1-\theta_{2\sigma(i)}/g))\ln(1+q(1-\theta_{2\sigma(i)}/g))}{2}- \\
& \frac{2+q(1-\theta_{1\sigma(i)}/g)+q(1-\theta_{2\sigma(i)}/g)}{2}\ln\frac{2+q(1-\theta_{1\sigma(i)}/g)+q(1-\theta_{2\sigma(i)}/g)}{2}\Bigg\}+ \\
& \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Bigg\{\frac{(1+q\mu_{1\sigma(j)}^{(k)})\ln(1+q\mu_{1\sigma(j)}^{(k)})+(1+q\mu_{2\sigma(j)}^{(k)})\ln(1+q\mu_{2\sigma(j)}^{(k)})}{2}- \\
& \frac{2+q\mu_{1\sigma(j)}^{(k)}+q\mu_{2\sigma(j)}^{(k)}}{2}\ln\frac{2+q\mu_{1\sigma(j)}^{(k)}+q\mu_{2\sigma(j)}^{(k)}}{2}\Bigg\}+ \\
& \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Bigg\{\frac{(1+q\nu_{1\sigma(j)}^{(k)})\ln(1+q\nu_{1\sigma(j)}^{(k)})+(1+q\nu_{2\sigma(j)}^{(k)})\ln(1+q\nu_{2\sigma(j)}^{(k)})}{2}- \\
& \frac{2+q\nu_{1\sigma(j)}^{(k)}+q\nu_{2\sigma(j)}^{(k)}}{2}\ln\frac{2+q\nu_{1\sigma(j)}^{(k)}+q\nu_{2\sigma(j)}^{(k)}}{2}\Bigg\}\Bigg),\quad q>0.
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
& CE'_{10}(\check{h}_1,\check{h}_2) \\
= & \Big(\frac{1}{|a_{1i}|T}\sum_{a_{ki}\in\check{h}_k}\Big(\frac{(1+q(\theta_{1\sigma(i)}/g))\ln(1+q(\theta_{1\sigma(i)}/g))+(1+q(\theta_{2\sigma(i)}/g))\ln(1+q(\theta_{2\sigma(i)}/g))}{2} \\
& -\frac{2+q(\theta_{1\sigma(i)}/g)+q(\theta_{2\sigma(i)}/g)}{2}\ln\frac{2+q(\theta_{1\sigma(i)}/g)+q(\theta_{2\sigma(i)}/g)}{2}\Big)^p\Big)^{1/p}+ \\
& \Big(\frac{1}{|a_{1i}|T}\sum_{a_{ki}\in\check{h}_k}\Big(\frac{(1+q(1-\theta_{1\sigma(i)}/g))\ln(1+q(1-\theta_{1\sigma(i)}/g))+(1+q(1-\theta_{2\sigma(i)}/g))\ln(1+q(1-\theta_{2\sigma(i)}/g))}{2} \\
& -\frac{2+q(\theta_{1\sigma(i)}/g)+q(1-\theta_{2\sigma(i)}/g)}{2}\ln\frac{2+q(1-\theta_{1\sigma(i)}/g)+q(1-\theta_{2\sigma(i)}/g)}{2}\Big)^p\Big)^{1/p}+ \\
& \Big(\frac{1}{|a_{1i}||lh(s_{\theta_{ki}})|T}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big(\frac{(1+q\mu_{1\sigma(j)}^{(k)})\ln(1+q\mu_{1\sigma(j)}^{(k)})+(1+q\mu_{2\sigma(j)}^{(k)})\ln(1+q\mu_{2\sigma(j)}^{(k)})}{2}- \\
& \frac{2+q\mu_{1\sigma(j)}^{(k)}+q\mu_{2\sigma(j)}^{(k)}}{2}\ln\frac{2+q\mu_{1\sigma(j)}^{(k)}+q\mu_{2\sigma(j)}^{(k)}}{2}\Big)^p\Big)^{1/p}+ \\
& \Big(\frac{1}{|a_{1i}||lh(s_{(\theta_{ki})})|T}\sum_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big(\frac{(1+q\nu_{1\sigma(j)}^{(k)})\ln(1+q\nu_{1\sigma(j)}^{(k)})+(1+q\nu_{2\sigma(j)}^{(k)})\ln(1+q\nu_{2\sigma(j)}^{(k)})}{2}- \\
& \frac{2+q\nu_{1\sigma(j)}^{(k)}+q\nu_{2\sigma(j)}^{(k)}}{2}\ln\frac{2+q\nu_{1\sigma(j)}^{(k)}+q\nu_{2\sigma(j)}^{(k)}}{2}\Big)^p\Big)^{1/p}.
\end{aligned} \tag{25}
$$

Here $q>0$, $p\geqslant 1$ and $T=(1+q)\ln(1+q)-(2+q)(\ln(2+q)-\ln 2)$.

Theorem 5. The measures defined in Eqs.(16)-(25) are linguistic hesitant intuitionisic fuzzy cross-entropy measures, which satisfy the conditions given in Definition 3.1.

Proof. We prove the Eq.(16) and other equations can be proved similarly.

According to Shannon's inequality, it is clear that $CE'_1(\check{h}_1,\check{h}_2)\geqslant 0$. If $\check{h}_1=\check{h}_2$, then $\forall\ x\in X$, $\widetilde{a}_{1\sigma(j)}=\widetilde{a}_{2\sigma(j)}$, $s_{\theta_{1\sigma(i)}}=s_{\theta_{2\sigma(i)}}$, $(\mu_{1\sigma(j)}^{(k)},\nu_{1\sigma(j)}^{(k)})=(\mu_{2\sigma(j)}^{(k)},\nu_{2\sigma(j)}^{(k)})$.

