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Soft points, *s*-relations and soft rough approximate operations

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Abstract

Soft set theory is a new mathematical tool to deal with uncertain problems. Since soft sets are defined by mappings and they lack "points", managing them is not convenient. In this paper, the concept of soft points is introduced and the relationship between soft points and soft sets is investigated. We prove that soft sets can be translated into soft point sets and may be expediently handled like ordinary sets. Moreover, we propose *s*-relations on soft sets. By means of soft points and these results, a pair of soft rough approximate operations is defined. Serial, reflexive, symmetric, transitive and Euclidean *s*-relations are characterized by using soft rough approximate operations. In addition, we research soft topologies induced by a reflexive *s*-relation on a special soft set and gives their structure.

Keywords: Soft sets; Soft points; Soft point sets; *s*-relations; Soft rough approximate operations; Soft topologies.

1. Introduction

Most of traditional methods for formal modeling, reasoning and computing are crisp, deterministic and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, fuzzy set theory [27], interval mathematics, and rough set theory [22], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [21]). For example, probability theory can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [21] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. [18,20,19] further studied soft set theory, used this theory to solve decision making problems and devoted fuzzy soft sets combining soft sets with fuzzy sets. Roy et al. [24] presented a fuzzy soft set theoretic approach towards decision making problems. Li et al. [12] investigated decision making based on intuitionistic fuzzy soft sets. Jiang et al. [10] extended soft sets with description logics. Aktas et al. [2] defined soft groups. Li et al. [16] proposed *L*-fuzzy soft sets based on complete Boolean lattices. Feng et al. [6,7] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [8] discussed relationships between soft sets and topologi-

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cal spaces. Shabir et al. [25] proposed soft topological spaces which are defined on the universe with a fixed set of parameters. Babitha et al. [5] introduced relations on soft sets. Li et al. [13,14] considered roughness of fuzzy soft sets and obtained the relationship among soft sets, soft rough sets and topologies. Li et al. [15] studied parameter reductions of soft coverings.

Rough set theory was proposed by Pawlak [22]. It is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions in rough set theory. They can also be seen as a closure operator and an interior operator of the topology induced by an equivalence relation on a universe. We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. Pawlak rough set theory has been extended to tolerance relations, similarity relations, binary relations [17,26,30].

Since soft sets are defined by mappings and then lack "points", managing them is not convenient. Thus, we try to attempt introducing the concept of "soft points" and deal with them as same as ordinary sets.

Feng et al. [7] proposed soft rough approximate operations. But the introduction of these operations seemed suddenly and disposing them is not convenient as soft sets lacks "points" and "soft points" are not proposed. In this paper, we introduce the concept of soft points, prove that soft sets can be translate into soft point sets and then it is convenient to deal with soft sets as same as ordinary sets. We propose s-relations on soft sets. By means of soft points and these results, soft rough approximate operations are defined. And because we do the above work, it is very convenient to deal with the operations introduced by us.

The organization of this paper is as follows: In Section 2,we briefly recall basic concepts about rough sets, soft sets and soft topological spaces. In Section 3, we introduce the concept of soft points and investigate the relationship between soft points and soft sets. In Section 4, we introduce the concepts of serial, reflexive, symmetric, transitive and Euclidean s-relations on soft sets, and investigate the relationships between these s-relations and soft point sets. In Section 5, we propose two soft rough approximate operations. In Section 6, we investigate soft topologies induced by a reflexive s-relation on a special soft set and give their structure. Section 7 concludes this paper and highlights the prospects for potential future development.

2. Overview of rough sets, soft sets and soft topological spaces

In this section, we briefly recall basic concepts about rough sets, soft sets and soft topological spaces.

Throughout this paper, U refers to an initial universe, E refers to the set of parameters and 2^U denotes the power set of U. We only consider the case where both U and E are nonempty finite sets.

2.1. Rough sets

Let R be an equivalence relation on U. The pair (U,R) is called a Pawlak approximation space. Using the equivalence relation R, one can define the following rough approximations:

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$
$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Then $R_*(X)$ and $R^*(X)$ called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

The Pawlak boundary region of X, defined by the difference between these Pawlak rough approximations, that is $Bnd_R(X) = R^*(X) - R_*(X)$. It can easily be seen that $R_*(X) \subseteq X \subseteq R^*(X)$.

A set is Pawlak rough if its boundary region is not empty. Otherwise, the set is crisp. Thus X is Pawlak rough if $R_*(X) \neq R^*(X)$.

We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. Pawlak rough set theory has been extended to binary relations [17,26,30].

Definition 2.1 ([30]) *Let R be a binary relation on U. The pair (U,R) is called a approximation space. Based on the approximation space (U,R), we define a pair of operations $\underline{R}$, $\overline{R}$: $2^U \longrightarrow 2^U$ as follows:*

$$\underline{R}(X) = \{x \in U : R(x) \subseteq X\},$$
$$\overline{R}(X) = \{x \in U : R(x) \cap X \neq \emptyset\},$$

where $X \in 2^U$ *and* $R(x) = \{y \in U : xRy\}$ *is the successor neighborhood of* x. *Then* $\underline{R}(X)$ *and* $\overline{R}(X)$ *are called the lower approximation and the upper approximation of* X, *respectively.*

X *is called a definable set if* $\underline{R}(X) = \overline{R}(X)$; X *is called a rough set if* $\underline{R}(X) \neq \overline{R}(X)$.

2.2. Soft sets

Definition 2.2 ([21]) *Let* $A \subseteq E$. *A pair* (f,A) *is called a soft set over* U, *if* f *is a mapping given by* $f : A \to 2^U$. *We denote* (f,A) *by* f_A.

In other words, a soft set over U is the parameterized family of subsets of the universe U. For $\varepsilon \in A$, $f(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set f_A. Obviously, a soft set is not a ordinary set.

Denote $S(U,E) = \{f_E : f_E \text{ is a soft set over } U\}$.

Definition 2.3 ([18]) *Let* $A,B \subseteq E$, $f_A \in S(U,A)$ *and* $g_B \in S(U,B)$.

(1) f_A *is called a soft subset of* g_B, *if* $A \subseteq B$ *and* $\forall\ \varepsilon \in A$, $f(\varepsilon) \subseteq g(\varepsilon)$. *We write* $f_A \widetilde{\subset} g_B$.

(2) f_A *is called a soft super set of* g_B, *if* $g_B \widetilde{\subset} f_A$. *We write* $f_A \widetilde{\supset} g_B$.

(3) f_A *and* g_B *are called soft equal, if* $A = B$ *and* $\forall\ \varepsilon \in A$, $f(\varepsilon) = g(\varepsilon)$. *We write* $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \widetilde{\subset} g_B$ and $f_A \widetilde{\supset} g_B$.

Definition 2.4 ([3, 18]) *Let* $A,B \subseteq E$, $f_A \in S(U,A)$ *and* $g_B \in S(U,B)$.

(1) $h_{A\cup B}$ *is called the union of* f_A *and* g_B, *if*

$$h(\varepsilon) = \begin{cases} f(\varepsilon), & \text{if } \varepsilon \in A - B, \\ g(\varepsilon), & \text{if } \varepsilon \in B - A, \\ f(\varepsilon) \cup g(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We write $f_A \widetilde{\cup} g_B = h_{A\cup B}$.

(2) $h_{A\cap B}$ *is called the soft intersection of* f_A *and* g_B, *if* $\forall\ \varepsilon \in A \cap B$, $h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon)$. *We write* $f_A \widetilde{\cap} g_B = h_{A\cap B}$.

Remark 2.5 *Let* $A,B,C \subseteq E$, $f_A \in S(U,A)$, $g_B \in S(U,B)$ *and* $h_C \in S(U,C)$. *Then*

(1) $f_A \widetilde{\cap} g_B \widetilde{\subset} f_A$ (*or* g_B) $\widetilde{\subset} f_A \widetilde{\cup} g_B$.

(2) *If* $h_C \widetilde{\subset} f_A$ *and* $h_C \widetilde{\subset} g_B$, *then* $h_C \widetilde{\subset} f_A \widetilde{\cap} g_B$.

(3) *If* $h_C \widetilde{\supset} f_A$ *and* $h_C \widetilde{\supset} g_B$, *then* $h_C \widetilde{\supset} f_A \widetilde{\cup} g_B$.

Definition 2.6 ([25]) *Let* $A \subseteq E$, $f_A, g_A, h_A \in S(U,A)$. h_A *is called the difference of* f_A *and* g_A, *if* $\forall\ \varepsilon \in A$, $h(\varepsilon) = f(\varepsilon) - g(\varepsilon)$. *We write* $h_A = f_A - g_A$.

Definition 2.7 ([3]) *Let* $A \subseteq E$, $f_A, g_A \in S(U,A)$. g_A *is called the relative complement of* f_A, *if* $\forall\ \varepsilon \in A$, $g(\varepsilon) = U - f(\varepsilon)$. *We write* $g_A = f'_A$ *or* $(f_A)'$.

Proposition 2.8 ([3]) *Let* $A \subseteq E$, $f_A, g_A \in S(U,A)$. *Then*

(1) $(f_A \widetilde{\cup} g_A)' = f'_A \widetilde{\cap} g'_A$.

(2) $(f_A \widetilde{\cap} g_A)' = f'_A \widetilde{\cup} g'_A$.

Remark 2.9 *Let* $A \subseteq E$, $f_A, g_A \in S(U,A)$. *Then*

(1) $(f'_A)' = f_A$.

(2) $f_A \widetilde{\subset} g_A \iff (f_A)' \widetilde{\supset} (g_A)'$.

Definition 2.10 ([25]) *Let* $X \in 2^U$. *The soft set* X_E *over* U *is defined by* $\forall\ \varepsilon \in E$, $X(\varepsilon) = X$.

In this paper, U_E and $\emptyset_E$ are also denoted by $\widetilde{U}$ and $\widetilde{\emptyset}$, respectively.

Remark 2.11 *Let* $f_A, g_A \in S(U,A)$. *Then*

(1) $U_A - f_A = f'_A$,

(2) $f_A \widetilde{\cap} g_A = \emptyset_A \iff f_A \widetilde{\subset} g'_A$,

(3) $f_A - g_A = f_A \widetilde{\cap} g'_A$.

2.3. Soft topological spaces

In what follows we consider problems on the universe U and the fixed set E of parameters.

Definition 2.12 ([25]) $\tau \subseteq S(U,E)$ *is called a soft topology over* U, *if* (*i*) $\widetilde{\emptyset}$, $\widetilde{U} \in \tau$; (*ii*) *the union of any number of soft sets in* τ *belongs to* τ; (*iii*) *the intersection of any two soft sets in* τ *belongs to* τ.

The triplet (U,τ,E) *is called a soft topological space over* U. *Every element of* τ *is called a soft open set in* U *and its relative complement is called a soft closed set in* U.

In this paper, the family of all soft closed sets is denoted by τ'.

Definition 2.13 ([25]) *Let (U,τ,E) be a soft topological space over U. $\forall\ f_E \in S(U,E)$, the soft closure of f_E is defined by*

$$cl(f_E) = \widetilde{\cap}\,\{g_E : f_E \,\widetilde{\subset}\, g_E \text{ and } g_E \in \tau'\}.$$

Definition 2.14 ([9]) *Let (U,τ,E) be a soft topological space over U. $\forall\ f_E \in S(U,E)$, the soft interior of f_E is defined by*

$$int(f_E) = \widetilde{\cup}\,\{g_E : g_E \,\widetilde{\subset}\, f_E \text{ and } g_E \in \tau\}.$$

Proposition 2.15 ([9]) *Let (U,τ,E) be a soft topological space over U. Then $\forall\ f_E \in S(U,E)$, $int(f_E) = \widetilde{U} - cl(\widetilde{U} - f_E)$.*

3. Soft points

In this section, we will introduce the concept of soft points and investigate the relationship between soft points and soft sets.