$$
\begin{aligned}
CE'_1(\check{h}_1,\check{h}_2) &= \max_{a_{ki}\in\check{h}_k}\Big\{\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{2\sigma(i)}}+(1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{2\sigma(i)}}\Big\}+ \\
&\quad \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big\{\mu_{1\sigma(j)}^{(k)}\log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}+\nu_{1\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}\Big\} \\
&= \max_{a_{ki}\in\check{h}_k}\Big\{\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{1\sigma(i)}}+(1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{1\sigma(i)}}\Big\}+ \\
&\quad \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\Big\{\mu_{1\sigma(j)}^{(k)}\log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{1\sigma(j)}^{(k)}}+\nu_{1\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}}\Big\} \\
&= \max_{a_{ki}\in\check{h}_k}\{0\}+\max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\{0\}=0.
\end{aligned}
$$

$$
\begin{aligned}
CE'_1(\check{h}_1^c,\check{h}_2^c) &= \max_{a_{ki}^c\in\check{h}_k^c}\Big\{\frac{g-\theta_{1\sigma(i)}}{g}\log_2\frac{2(g-\theta_{1\sigma(i)})}{(g-\theta_{1\sigma(i)})+(g-\theta_{2\sigma(i)})}+(1-\frac{g-\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-(g-\theta_{1\sigma(i)}))}{2g-(g-\theta_{1\sigma(i)})-(g-\theta_{2\sigma(i)})}\Big\} \\
&\quad +\max_{(\nu_{ij}^{(k)},\mu_{ij}^{(k)})\in lh(s_{\theta_{ij}})^c}\Big\{\nu_{1\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{2\sigma(j)}^{(k)}}+\mu_{1\sigma(j)}^{(k)}\log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{2\sigma(j)}^{(k)}}\Big\},
\end{aligned}
$$

$$\begin{aligned} CE'_1(\check{h}_1^c,\check{h}_2^c) &= \max_{a_{ki}\in\check{h}_k}\left\{(1-\frac{\theta_{1\sigma(i)}}{g})\log_2\frac{2(g-\theta_{1\sigma(i)})}{2g-\theta_{1\sigma(i)}-\theta_{1\sigma(i)}}+\frac{\theta_{1\sigma(i)}}{g}\log_2\frac{2\theta_{1\sigma(i)}}{\theta_{1\sigma(i)}+\theta_{1\sigma(i)}}\right\}+ \\ &\quad \max_{(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in lh(s_{\theta_{ij}})}\left\{\mu_{1\sigma(j)}^{(k)}\log_2\frac{2\mu_{1\sigma(j)}^{(k)}}{\mu_{1\sigma(j)}^{(k)}+\mu_{1\sigma(j)}^{(k)}}+\nu_{1\sigma(j)}^{(k)}\log_2\frac{2\nu_{1\sigma(j)}^{(k)}}{\nu_{1\sigma(j)}^{(k)}+\nu_{1\sigma(j)}^{(k)}}\right\} \\ &= CE'_1(\check{h}_1,\check{h}_2). \end{aligned}$$

The proof is thus complete.

Several linguistic hesitant intuitionisic fuzzy cross-entropy measures have been introduced in this paper, each cross-entropy measure has its own characteristics and emphasis. Decision makers can select the cross-entropy measure according to the real needs and his preference.

The proposed cross-entropy measures are the degrees of discrimination of $\check{h}_1$ from $\check{h}_2$. However, $CE'_i(\check{h}_1,\check{h}_2)$ $(i=1,2,...,10)$ are not symmetric with respect to their arguments. Symmetric form of cross-entropy measures for LHIFEs can be got by modifying the Eqs. (16)-(25) as follows:

$$CE^*_i(\check{h}_1,\check{h}_2)=CE'_i(\check{h}_1,\check{h}_2)+CE'_i(\check{h}_2,\check{h}_1). \quad (26)$$

Example: Let $\check{h}_1=\{(s_5,(0.7,0.2),(0.6,0.3)),(s_6,(0.6,0.2),(0.5,0.4)),(s_7,(0.5,0.3),(0.6,0.4))\}$, $\check{h}_2=\{(s_6,(0.8,0.1),(0.7,0.3)),(s_7,(0.6,0.2),(0.5,0.3)),(s_8,(0.5,0.2),(0.5,0.3))\}$ be two patterns and $\check{h}=\{(s_4,\ (0.7,0.1),\ (0.6,0.2)),(s_5,(0.7,0.2),(0.6,0.3)),(s_6,(0.6,0.2),\ (0.6,\ 0.3))\}$ be a sample. $p=q=2$, then the following results can be got:

$CE^*_1(\check{h}_1,\check{h})=0.2636$, $CE^*_1(\check{h}_2,\check{h})=0.3359$,
$CE^*_2(\check{h}_1,\check{h})=0.0480$, $CE^*_2(\check{h}_2,\check{h})=0.0978$,
$CE^*_3(\check{h}_1,\check{h})=0.3152$, $CE^*_3(\check{h}_2,\check{h})=0.4811$,
$CE^*_4(\check{h}_1,\check{h})=0.0571$, $CE^*_4(\check{h}_2,\check{h})=0.1227$,
$CE^*_5(\check{h}_1,\check{h})=0.6137$, $CE^*_5(\check{h}_2,\check{h})=0.8592$,
$CE^*_6(\check{h}_1,\check{h})=0.5824$, $CE^*_6(\check{h}_2,\check{h})=0.7987$,
$CE^*_7(\check{h}_1,\check{h})=0.1867$, $CE^*_7(\check{h}_2,\check{h})=0.2969$,
$CE^*_8(\check{h}_1,\check{h})=0.0394$, $CE^*_8(\check{h}_2,\check{h})=0.0810$,
$CE^*_9(\check{h}_1,\check{h})=0.0840$, $CE^*_9(\check{h}_2,\check{h})=0.1322$,
$CE^*_{10}(\check{h}_1,\check{h})=0.0432$, $CE^*_{10}(\check{h}_2,\check{h})=0.0830$.

From the above results we can see that the sample $\check{h}$ belongs to the pattern $\check{h}_1$ by using all the above cross-entropy measures.

4. New MADM methods based on the cross-entropy measures of LHIFEs

Considering a multiple attribute decision making problem, let $A=\{A_1,A_2,...,A_m\}$ be a finite alternative set and $C=\{C_1,C_2,...,C_n\}$ be a finite attribute set. The decision makers evaluate alternatives with respect to attributes with linguistic terms and intuitionisic fuzzy memberships. If two or more decision makers gave the same LIFE in evaluating the same alternative with respect to some attribute, it is counted only once. The linguistic hesitant intuitionistic fuzzy decision matrix can be got as $\widetilde{D}=(\check{h}_{ij})_{m\times n}$, where $\check{h}_{ij}=\{(s_{\theta_{ij}},lh(s_{\theta_{ij}})|(\mu_{ij}^{(t)},\nu_{ij}^{(t)})\in lh(s_{\theta_{ij}}),t=1,2,...,l_{ij}\}$ is LHIFE. $\{s_{\theta_{ij}}\}$ is the linguistic terms given by experts to evaluate alternative A_i with respect to the attribute C_j, $\mu_{ij}^{(t)}$ indicates the degree of linguistic term $s_{\theta_{ij}}$ satisfying the attribute C_j and $\nu_{ij}^{(t)}$ indicates the degree of linguistic term $s_{\theta_{ij}}$ dissatisfying the attribute C_j, such that $\mu_{ij}^{(t)},\nu_{ij}^{(t)}\in[0,1]$ and $\mu_{ij}^{(t)}+\nu_{ij}^{(t)}\leqslant 1$.

Different LHIFEs may have different number of LIFEs and different LIFEs may have different number of intuitionistic fuzzy memberships. In order to define cross-entropy measures more accurately, we extend LHIFEs according to the risk attitudes of decision makers until all LHIFEs have the same number of LIFEs and all LIFEs have the same number of intuitionistic fuzzy memberships. If the decision maker is risk-seeking, the largest LIFE and largest intuitionistic fuzzy membership can be added; if the decision maker is risk-averse, the smallest LIFE and smallest intuitionistic fuzzy membership can be added; if the decision maker is risk-neutral, the average LIFE and average intuitionistic fuzzy membership can be added.

In some decision making process, information for attribute weight is partly known or unknown completely due to decision time pressure, decision makers' lack of knowledge and expertise, complicated decision problems, etc. Generally, partly known attribute information can be expressed as a subset of the following relations: a weak ranking: $\{w_i \geqslant w_j\}, i \neq j$; a strict ranking: $\{w_i - w_j \geqslant \varepsilon_i (> 0)\}, i \neq j$; a ranking with multiples: $\{w_i \geqslant \alpha_i w_j\}, 0 \leqslant \alpha_i \leqslant 1, i \neq j$; an interval form: $\{\beta_j \leqslant w_j \leqslant \beta_j + \varepsilon_j\}, 0 \leqslant \beta_j < \beta_j + \varepsilon_j \leqslant 1$; a ranking of differences: $\{w_i - w_j \geqslant w_k - w_l\}$, for $i \neq j \neq k \neq l$. For a specific decision problem, attribute weight information can be described as a subset of the above relationships. The corresponding attribute weight information set can be denoted as H. According to information theory, an attribute should be assigned a larger weight if its evaluation values have obvious differences since it plays an important role in the priority procedure. Otherwise, it should be assigned a smaller weight[51]. Since several linguistic hesitant intuitionistic fuzzy cross-entropy measures have been developed, we only choose one cross-entropy measure for the convenience of calculation and analysis and we can calculate similarly if other cross-entropy measures are chosen. Then the deviation value d_j of attribute C_j can be calculated as

$$d_j = \sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj}). \tag{27}$$

The weighted deviation value can be calculated as

$$\begin{aligned} d &= \textstyle\sum_{j=1}^{n} w_j d_j \\ &= \textstyle\sum_{j=1}^{n} w_j (\sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj})) \\ &= \textstyle\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} w_j CE^*(\check{h}_{ij}, \check{h}_{kj}). \end{aligned} \tag{28}$$

A reasonable weight vector $w = (w_1, w_2, ..., w_n)$ should make the weighted deviation value as large as possible to differentiate the problem characteristics more effectively[51]. Thus we set up the following programming model to determine optimal attribute weights if attribute weights are partly known.

$$\begin{array}{ll} \text{(M-1)} & \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} w_j CE^*(\check{h}_{ij}, \check{h}_{kj}) \\ \text{s.t.} & w \in H, \\ & w_j \geqslant 0, j = 1, 2, ..., n, \\ & w_1 + w_2 + ... + w_n = 1. \end{array}$$

The model (M-1) is a linear programming model, which can be solved easily by using many existing methods.

If information for attribute weights is unknown completely, we set up the following model

$$\begin{array}{ll} \text{(M-2)} & \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} w_j CE^*(\check{h}_{ij}, \check{h}_{kj}) \\ \text{s.t.} & \sum_{j=1}^{n} w_j^2 = 1, \\ & w_j \geqslant 0, j = 1, 2, ..., n. \end{array}$$

In order to solve model (M-2), we construct the following Lagrange function:

$$L(W, \lambda) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} w_j CE^*(\check{h}_{ij}, \check{h}_{kj}) + \frac{\lambda}{2} \Big(\sum_{j=1}^{n} w_j^2 - 1 \Big), \tag{29}$$

where λ is the Lagrange multiplier. Calculate the differentiation of Eq. (29) with respect to w_j $(j = 1, 2, ..., n)$ and λ, and set these partial derivatives equal to zeros to get

$$\begin{cases} \frac{\partial L}{\partial w_j} = \sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj}) + \lambda w_j = 0, \\ \frac{\partial L}{\partial \lambda} = \sum_{j=1}^{n} w_j^2 - 1 = 0. \end{cases} \tag{30}$$

By solving Eq.(30), we can get the formula for calculating attribute weights as

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj})}{\sqrt{\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj}) \right)^2}}, \ j = 1, 2, ..., n. \tag{31}$$

Normalize the attribute weights to get

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj})}{\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} CE^*(\check{h}_{ij}, \check{h}_{kj})}, \ j = 1, 2, ..., n. \tag{32}$$

Based on the above analysis, we can present an effective approach to solve the multiple attribute decision making problem with linguistic hesitant intuitionistic fuzzy information.