3.1. The concept of soft points

In this subsection we define soft points, which originate from the concept of fuzzy points (see [11,23]).

Definition 3.1 *Let $f_E^* \in S(U,E)$. f_E^* is called a soft point over U, if there exist $e \in E$ and $x \in U$ such that*

$$f^*(\varepsilon) = \begin{cases} \{x\}, & \text{if } \varepsilon = e, \\ \emptyset, & \text{if } \varepsilon \in E - \{e\}. \end{cases}$$

We denote f_E^* by $(x_e)_E$.

In this case, x is called the support point of $(x_e)_E$, $\{x\}$ is called the support point set of $(x_e)_E$ and e is called the expressive parameter of $(x_e)_E$.

Example 3.2 *Let $U = \{x_1,x_2,x_3,x_4,x_5\}$ and $E = \{e_1,e_2,e_3,e_4\}$. We define $f^*(e_1) = \emptyset$, $f^*(e_2) = \emptyset$, $f^*(e_3) = \{x_5\}$, $f^*(e_4) = \emptyset$.*

Then f_E^* is a soft point over U. We denote f_E^* by $((x_5)_{e_3})_E$, where x_5 is the support point of $((x_5)_{e_3})_E$, $\{x_5\}$ is the support point set of $((x_5)_{e_3})_E$ and e_3 is the expressive parameter of $((x_5)_{e_3})_E$.

For $f_E \in S(U,E)$, denote

$\mathscr{F}(E) = \{(x_e)_E : x \in f(e) \text{ and } e \in E\}$,

$P(U,E) = \{(x_e)_E : (x_e)_E$ is a soft points over U$\}$.

Remark 3.3 (1) $(x_e)_E \in \mathscr{F}(E) \iff x \in f(e)$ *and* $e \in E$.

(2) $|\,\mathscr{F}(E)\,| = \sum_{e\in E} |\,f(e)\,|$.

(3) *If $f_E = (x_e)_E$, then $\mathscr{F}(E) = \{(x_e)_E\}$.*

Example 3.4 *Let $U = \{x_1,x_2,x_3,x_4,x_5\}$ and $E = \{e_1,e_2,e_3,e_4\}$. We define $f(e_1) = \{x_1,x_4\}$, $f(e_2) = U$, $f(e_3) = \{x_5\}$, $f(e_4) = \emptyset$. Then*

$\mathscr{F}(E) = \{((x_1)_{e_1})_E, ((x_4)_{e_1})_E, ((x_1)_{e_2})_E, ((x_2)_{e_2})_E, ((x_3)_{e_2})_E, ((x_4)_{e_2})_E, ((x_5)_{e_2})_E, ((x_5)_{e_3})_E\}$ *and*

$P(U,E) = \{((x_i)_{e_j})_E : 1 \leqslant i \leqslant 5, 1 \leqslant j \leqslant 4\}$.

To illustrate the fact that the soft contain relation, the soft intersection operation, the soft union operation and the soft difference operation on two soft sets can be be translated into the contain relation, the intersection operation, the union operation and the difference operation on two soft point sets (i.e., two ordinary sets), respectively, we give the following Proposition 3.5.

Proposition 3.5 *Let $f_E, g_E, h_E \in S(U,E)$.*

(1) *If $g_E \,\widetilde{\subset}\, f_E$, then $\mathscr{G}(E) \subseteq \mathscr{F}(E)$.*

(2) *If $f_E = g_E \,\widetilde{\cap}\, h_E$, then $\mathscr{F}(E) = \mathscr{G}(E) \cap \mathscr{H}(E)$.*

(3) *If $f_E = g_E \,\widetilde{\cup}\, h_E$, then $\mathscr{F}(E) = \mathscr{G}(E) \cup \mathscr{H}(E)$.*

(4) *If $f_E = g_E - h_E$, then $\mathscr{F}(E) = \mathscr{G}(E) - \mathscr{H}(E)$.*

Proof. (1) This is obvious.

(2) Let $(x_e)_E \in \mathscr{F}(E)$. Then $x \in f(e)$. Since $f_E = g_E \,\widetilde{\cap}\, h_E$, we have $x \in g(e)$ and $x \in h(e)$. Thus $(x_e)_E \in \mathscr{G}(E)$ and $(x_e)_E \in \mathscr{F}(E)$. Hence $(x_e)_E \in \mathscr{G}(E) \cap \mathscr{H}(E)$. Conversely, the proof is similar.

(3) The proof is similar to (2).

(4) The proof is similar to (2). □

Proposition 3.6 (1) *If $f_E = U_E$, then $P(U,E) = \mathscr{F}(E)$.*

(2) $P(U,E) = \cup\,\{\mathscr{F}(E) : f_E \in S(U,E)\}$.

Proof. (1) This is obvious.

(2) Let $f_E \in S(U,E)$. Since $f_E \,\widetilde{\subset}\, U_E$, by Proposition 3.5 and (1), $\mathscr{F}(E) \subseteq P(U,E)$. Thus $P(U,E) \supseteq \cup\{\mathscr{F}(E) : f_E \in S(U,E)\}$.

Conversely, since $U_E \in S(U,E)$, by (1), we have $P(U,E) \subseteq \cup\{\mathscr{F}(E) : f_E \in S(U,E)\}$.

Hence $\mathscr{F}(E) = \cup\{\mathscr{F}(E) : f_E \in S(U,E)\}$. □

3.2. Soft points and soft sets

In this subsection, we will investigate the relationship between soft points and soft sets.

Definition 3.7 *Let* $f_E \in S(U,E)$ *and* $(x_e)_E \in P(U,E)$. *We define* $(x_e)_E \,\widetilde{\in}\, f_E$ *by* $(x_e)_E \,\widetilde{\subset}\, f_E$.

Note that $(x_e)_E \,\widetilde{\notin}\, f_E$, if $(x_e)_E \,\widetilde{\not\subset}\, f_E$.

Remark 3.8 (1) $(x_e)_E = (x'_{e'})_E \Leftrightarrow x = x'$ *and* $e = e'$.

(2) $(x_e)_E \,\widetilde{\in}\, f_E \Leftrightarrow x \in f(e)$ *and* $e \in E \Leftrightarrow (x_e)_E \in \mathscr{F}(E)$.

(3) $(x_e)_E \,\widetilde{\in}\, f_E$ *and* $f_E \,\widetilde{\subset}\, g_E \Rightarrow (x_e)_E \,\widetilde{\in}\, g_E$.

(4) $(x_e)_E \,\widetilde{\in}\, (x_e)_E$.

(5) $(x_e)_E \,\widetilde{\in}\, f_E \Leftrightarrow (x_e)_E \,\widetilde{\notin}\, f'_E$.

Theorem 3.9 *Let* $f_E \in S(U,E)$. *Then* $f_E = \widetilde{\cup}\, \mathscr{F}(E)$.

Proof. Denote $h_E = \widetilde{\cup}\, \mathscr{F}(E)$. Then $h_E = \widetilde{\cup}\, \{(x_e)_E : x \in f(e)$ *and* $e \in E\}$. Thus

$$h_E = \widetilde{\bigcup_{e\in E}}\ \widetilde{\bigcup_{x\in f(e)}} (x_e)_E.$$

$\forall\, \varepsilon \in E$,

$h(\varepsilon) = \bigcup\limits_{e\in E} \bigcup\limits_{x\in f(e)} x_e(\varepsilon) = (\bigcup\limits_{x\in f(\varepsilon)} x_\varepsilon(\varepsilon))$
$\bigcup(\bigcup\limits_{e\in E-\{\varepsilon\}} \bigcup\limits_{x\in f(e)} x_e(\varepsilon)) = (\bigcup\limits_{x\in f(\varepsilon)} \{x\}) \bigcup\, \emptyset = f(\varepsilon)$.

This shows $h_E = f_E$. Hence $f_E = \widetilde{\cup}\, \mathscr{F}(E)$. □

Remark 3.10 *Theorem 3.9 reveals the fact that a soft set can be translated into a soft point set and vice versa.*

Theorem 3.11 *Let* $f_E, g_E \in S(U,E)$. *Then*

(1) $f_E \,\widetilde{\subset}\, g_E \Leftrightarrow \mathscr{F}(E) \subseteq \mathscr{G}(E)$.

(2) $f_E = g_E \Leftrightarrow \mathscr{F}(E) = \mathscr{G}(E)$.

Proof. These hold by Proposition 3.5 and Theorem 3.9. □

Remark 3.12 *Theorem 3.11 illustrates that the soft contain relation and the soft equal relation can be respectively translated into the contain relation and the equal relation on two soft point sets (i.e., two ordinary sets) and vice versa.*

When we study some problems of soft sets by using soft points in this paper, we will abide by the following logic thinking: firstly, the soft contain relation, the soft intersection operation, the soft union operation and the soft difference operation on soft sets are translated into the contain relation, the intersection operation, the union operation and the difference operation on soft point sets by Proposition 3.5, respectively; secondly, the relations and operations on ordinary sets (i.e., soft point sets) are realized; thirdly, the results of the relations and operations on ordinary sets are translated into the results on soft sets by Theorem 3.9.

4. *s*-relations on soft sets

In this section, we introduce the concepts of serial, reflexive, symmetric, transitive and Euclidean *s*-relations on soft sets, and investigate the relationships between these *s*-relations and soft point sets.

Definition 4.1 ([5]) *Let* $A, B \subseteq E$, $f_A \in S(U,A)$ *and* $g_B \in S(U,B)$. $h_{A\times B}$ *is called the cartesian product of* f_A *and* g_B, *if* $\forall\, (a,b) \in A \times B$, $h(a,b) = f(a) \times g(b)$. *We write* $h_{A\times B} = f_A \times g_B$.

Definition 4.2 ([5]) *Let* $A, B \subseteq E$, $f_A \in S(U,A)$ *and* $g_B \in S(U,B)$.

(1) R *is called a relation from* f_A *to* g_B, *if* $R \,\widetilde{\subset}\, f_A \times g_B$.

(2) R *is called a relation on* f_A, *if* $R \,\widetilde{\subset}\, f_A \times f_A$.

In other words, a relation R from f_A to g_B is of the form l_P, where $P \subseteq A \times B$ and $\forall\, (a,b) \in P$,

$l(a,b) \subseteq f(a) \times g(b)$.

Definition 4.3 *Let $f_E \in S(U,E)$. R is called a surjective relation (brief. s-relation) on f_E, if there exists a soft set $l_{E\times E}$ over $U \times U$ such that $R = l_{E\times E} \widetilde{\subset} f_E \times f_E$.*

Remark 4.4 *R is a s-relation on $f_E \Rightarrow R$ is a relation on f_E.*

Example 4.5 *Let $U = \{x_1,x_2,x_3,x_4,x_5\}$ and $E = \{e_1,e_2\}$. We define $f(e_1) = \{x_1,x_3,x_5\}$, $f(e_2) = \{x_2,x_4\}$. Then $f_E \in S(U,E)$ and $E \times E = \{(e_1,e_1),(e_1,e_2),(e_2,e_1),(e_2,e_2)\}$.*

Let $h_{E\times E} = f_E \times f_E$. Then

$$h(e_1,e_1) = f(e_1)\times f(e_1),\ h(e_1,e_2) = f(e_1)\times f(e_2),$$

$$h(e_2,e_1) = f(e_2)\times f(e_1) \text{ and } h(e_2,e_2) = f(e_2)\times f(e_2).$$