Algorithm I

Step 1. Construct the decision matrix $\widetilde{D} = (\check{h}_{ij})_{m \times n}$. Multiple decision makers evaluate the alternatives with respect to attributes with linguistic terms and intuitionistic fuzzy memberships. Then linguistic hesitant intuitionistic fuzzy elements are formed as $\check{h}_{ij}$.

Step 2. If attribute weights are known completely, go to Step 3 directly. In order to calculate the cross-entropy measure more accurately, we extend the decision matrix according to the risk attitude of decision makers and the decision matrix $\widetilde{D}' = (\check{h}'_{ij})_{m\times n}$ can be got. If attribute weights are known partly, we can solve model (M-1) to obtain them; if attribute weights are completely unknown, we can calculate them by using Eq.(32).

Step 3. Calculate the collective evaluation values of alternatives by using the decision matrix $\widetilde{D} = (\check{h}_{ij})_{m\times n}$. We can calculate by using the LHIFWA operator, the LHIFWG operator or the GLHIFWA operator.

Step 4. Calculate the scores $S(\check{h}_i)$ $(i = 1,2,...,m)$ and the accuracy degrees $A(\check{h}_i)$ $(i = 1,2,...,m)$ of alternatives' collective evaluation values $\check{h}_i$ $(i = 1,2,...,m)$ by using the score function and accuracy function.

$$\begin{aligned} & S(\check{h}_i) \\ = & \tfrac{1}{|\check{h}_i|}\Big(\sum \tfrac{\theta_i}{g|lh(s_{\theta_i})|}\sum_{(\alpha_i^{(k)},\beta_i^{(k)})\in lh(s_{\theta_i})}(\mu_i^{(k)}-\nu_i^{(k)})\Big), \end{aligned} \tag{33}$$

$$\begin{aligned} & A(\check{h}_i) \\ = & \tfrac{1}{|\check{h}_i|}\Big(\sum \tfrac{\theta_i}{g|lh(s_{\theta_i})|}\sum_{(\alpha_i^{(k)},\beta_i^{(k)})\in lh(s_{\theta_i})}(\mu_i^{(k)}+\nu_i^{(k)})\Big), \end{aligned} \tag{34}$$

where $|\ \check{h}_i\ |$ and $|\ lh(s_{\theta(i)})\ |$ are the cardinalities of $\check{h}_i$ and $lh(s_{\theta_i})$, respectively. $\check{h}_i = \{(s_{\theta_i}, lh(s_{\theta_i}))\}$, $lh(s_{\theta_i}) = \{(\mu_i^{(k)}, \nu_i^{(k)})\}$.

Step 5. Rank $\check{h}_i$ according the method given in Definition 2.6 and rank alternatives accordingly.

In LHIFEs, intuitionistic fuzzy memberships and nonmemberships have been considered besides linguistic evaluation values. Hence, aggregation of linguistic hesitant intuitionistic fuzzy information is more complex than linguistic hesitant fuzzy sets. By calculating the number of basic operations at each step, we can get the worst-case time complexity of our algorithm is $O(m^2nlt)$, where m is the number of alternatives, n is the number of attributes, l is the largest number of LIFEs in LHIFEs, t is the largest number of intuitionistic fuzzy values in LIFEs. Then the complexity of algorithm can tell us the new algorithm is an efficient and practical polynomial-time algorithm for solving multiple attribute decision making problems.

TOPSIS method was developed by Hwang and Yong[52], which is based on the principle that the optimal alternative should have the shortest distance from the positive ideal solution and at the same time have the farthest distance from the negative ideal solution. From the risk viewpoint, decision makers are risk-averse since they choose the alternative which is not only making as much profit as possible, but also avoiding as much risk as possible. In the following, we present a new ranking method based on the cross-entropy measures and the idea of TOPSIS.

Algorithm II

Step 1. As for Algorithm I.

Step 2. As for Algorithm I.

Step 3. Determine the linguistic hesitant intuitionistic fuzzy positive-ideal solution (LHIFPIS) as

$$\check{h}^+ = \{(s_g, \{(1,0),...,(1,0)\})\},$$

and linguistic hesitant intuitionistic fuzzy negative-ideal solution (LHIFNIS) as

$$\check{h}^- = \{(s_1, \{(0,1),...,(1,0)\})\}.$$

Each $\check{h}^+$ and $\check{h}^-$ have the same number of LHIFEs and LIFEs as $\check{h}'_{ij}$.

Step 4. Calculate the symmetric cross-entropy of $\check{h}'_{ij}$ from $\check{h}^+$ and $\check{h}^-$ as G^+_{ij}, G^-_{ij}, respectively.

$$G^+_{ij} = C^*_2(\check{h}'_{ij}, \check{h}^+) = C_2(\check{h}'_{ij}, \check{h}^+) + C_2(\check{h}^+, \check{h}'_{ij}), \tag{35}$$

$$G^-_{ij} = C^*_2(\check{h}'_{ij}, \check{h}^-) = C_2(\check{h}'_{ij}, \check{h}^-) + C_2(\check{h}^-, \check{h}'_{ij}). \tag{36}$$

Step 4. Determine the relative closeness by using the following equation

$$G_{ij} = \frac{G^-_{ij}}{G^+_{ij} + G^-_{ij}},\ i = 1,2,...,m, j = 1,2,...,n. \tag{37}$$

Then the weighted relative closeness coefficients of alternatives can be calculated as follows

$$G_i = \sum_{j=1}^{n} w_j G_{ij},\ i = 1,2,...,m. \tag{38}$$

Step 5. Rank alternatives according to the ranking of G_i $(i = 1,2,...,m)$. The larger the G_i, the better the alternative A_i.

5. Numerical example

A numerical example adapted from Chen and Yang[53] is presented to illustrate efficiency and practical advantages of the proposed procedure.

Suppose that there is an architecture company wanting to select a company to supply an important material, such as cement. Experts from different departments have been invited and they mainly consider the following four attributes: C_1- the price of product, $C_2-$the quality of product, C_3-delivery time, C_4-risk. After pre-evaluation, there are still five alternatives A_i $(i=1,2,...,5)$ left for further evaluation. We use the new algorithms to rank alternatives.