(1) Define $l : E \times E \to 2^{U\times U}$ by

$$l(e_1,e_1) = \{(x_1,x_1),(x_1,x_3),(x_1,x_5),(x_3,x_3),(x_3,x_5),(x_5,x_5)\},$$

$$l(e_1,e_2) = f(e_1)\times f(e_2),$$

$$l(e_2,e_1) = \{(x_2,x_1),(x_2,x_3),(x_2,x_5),(x_4,x_3),(x_4,x_5)\}$$

and

$$l(e_2,e_2) = f(e_2)\times f(e_2).$$

Then

$$l(e_1,e_1) \subseteq h(e_1,e_1),\ l(e_1,e_2) \subseteq h(e_1,e_2),$$

$$l(e_2,e_1) \subseteq h(e_2,e_1) \text{ and } l(e_2,e_2) \subseteq h(e_2,e_2).$$

So $l_{E\times E} \widetilde{\subset} f_E \times f_E$

Put $R_1 = l_{E\times E}$. Then R_1 is a s-relation on f_E.

(2) Put $P = \{(e_1,e_1),(e_1,e_2)\}$. Then $P \subsetneq E \times E$. Define $k : P \to 2^{U\times U}$ by

$$k(e_1,e_1) = f(e_1)\times f(e_1) \text{ and } k(e_1,e_2) = f(e_1)\times f(e_2).$$

Put $R_2 = k_P$. Since $R_2 \widetilde{\subset} f_E \times f_E$, R_2 is a relation on f_E. But R_2 is not a s-relation on f_E.

Since soft sets can be translated into soft point sets, every relation on a soft set can be translated into a relation on a soft point set. We introduce the following Definition 4.6 for this reason.

Definition 4.6 *Let R be a s-relation on $f_E \in S(U,E)$. Define a relation R^* on $\mathscr{F}(E)$ as follows: for any $(x_e)_E, (x'_{e'})_E \in \mathscr{F}(E)$,*

$$(x_e)_E R^* (x'_{e'})_E \Leftrightarrow (x_e)_E \times (x'_{e'})_E \widetilde{\subset} R.$$

Then R^* is called the relation induced by R.

Remark 4.7 (1) $(x_e)_E \times (x'_{e'})_E \widetilde{\subset} f_E \times f_E \Leftrightarrow x \in f(e)$ *and* $x' \in f(e')$.

(2) Let $R = l_{E\times E} \widetilde{\subset} f_E \times f_E$. Then

$$(x_e)_E R^* (x'_{e'})_E \Leftrightarrow (x,x') \in l(e,e') \Longrightarrow x \in f(e),\ x' \in f(e').$$

Definition 4.8 *Let R be a s-relation on f_E. R is called serial (resp. reflexive, symmetric, transitive, Euclidean), if R^* is serial (resp. reflexive, symmetric, transitive, Euclidean).*

Let $f_E \in S(U,E)$. Denote $S_f(U,E) = \{g_E \in S(U,E) : g_E \widetilde{\subset} f_E\}$.

Let R be a s-relation on f_E and R^* the relation induced by R. $\forall\ (x_e)_E \in \mathscr{F}(E)$, $g_E \in S_f(X,E)$, put

$$R^*((x_e)_E) = \{(x'_{e'})_E \in \mathscr{F}(E) : (x_e)_E R^* (x'_{e'})_E\},$$

$$P_g(f_E,R) = \{(x_e)_E \in \mathscr{F}(E) : R^*((x_e)_E) \subseteq \mathscr{G}(E)\},$$

$$P^g(f_E,R) = \{(x_e)_E \in \mathscr{F}(E) : R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset\}.$$

Remark 4.9 *Let R be a s-relation on $f_E \in S(U,E)$ and R^* the relation induced by R. Then*

(1) R is serial $\Leftrightarrow \forall\ (x_e)_E \in \mathscr{F}(E)$, $R^*((x_e)_E) \neq \emptyset$.

(2) R is reflexive $\Leftrightarrow \forall\ (x_e)_E \in \mathscr{F}(E)$, $(x_e)_E \in R^*((x_e)_E)$.

(3) R is symmetric $\Leftrightarrow \forall\ (x_e)_E, (x'_{e'})_E \in \mathscr{F}(E)$, $(x'_{e'})_E \in R^*((x_e)_E)$ implies $(x_e)_E \in R^*((x'_{e'})_E)$.

(4) R is transitive $\Leftrightarrow \forall\ (x_e)_E, (x'_{e'})_E, (x''_{e''})_E \in \mathscr{F}(E)$, $(x_e)_E \in R^*((x'_{e'})_E)$ and $(x'_{e'})_E \in R^*((x''_{e''})_E)$ implies $(x_e)_E \in R^*((x''_{e''})_E)$

$\Leftrightarrow \forall\ (x_e)_E, (x'_{e'})_E \in \mathscr{F}(E)$, $(x'_{e'})_E \in R^*((x_e)_E)$ implies $R^*((x'_{e'})_E) \subseteq R^*((x_e)_E)$.

(5) *R is Euclidean* $\Leftrightarrow \forall\ (x_e)_E, (x'_{e'})_E, (x''_{e''})_E \in \mathscr{F}(E)$, $(x'_{e'})_E \in R^*((x_e)_E)$ *and* $(x''_{e''})_E \in R^*((x_e)_E)$ *implies* $R^*((x''_{e''}) \subseteq R^*((x'_{e'})_E)$

$\Leftrightarrow \forall\ (x_e)_E, (x'_{e'})_E \in \mathscr{F}(E)$, $(x'_{e'})_E \in R^*((x_e)_E)$ *implies* $R^*((x_e)_E) \subseteq R^*((x'_{e'})_E)$.