Step 1. The experts evaluate alternatives A_i $(i=1,2,...,5)$ with respect to attributes C_j $(j=1,2,...,4)$ with linguistic terms and intuitionistic fuzzy memberships. The decision matrix is formed as $\widetilde{D}=(\check{h}_{ij})_{4\times 5}$ in Table 1.

Step 2. Assume attribute weights are known completely as $w_1=(0.15,0.20,0.30,0.35)$.

Step 3. Calculate alternatives' collective evaluation values. For example, we calculate $\check{h}_1$ by using the LHIFWA operator as follows

$$\check{h}_1 = \Big\{\big(s_{5.15},\{(0.6206,0.1803),(0.6418,0.1803)\}\big),\big(s_{5.30},\{(0.6331,0.1726),(0.6536,0.1726)\}\big),\big(s_{5.45},\{(0.5864,0.2219),(0.6095,0.2219)\}\big),\big(s_{5.60},\{(0.6000,0.2125),(0.6224,0.2125)\}\big)\Big\}.$$

Similarly, we can calculate other $\check{h}_i(i=2,3,...,5)$.

Step 4. Calculate scores of collective evaluation values $\check{h}_i$ $(i=1,2,...,5)$ to get $S(\check{h}_1)=0.2527, S(\check{h}_2)=0.2493, S(\check{h}_3)=0.2193, S(\check{h}_4)=0.2505, S(\check{h}_5)=0.2601$. Rank $S(\check{h}_i)$ $(i=1,2,...,5)$ to get

$$S(\check{h}_5)>S(\check{h}_1)>S(\check{h}_4)>S(\check{h}_2)>S(\check{h}_3).$$

Step 5. Rank $\check{h}_i$ according to the ranking of $S(\check{h}_i)$ to get

$$\check{h}_5>\check{h}_1>\check{h}_4>\check{h}_2>\check{h}_3.$$

Then alternatives can be ranked accordingly as

$$A_5\succ A_1\succ A_4\succ A_2\succ A_3.$$

The optimal alternative is A_5.

We can use the LHIFWG operator or the GLHIFWA$_\lambda$ operator in Step 3 and other steps are the same as above. Then results can be obtained as shown in Table 3. If attribute weights are partly known or completely unknown, we need to calculate them first. Assume decision makers are risk-averse, then the smallest intuitionistic fuzzy value and LIFE can be added to extend the decision matrix as $\widetilde{D}'$, which is shown in Table 2. Though several cross-entropy measures have been introduced in this paper, we only choose one to calculate attribute weights for space limit. Here we use cross-entropy measures CE_2^* and similar results can be got if other cross-entropy measures are used. If attribute weights are known partly, we can set up the following Model (M-3) to determine them as $w_2=(0.2782,0.3062,0.2279,0.1877)$. If attribute weights are unknown completely, we determine them by using Eq.(33) to get $w_3=(0.30,0.15,0.35,0.20)$. Other steps can be calculated similarly as that of completely known attribute weights method above and results are shown in Table 3. If attribute weight vector w_2 is used, A_2 becomes the optimal alternative and A_5 becomes the sub-optimal alternative in most case. For w_3, A_1 becomes the optimal alternative and A_5 becomes the sub-optimal alternative.

$$\begin{array}{ll}(\text{M-3}) & 1.9957w_1+2.1970w_2+1.6348w_3\\ & +1.3467w_4\\ \text{s.t.} & 0.15\leqslant w_1\leqslant 0.30, 0.10\leqslant w_2\leqslant 0.25,\\ & 0.20\leqslant w_3\leqslant 0.35, 0.10\leqslant w_4\leqslant 020,\\ & 2w_2\leqslant w_3, w_1+w_2+...+w_4=1.\end{array}$$

If Algorithm II is used to rank alternatives, the first two steps are the same as that of Algorithm I. In Step 3, we determine the LHIFPIS $\check{h}^+$ and LHIFNIS $\check{h}^-$ as $\check{h}^+=\{\check{a}^+\}, \check{a}^+=(s_9,\{(1,0),(1,0)\})$, $\check{h}^-=\{\check{a}^-\}, \check{a}^-=(s_1,\{(0,1),(0,1)\})$. In Step 4, we calculate the symmetric cross-entropy of $\check{h}_{ij}$ from $\check{h}^+$ and $\check{h}^-$. For example, we calculate G_{ij}^+ and G_{ij}^- by using Eqs.(35)-(36) if CE_2^* is chosen. The relative closeness coefficients can be calculated by using Eq. (37). For completely known attribute weight vector $w_1=(0.15,0.20,0.30,0.35)$, the weighted relative closeness coefficients can be calculated by using Eq.(38) as $G_1=0.5588, G_2=0.5414, G_3=0.4701, G_4=$

Table 1: Decision matrix $\widetilde{D}$.

	C_1	C_2
A_1	$\{(s_6,(0.5,0.4)),(s_7,(0.6,0.3))\}$	$\{(s_2,(0.6,0.2),(0.7,0.2))\}$
A_2	$\{(s_3,(0.5,0.2),(0.5,0.3))\}$	$\{(s_7,(0.7,0.1)),(s_8,(0.8,0.2))\}$
A_3	$\{(s_4,(0.6,0.1))\}$	$\{(s_5,(0.7,0.3),(0.5,0.4))\}$
A_4	$\{(s_2,(0.6,0.2),(0.5,0.3))\}$	$\{(s_6,(0.8,0.2))\}$
A_5	$\{(s_5,(0.7,0.3)),(s_6,(0.6,0.2))\}$	$\{(s_3,(0.5,0.2),(0.6,0.3))\}$
	C_3	C_4
A_1	$\{(s_7,(0.7,0.1)),(s_8,(0.6,0.2))\}$	$\{(s_5,(0.6,0.2))\}$
A_2	$\{(s_6,(0.6,0.3),(0.7,0.2))\}$	$\{(s_4,(0.7,0.2)),(s_5,(0.5,0.3),(0.6,0.4))\}$
A_3	$\{(s_2,(0.5,0.4)),(s_3,(0.5,0.2))\}$	$\{(s_8,(0.6,0.1),(0.7,0.2))\}$
A_4	$\{(s_3,(0.7,0.2),(0.6,0.3))\}$	$\{(s_6,(0.6,0.2)),(s_7,(0.7,0.1))\}$
A_5	$\{(s_7,(0.8,0.2)),(s_8,(0.6,0.3))\}$	$\{(s_4,(0.5,0.2))\}$

Table 2: Extended decision matrix $\widetilde{D}'$.

	C_1	C_2
A_1	$\{(s_6,(0.5,0.4),(0.5,0.4)),(s_7,(0.6,0.3),(0.5,0.4))\}$	$\{(s_2,(0.6,0.2),(0.7,0.2)),(s_2,(0.5,0.4),(0.5,0.4))\}$
A_2	$\{(s_3,(0.5,0.2),(0.5,0.3)),(s_2,(0.5,0.4),(0.5,0.4))\}$	$\{(s_7,(0.7,0.1),(0.5,0.4)),(s_8,(0.8,0.2),(0.5,0.4))\}$
A_3	$\{(s_4,(0.6,0.1),(0.5,0.4)),(s_2,(0.5,0.4),(0.5,0.4))\}$	$\{(s_5,(0.7,0.3),(0.5,0.4)),(s_2,(0.5,0.4),(0.5,0.4))\}$
A_4	$\{(s_2,(0.6,0.2),(0.5,0.3)),(s_2,(0.5,0.4),(0.5,0.4))\}$	$\{(s_6,(0.8,0.2),(0.5,0.4)),(s_2,(0.5,0.4),(0.5,0.4))\}$
A_5	$\{(s_5,(0.7,0.3),(0.5,0.4)),(s_6,(0.6,0.2),(0.5,0.4))\}$	$\{(s_3,(0.5,0.2),(0.6,0.3)),(s_2,(0.5,0.4),(0.5,0.4))\}$
	C_3	C_4
A_1	$\{(s_7,(0.7,0.1),(0.5,0.4)),(s_8,(0.6,0.2),(0.5,0.4))\}$	$\{(s_5,(0.6,0.2),(0.5,0.4)),(s_2,(0.5,0.4),(0.5,0.4))\}$
A_2	$\{(s_6,(0.6,0.3),(0.7,0.2)),(s_2,(0.5,0.4),(0.5,0.4))\}$	$\{(s_4,(0.7,0.2),(0.5,0.4)),(s_5,(0.5,0.3),(0.6,0.4))\}$
A_3	$\{(s_2,(0.5,0.4),(0.5,0.4)),(s_3,(0.5,0.2),(0.5,0.4))\}$	$\{(s_8,(0.6,0.1),(0.7,0.2)),(s_2,(0.5,0.4),(0.5,0.4))\}$
A_4	$\{(s_3,(0.7,0.2),(0.6,0.3)),(s_2,(0.5,0.4),(0.5,0.4))\}$	$\{(s_6,(0.6,0.2),(0.5,0.4)),(s_7,(0.7,0.1),(0.5,0.4))\}$
A_5	$\{(s_7,(0.8,0.2),(0.5,0.4)),(s_8,(0.6,0.3),(0.5,0.4))\}$	$\{(s_4,(0.5,0.2),(0.5,0.4)),(s_2,(0.5,0.4),(0.5,0.4))\}$

$0.5273, G_5 = 0.5332$. Then we can rank the relative closeness coefficients as $G_1 > G_2 > G_5 > G_4 > G_3$. Alternatives can be ranked accordingly as $A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$ and the optimal alternative is A_1. If other cross-entropy measures are used, we can calculate similarly and results are shown in Table 4, where $p = q = 2$. For attribute weights w_2 and w_3, we can get results as in Table 5 and Table 6, respectively.

In Table 4, it can be seen that A_2 becomes the optimal alternative if cross-entropy measures CE_9^* and CE_{10}^* are used, which is quite different from other results. However, in most cases, the ranking is $A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$ and A_1 becomes the best alternative and A_3 becomes the worst alternative. The subtle ranking differences are due to the different information fusion mechanisms. In Table 5, A_1 becomes the optimal alternative if CE_1^*, CE_2^*, CE_4^*, CE_7^* are used and A_2 becomes the best alternative for other cross-entropy measures. A_3 is still the worst alternative. In Table 6, we can get the same ranking $A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$ for all the cross-entropy measures.

Since different cross-entropy measures may produce different ranking results and each cross-entropy has its own characteristics and emphasis, different cross-entropy measures can provide different views of the decision problem. Decision makers can choose the corresponding cross-entropy measure according to real needs, decision makers' preference and interests.

In order to illustrate practical advantages of the new method, we compare it with the method of Peng et al.[26]. In Peng et al.'s method, only hesitant intuitionistic fuzzy values are considered. If linguistic terms are omitted, CE_i^* $(i = 1, 2, ..., 10)$ reduce to

Table 3: The results of different aggregation operators with different attribute weights.

		$S(\check{h}_1)$	$S(\check{h}_2)$	$S(\check{h}_3)$	$S(\check{h}_4)$	$S(\check{h}_5)$	Rankings
w_1	LHIFWA	0.2527	0.2493	0.2193	0.2505	0.2126	$A_1 \succ A_4 \succ A_2 \succ A_3 \succ A_5$
	LHIFWG	0.2269	0.2285	0.1901	0.2193	0.1884	$A_2 \succ A_1 \succ A_4 \succ A_3 \succ A_5$
	GLHIFWA$_2$	0.2709	0.2626	0.2445	0.2724	0.2283	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$
w_2	LHIFWA	0.2258	0.2599	0.2018	0.2278	0.2057	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
	LHIFWG	0.1949	0.2297	0.1818	0.1922	0.1838	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
	GLHIFWA$_2$	0.2500	0.2791	0.2212	0.2525	0.2207	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
w_3	LHIFWA	0.2630	0.2215	0.1876	0.1887	0.2442	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$
	LHIFWG	0.2390	0.1987	0.1654	0.1614	0.2201	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
	GLHIFWA$_2$	0.2787	0.2363	0.2100	0.2118	0.2582	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

Table 4: The results with known attribute weight vector w_1.