Lemma 4.10 *Let R be a s-relation on* $f_E \in S(U,E)$. *Then* $\forall\ g_E, h_E \in S_f(U,E)$,

(1) $P_f(f_E,R) = \mathscr{F}(E)$.

(2) *a) R is serial* $\Rightarrow P_g(f_E,R) \subseteq P^g(f_E,R)$.

b) R is reflexive $\Rightarrow P_g(f_E,R) \subseteq \mathscr{G}(E) \subseteq P^g(f_E,R)$.

(3) *a)* $g_E \widetilde{\subset} h_E \Rightarrow P_g(f_E,R) \subseteq P_h(f_E,R)$;

b) $g_E \widetilde{\subset} h_E \Rightarrow P^g(f_E,R) \subseteq P^h(f_E,R)$.

(4) *a)* $P^l(f_E,R) = P^g(f_E,R) \cup P^h(f_E,R)$ *where* $l_E = g_E \widetilde{\cup} h_E$;

b) $P_l(f_E,R) = P_g(f_E,R) \cap P_h(f_E,R)$ *where* $l_E = g_E \widetilde{\cap} h_E$.

Proof. (1) This is obvious.

(2) *a*) Let $(x_e)_E \in P_g(f_E,R)$. Thus $R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Since R is serial, by Remark 4.9, $R^*((x_e)_E) \neq \emptyset$. This implies $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$. So $(x_e)_E \in P^g(f_E,R)$. Thus $P_g(f_E,R) \subseteq P^g(f_E,R)$.

b) Let $(x_e)_E \in P_g(f_E,R)$. Then $R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Since R is reflexive, by Remark 4.9, we have $(x_e)_E \in R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Thus $P_g(f_E,R) \subseteq \mathscr{G}(E)$. Since $(x_e)_E \in R^*((x_e)_E)$ and $(x_e)_E \in \mathscr{G}(E)$, $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$. Thus $\mathscr{G}(E) \subseteq P^g(f_E,R)$.

(3) *a*) Let $(x_e)_E \in P_g(f_E,R)$. Then $R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Since $g_E \widetilde{\subset} h_E$, $\mathscr{G}(E) \subseteq \mathscr{H}(E)$ and $R^*((x_e)_E) \subseteq \mathscr{H}(E)$. Thus $(x_e)_E \in P_h(f_E,R)$. Hence $P_g(f_E,R) \subseteq P_h(f_E,R)$.

b) The proof is similar to *a*).

(4) *a*) Let $(x_e)_E \in P^l(f_E,R)$. Then $R^*((x_e)_E) \cap \mathscr{L}(E) \neq \emptyset$. Since $l_E = g_E \widetilde{\cup} h_E$, by Proposition 3.5, $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$ and $R^*((x_e)_E) \cap \mathscr{H}(E) \neq \emptyset$. Thus $(x_e)_E \in P^g(f_E,R)$ and $(x_e)_E \in P^h(f_E,R)$. Hence $P^l(f_E,R) \subseteq P^g(f_E,R) \cup P^h(f_E,R)$.

Conversely, this is obvious.

b) The proof is similar to *a*). □

5. Soft rough approximate operations

In this section, we propose two soft rough approximate operations. Serial, reflexive, symmetric, transitive and Euclidean *s*-relations are characterized by using them.

Definition 5.1 *Let R be a s-relation on* $f_E \in S(U,E)$. *Then the pair* $P = (f_E,R)$ *is called a soft approximation space. Based on P, we define the following operations* $\underline{apr}_P, \overline{apr}_P : S_f(U,E) \to S_f(U,E)$ *by*

$$\underline{apr}_P(g_E) = \widetilde{\cup}\, P_g(f_E,R),\ \ \overline{apr}_P(g_E) = \widetilde{\cup}\, P^g(f_E,R),$$

where $g_E \in S_f(U,E)$. *Then,* $\underline{apr}_P$ *and* $\overline{apr}_P$ *are called the soft P-lower approximation operator and the soft P-upper approximation operator on* f_E, *respectively;* $\underline{apr}_P(g_E)$ *and* $\overline{apr}_P(g_E)$ *are called the soft P-lower approximation of* g_E *and the soft P-upper approximation of* g_E, *respectively.*

g_E *is called a soft P-definable set if* $\underline{apr}_P(g_E) = \overline{apr}_P(g_E)$; g_E *is called a soft P-rough set if* $\underline{apr}_P(g_E) \neq \overline{apr}_P(g_E)$.

Remark 5.2 *In* [7], *Feng et al. proposed two operations* $\underline{apr}_P, \overline{apr}_P : 2^U \to 2^U$ *by*

$$\underline{apr}_P(X) = \{u \in U : \exists\ e \in E,\ s.t.\ u \in f(e) \subseteq X\},$$

$$\overline{apr}_P(X) = \{u \in U : \exists\ e \in E,\ s.t.\ u \in f(e)\ \ and\ f(e) \cap X \neq \emptyset\}.$$

where $X \in 2^U$, $P = (U, f_E)$ *and* $f_E \in S(U,E)$.

Lemma 5.3 *Let R be a s-relation on* $f_E \in S(U,E)$. *Then* $\forall\ g_E, h_E \in S_f(U,E)$, *we have*

(1) $h_E = \underline{apr}_P(g_E) \Leftrightarrow \mathscr{H}_E = P_g(f_E,R)$.

(2) $h_E = \overline{apr}_P(g_E) \Leftrightarrow \mathscr{H}_E = P^g(f_E,R)$.

Proof. (1) Sufficiency. This holds by Theorem 3.11.

Necessity. Denote $P_g(f_E,R) = \{(y_a)_E : y \in X\ and\ a \in A\}$ where $X \subseteq U$ and $A \subseteq E$.

Let $(x_e)_E \in \mathscr{H}_E$. Then $x \in h(e) = \cup\, \{y_a(e) : y \in X\ and\ a \in A\}$.

We claim that $e \in A$. Otherwise, $y_a(e) = \emptyset\ \forall\ y \in X$ and $a \in A$. Then $h(e) = \cup\, \{y_a(e) : y \in X\ and\ a \in A\} = \emptyset$, a contradiction.

Thus $h(e) = \cup \{y_e(e) : y \in X\} = \cup \{\{y\} : y \in X\} = X$. This implies $x \in X$. So $(x_e)_E \in P_g(f_E, R)$.

Conversely, $(x_e)_E \in P_g(f_E, R)$. Then $x \in X$ and $e \in A$. Note that $h(e) = \cup \{y_a(e) : y \in X \text{ and } a \in A\} = \cup \{y_e(e) : y \in X\} = \cup \{\{y\} : y \in X\} = X$. So $x \in h(e)$. This implies $(x_e)_E \in \mathscr{H}_E$.

Hence $\mathscr{H}_E = P_g(f_E, R)$.

(2) The proof is similar to (1). □

Lemma 5.4 *Let $(f_\alpha)_E \in S(U,E)$ for $\alpha \in A \cup B$. Then*
$\widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in A \cup B\} = (\widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in A\}) \widetilde{\cup} (\widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in B\})$.

Proof. Denote $C = A \cup B$, $f_E^C = \widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in C\}$, $f_E^A = \widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in A\}$, $f_E^B = \widetilde{\cup}\ \{(f_\alpha)_E : \alpha \in B\}$ and $g_E = f_E^A \widetilde{\cup} f_E^B$.

Then $f^C(e) = \cup\ \{f_\alpha(e) : \alpha \in C\}\ \forall\ e \in E$ and $g(e) = f^A(e) \cup f^B(e)\ \forall\ e \in E$. Thus $g(e) = (\cup\ \{f_\alpha(e) : \alpha \in A\}) \cup (\cup\ \{f_\alpha(e) : \alpha \in B\}) = \cup \{f_\alpha(e) : \alpha \in A \cup B\} = \cup \{f_\alpha(e) : \alpha \in C\} = f^C(e)$. □

Lemma 5.5 *Let R be a s-relation on f_E, $(x_e)_E \in S_f(U,E)$. Denote $h_E = \overline{apr}_P((x_e)_E)$. Then $\mathscr{H}(E) = \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}$.*

Proof. Denote $g_E = (x_e)_E$. Then $h_E = \overline{apr}_P(g_E)$.

Let $(y_\varepsilon)_E \in \mathscr{H}(E)$. By Lemma 5.3, $(y_\varepsilon)_E \in P^g(f_E, R)$. This implies $R^*((y_\varepsilon)_E) \cap \mathscr{G}(E) \neq \emptyset$. By Remark 3.3, $\mathscr{G}(E) = \{(x_e)_E\}$. So $(x_e)_E \in R^*((y_\varepsilon)_E)$. Thus $(y_\varepsilon)_E \in \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}$.

Conversely, let $(y_\varepsilon)_E \in \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}$. Then $(x_e)_E \in R^*((y_\varepsilon)_E)$. So $\{(x_e)_E\} = R^*((y_\varepsilon)_E) \cap \mathscr{G}(E) \neq \emptyset$. This implies $(y_\varepsilon)_E \in P^g(f_E, R)$. By Lemma 5.3, $(y_\varepsilon)_E \in \mathscr{H}(E)$.

Therefore, $\mathscr{H}(E) = \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}$. □

Proposition 5.6 *Let R be a s-relation on $f_E \in S(U,E)$. Then $\forall\ g_E, h_E \in S_f(U,E)$,*

(1) *If $g_E \widetilde{\subset} h_E$, then*

a) $\underline{apr}_P(g_E) \widetilde{\subset} \underline{apr}_P(h_E)$;

b) $\overline{apr}_P(g_E) \widetilde{\subset} \overline{apr}_P(h_E)$.

(2) *a)* $\underline{apr}_P(g_E \widetilde{\cap} h_E) = \underline{apr}_P(g_E) \widetilde{\cap} \underline{apr}_P(h_E)$;

b) $\overline{apr}_P(g_E \widetilde{\cup} h_E) = \overline{apr}_P(g_E) \widetilde{\cup} \overline{apr}_P(h_E)$.

Proof. (1) These hold by Lemma 4.10 and Theorem 3.11.

(2) *a)* Denote $q_E = \underline{apr}_P(g_E)$, $p_E = \underline{apr}_P(h_E)$, $k_E = q_E \widetilde{\cap} p_E$, $l_E = g_E \widetilde{\cap} h_E$ and $w_E = \underline{apr}_P(l_E)$. By Proposition 3.5 and Lemma 4.10, $\mathscr{K}(E) = \mathscr{Q}(E) \cap \mathscr{P}(E)$ and $P_l(f_E, R) = P_g(f_E, R) \cap P_h(f_E, R)$.

Let $(x_e)_E \in \mathscr{K}(E)$. Then $(x_e)_E \in \mathscr{Q}(E)$ and $(x_e)_E \in \mathscr{P}(E)$. By Lemma 5.3, $(x_e)_E \in P_g(f_E, R)$ and $(x_e)_E \in P_h(f_E, R)$. Thus $(x_e)_E \in P_l(f_E, R)$. By Lemma 5.3, $(x_e)_E \in \mathscr{W}(E)$. By Theorem 3.11, $\underline{apr}_P(g_E) \widetilde{\cap} \underline{apr}_P(h_E) \widetilde{\subset} \underline{apr}_P(g_E \widetilde{\cap} h_E)$.

Conversely, $\underline{apr}_P(g_E \widetilde{\cap} h_E) \widetilde{\subset} \underline{apr}_P(g_E) \widetilde{\cap} \underline{apr}_P(h_E)$ is obvious.

b) This holds by Lemma 4.10 and Lemma 5.4. □

Proposition 5.7 *Let R be a s-relation on f_E. Then the following are equivalent.*

(1) *R is serial;*

(2) *$\forall\ g_E \in S_f(U,E)$, $\underline{apr}_P(g_E) \widetilde{\subset} \overline{apr}_P(g_E)$.*

Proof. (1) ⇒ (2) holds by Lemma 4.10 and Theorem 3.11.