	G_1	G_2	G_3	G_4	G_5	Rankings	Best alternative
CE_1^*	0.5588	0.5414	0.4701	0.5273	0.5332	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_2^*	0.6048	0.5943	0.4865	0.5395	0.5698	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_3^*	0.5410	0.5349	0.4696	0.5133	0.5241	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_4^*	0.5808	0.5670	0.4740	0.5465	0.5439	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$	A_1
CE_5^*	0.5321	0.5295	0.4633	0.4990	0.5162	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_6^*	0.5264	0.5213	0.4515	0.4913	0.5114	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_7^*	0.6046	0.5895	0.4876	0.5670	0.5751	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_8^*	0.5512	0.5470	0.4763	0.5153	0.5341	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_9^*	0.4120	0.4171	0.3913	0.4009	0.3940	$A_2 \succ A_1 \succ A_4 \succ A_5 \succ A_3$	A_2
CE_{10}^*	0.3854	0.3874	0.3713	0.3676	0.3644	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	A_2

intuitionistic hesitant fuzzy cross-entropy measures $CE_i^{\dagger}$ $(i = 1,2,...,10)$. Extend hesitant intuitionistic fuzzy evaluation elements according to the risk attitude of decision makers. Assume decision makers are risk-averse and the minimum intuitionistic fuzzy value is added until all the hesitant intuitionisic fuzzy elements have the same number of intuitionisic fuzzy values and the extended decision matrix $D = (h'_{ij})_{5\times 4}$ is formed. Determine the hesitant intuitionistic fuzzy positive ideal solution h^+ and the hesitant intuitionistic fuzzy negative ideal solution h^- as $h^+ = \{a^+\}, a^+ = ((1,0),(1,0)), h^- = \{a^-\}, a^- = ((0,1),(0,1))$. Calculate the symmetric cross-entropy measures of h'_{ij} from h^+ and h^- as G^+_{ij}, G^-_{ij}, respectively. Assume weight vector of attributes is also $w_1 = (0.15, 0.20, 0.30, 0.35)$ to facilitate comparison. We can calculate weighted relative closeness coefficients of alternatives by using Eq.(38) and results are shown in Table 7. From the results we can see different ranking results can be got in the proposed method and Peng et al.[26] method. In the proposed method, A_1 is the optimal alternative in most case and A_2 is optimal alternative in CE_9^* and CE_{10}^*. In Peng et al.[26] method, A_2, A_3 and A_4 become the optimal alternative for different cross-entropy measures. Different ranking results due to different decision information. Comparing evaluation information in Peng el al.'s method with that in the proposed algorithm, linguistic terms have been omitted in Peng et al.'s method. If all decision makers use the same linguistic term in evaluation in the proposed method, we can got the same ranking results in the two methods. Since different linguistic terms can be used in the proposed method, more information has been used and more accurate evaluation values can be got. Different evaluation information has been used in two methods and different results are reasonable. We further compare it with the method of Yang et al.[54]. In Yang et al.'s method, each linguistic term only has one intuitionistic fuzzy membership. Hence we first aggregate the intuition-

Table 5: The results with completely unknown attribute weight vector w_2.

	G_1	G_2	G_3	G_4	G_5	Rankings	Best alternative
CE_1^*	0.5572	0.5480	0.4558	0.4877	0.5339	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_2^*	0.4756	0.4613	0.3137	0.3603	0.4364	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_3^*	0.5392	0.5400	0.4575	0.4828	0.5239	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$	A_2
CE_4^*	0.5819	0.5689	0.4566	0.5012	0.5419	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_5^*	0.5267	0.5339	0.4492	0.4656	0.5149	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$	A_2
CE_6^*	0.5442	0.5497	0.4629	0.4824	0.5321	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$	A_2
CE_7^*	0.6078	0.5929	0.4696	0.5169	0.5798	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$	A_1
CE_8^*	0.5463	0.5520	0.4615	0.4796	0.5327	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$	A_2
CE_9^*	0.3955	0.4178	0.3797	0.3946	0.3889	$A_2 \succ A_1 \succ A_4 \succ A_5 \succ A_3$	A_2
CE_{10}^*	0.3727	0.3902	0.3531	0.3576	0.3586	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$	A_2

Table 6: The results with partly known attribute weight vector w_3.

	G_1	G_2	G_3	G_4	G_5	Rankings	Best alternative
CE_1^*	0.6049	0.5051	0.4501	0.4769	0.5807	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_2^*	0.5457	0.3975	0.3085	0.3498	0.5026	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_3^*	0.5764	0.5076	0.4501	0.4742	0.5601	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_4^*	0.6360	0.5213	0.4482	0.4876	0.6005	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_5^*	0.5691	0.5010	0.4418	0.4546	0.5533	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_6^*	0.5843	0.5139	0.4536	0.4703	0.5686	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_7^*	0.6660	0.5408	0.4617	0.5033	0.6380	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_8^*	0.5914	0.5160	0.4536	0.4684	0.5741	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_9^*	0.4158	0.4029	0.3623	0.3789	0.4099	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1
CE_{10}^*	0.3939	0.3774	0.3374	0.3425	0.3801	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	A_1

istic fuzzy memberships into a collective one by using intuitionistic fuzzy averaging operator (IFA) if a linguistic term has several intuitionistic fuzzy memberships. $\mathrm{IFA}(\alpha_1,\alpha_2,...,\alpha_n) = \sum_{j=1}^{n} \frac{1}{n}\alpha_j = (1 - \prod_{j=1}^{n}(1-\mu_j^{(k)})^{\frac{1}{n}}, \prod_{j=1}^{n}(\nu_j)^{\frac{1}{n}})$, $\alpha_j = (\mu_j,\nu_j)$. The decision matrix degenerates to $\widehat{D} = (\widehat{h}_{ij})_{5\times 4}$. Then we aggregate the evaluation values by using the hesitant intuitionistic fuzzy linguistic weighted averaging (HIFLWA) operator, $\mathrm{HIFLWA}_w(\widehat{h}_1,\widehat{h}_2,...,\widehat{h}_n) = \sum_{j=1}^{n} w_j\widehat{h}_j = \bigcup_{a_i\in\widehat{h}_i} \{(s_{(\sum_{j=1}^{n} w_j\theta_j)},(1 - \prod_{j=1}^{n}(1 - \mu_j^{(k)})^{w_j}, \prod_{j=1}^{n}(\nu_j^{(k)})^{w_j}))\}$, the hesitant intuitionistic fuzzy linguistic weighted geometric (HIFLWG) operator $\mathrm{HIFLWG}_w(\widehat{h}_1,\widehat{h}_2,...,\widehat{h}_n) = \prod_{j=1}^{n}(\widehat{h}_j)^{w_j} = \bigcup_{a_i\in\widehat{h}_i} \{(s_{(\prod_{j=1}^{n}(\theta_j)^{w_j})},(\prod_{j=1}^{n}(\mu_j^{(k)})^{w_j}, 1 - \prod_{j=1}^{n}(1 - \nu_j^{(k)})^{w_j}))\}$, or the generalized hesitant intuitionistic fuzzy linguistic weighted averaging (GHIFLWA) operator $\mathrm{GHIFLWA}_w(\widehat{h}_1,\widehat{h}_2,...,\widehat{h}_n) = (\sum_{j=1}^{n} w_j(\widehat{h}_j)^{\lambda})^{1/\lambda} = \bigcup_{a_i\in\widehat{h}_i} \{(s_{(\sum_{j=1}^{n} w_j(\theta_j)^{\lambda})^{1/\lambda}},((1 - \prod_{j=1}^{n}(1-(\mu_j^{(k)})^{\lambda})^{w_j})^{1/\lambda}, 1-(1-\prod_{j=1}^{n}(1-(1-\nu_j^{(k)})^{\lambda})^{w_j})^{1/\lambda}))\}$. The weight vector of attributes is also taken as $w_1 = (0.15, 0.20, 0.30, 0.35)$ to facilitate comparison. The results are shown in Table 8. From the results we can see that we can get similar ranking results in the proposed method and Yang et al.'s method[54]. There is little difference in values to be aggregated since different intuitionistic fuzzy values in the proposed method are replaced by the average one. For space limit, we only present a simple example to illustrate the new algorithm. The results for other large-scale complex decision problems may have a greater difference. Since different intuitionistic fuzzy values can be used to model hesitation and uncertainty, the proposed method is more flexible and accurate.

Table 7: The results of Peng et al.[29] method with known attribute weights w_1.

	G_1	G_2	G_3	G_4	G_5	Rankings	Best alternative
$CE_1^\dagger$	0.6797	0.6785	0.6763	0.6955	0.6685	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_4$	A_4
$CE_2^\dagger$	0.6654	0.6799	0.6579	0.6803	0.6496	$A_4 \succ A_2 \succ A_1 \succ A_3 \succ A_5$	A_4
$CE_3^\dagger$	0.6079	0.6222	0.6008	0.6196	0.6009	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$	A_2
$CE_4^\dagger$	0.6636	0.6551	0.6660	0.6637	0.6111	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$	A_3
$CE_5^\dagger$	0.5600	0.5732	0.5541	0.5719	0.5538	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$	A_2
$CE_6^\dagger$	0.6068	0.6204	0.6007	0.6180	0.6000	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$	A_2
$CE_7^\dagger$	0.7206	0.7334	0.7517	0.7276	0.7376	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
$CE_8^\dagger$	0.6043	0.6272	0.6196	0.6271	0.6128	$A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$	A_2
$CE_9^\dagger$	0.7280	0.7380	0.7284	0.7455	0.7205	$A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$	A_4
$CE_{10}^\dagger$	0.7046	0.7242	0.6949	0.7221	0.6922	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$	A_2

Table 8: The results of Yang et al.[68] with weights w_1.

	$S(\widehat{h_1})$	$S(\widehat{h_2})$	$S(\widehat{h_3})$	$S(\widehat{h_4})$	$S(\widehat{h_5})$	Rankings
HIFLWA	0.2528	0.2499	0.2208	0.2420	0.2128	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$
HIFLWG	0.2275	0.2302	0.1936	0.2021	0.1890	$A_2 \succ A_1 \succ A_4 \succ A_3 \succ A_5$
GHIFLWA$_2$	0.2709	0.2628	0.2454	0.2650	0.2282	$A_1 \succ A_4 \succ A_2 \succ A_3 \succ A_5$

From the above analysis we can see the proposed approaches have the following advantages. First, LHIFEs have been used to evaluate alternatives, which are more flexible since each LHIFE has several linguistic evaluation values and each linguistic evaluation value has several intuitionistic fuzzy memberships. The inherent fuzzy thought of the decision makers have been retained, which can guarantee accuracy of final results. Second, the cross-entropy measures are very important in decision making and we have found few study based on the linguistic hesitant intuitionistic fuzzy information. The new proposed cross-entropy measures can include the advantages of intuitionistic fuzzy cross-entropy measures and hesitant fuzzy cross-entropy measures. Finally, the proposed approaches can provide useful and flexible way to deal with multiple attribute decision making problem with different attribute weight situations including attribute weights partly known, completely known and completely unknown.

6. Conclusions

In this paper, some linguistic hesitant intuitionistic fuzzy cross-entropy measures have been proposed, which have the advantages of the intuitionistic fuzzy cross-entropy measures and hesitant fuzzy cross-entropy measures. We first introduce some aggregation operators including LHIFWA operator, LHIFWG operator and the GLHIFWA operator. Then we propose several linguistic hesitant intuitionistic fuzzy cross-entropy measures. The properties of new cross-entropy measures have been studied. Two new multiple attribute decision making methods have been proposed based on the proposed cross-entropy measures, in which attribute values are given as linguistic hesitant intuitionistic fuzzy elements. The supplier selection problem has been presented to illustrate feasibility and practical advantages of the new methods. The prominent feature of the new methods is that they can provide a flexible and useful way to deal with decision making problems within linguistic hesitant intuitionisic fuzzy environment.

Further improvements of our algorithm might include the application of our new method to more complex multiple attribute decision making problem in reality, such as the personnel selection, the product selection, and the environment evaluation, etc.

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