(2) ⇒ (1). Let $g_E \in S_f(U,E)$.

Denote $h_E = \underline{apr}_P(g_E)$ and $l_E = \overline{apr}_P(g_E)$.

Suppose $\forall\ (x_e)_E \in \mathscr{F}(E)$, $R^*((x_e)_E) = \emptyset$. Then $R^*((x_e)_E) \subseteq \mathscr{G}(E)\ \forall\ g_E \in S_f(U,E)$. This implies $(x_e)_E \in P_g(f_E, R)$. By Lemma 5.3, $(x_e)_E \in \mathscr{H}(E)$. Since $h_E \widetilde{\subset} l_E$, $\mathscr{H}(E) \subseteq \mathscr{L}(E)$ and $(x_e)_E \in \mathscr{L}(E)$. But $R^*((x_e)_E) \cap \mathscr{G}(E) = \emptyset$. Thus $(x_e)_E \notin P^g(f_E, R)$. By Lemma 5.3, $(x_e)_E \notin \mathscr{L}(E)$, a contradiction. Hence $R^*((x_e)_E) \neq \emptyset$. □

Proposition 5.8 *Let R be a s-relation on f_E. Then the following are equivalent.*

(1) *R is reflexive;*

(2) *$\forall\ g_E \in S_f(U,E)$,*
$\underline{apr}_P(g_E) \widetilde{\subset} g_E \widetilde{\subset} \overline{apr}_P(g_E)$.

Proof. (1) ⇒ (2) holds by Lemma 4.10 and Theorem 3.11.

(2) ⇒ (1). Let $g_E \in S_f(U,E)$. Denote $g_E = (x_e)_E$ and $h_E = \overline{apr}_P(g_E)$.

By (2), $g_E \widetilde{\subset} h_E$. Then $\mathscr{G}(E) \subseteq \mathscr{H}(E)$. This implies $(x_e)_E \in \mathscr{H}(E)$. By Lemma 5.3, $(x_e)_E \in P^g(f_E, R)$. Thus $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$.

So $R^*((x_e)_E) \cap \mathscr{G}(E) = (x_e)_E$. Hence $(x_e)_E \in R^*((x_e)_E)$. □

Proposition 5.9 *Let R be reflexive on $f_E \in S(U,E)$. Then*

(1) $\underline{apr}_P(f_E) = \overline{apr}_P(f_E) = f_E$.

(2) $\underline{apr}_P(\widetilde{\emptyset}) = \overline{apr}_P(\widetilde{\emptyset}) = \widetilde{\emptyset}$.

Proof. (1) By Lemma 4.10, $\underline{apr}_P(f_E) \,\widetilde{\subset}\, \overline{apr}_P(f_E)$.

Conversely, since $P^f(f_E,R) \subseteq \mathscr{F}(E)$, then $\overline{apr}_P(f_E) \,\widetilde{\subset}\, f_E$. By Lemma 4.10, $f_E = \underline{apr}_P(f_E)$. Thus $\overline{apr}_P(f_E) \,\widetilde{\subset}\, \underline{apr}_P(f_E)$. Therefore, $\underline{apr}_P(f_E) = \overline{apr}_P(f_E) = f_E$.

(2) This is obvious. □

Proposition 5.10 *Let R be a s-relation on f_E. Then the following are equivalent.*

(1) *R is symmetric;*

(2) $\forall\, g_E \in S_f(U,E)$,

$\overline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, g_E \,\widetilde{\subset}\, \underline{apr}_P(\overline{apr}_P(g_E))$.

Proof. (1) $\Rightarrow$ (2). Let $g_E \in S_f(U,E)$. Denote

$k_E = \underline{apr}_P(g_E)$, $w_E = \overline{apr}_P(k_E)$, $h_E = \overline{apr}_P(g_E)$ and $l_E = \underline{apr}_P(h_E)$.

Suppose $\mathscr{W}(E) - \mathscr{G}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{W}(E) - \mathscr{G}(E)$. Then $(x_e)_E \notin \mathscr{G}(E)$ and $(x_e)_E \in \mathscr{W}(E)$. By Lemma 5.3, $(x_e)_E \in P^k(f_E,R)$. This implies $R^*((x_e)_E) \cap \mathscr{K}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) \cap \mathscr{K}(E)$. Then $(x'_{e'})_E \in \mathscr{K}(E)$. By Lemma 5.3, $(x_e)_E \in P_g(f_E,R)$. This implies $R^*((x'_{e'})_E \subseteq \mathscr{G}(E)$. Since R is symmetric, $(x_e)_E \in R^*((x'_{e'})_E)$. Thus $(x_e)_E \in \mathscr{G}(E)$, a contradiction. Hence $\mathscr{W}(E) \subseteq \mathscr{G}(E)$. By Theorem 3.11, $w_E \,\widetilde{\subset}\, g_E$.

Therefore, $\overline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, g_E$.

Suppose $\mathscr{G}(E) - \mathscr{L}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{G}(E) - \mathscr{L}(E)$. Then $(x_e)_E \in \mathscr{G}(E)$ and $(x_e)_E \notin \mathscr{L}(E)$. By Lemma 5.3, $(x_e)_E \notin P_h(f_E,R)$. This implies $R^*((x_e)_E) \nsubseteq \mathscr{H}(E)$. Thus $R^*((x_e)_E) - \mathscr{H}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) - \mathscr{H}(E)$. Then $(x'_{e'})_E \notin \mathscr{H}(E)$. By Lemma 5.3, $(x'_{e'})_E \notin P^g(f_E,R)$. This implies $R^*((x'_{e'})_E) \cap \mathscr{G}(E) = \emptyset$. Since $(x'_{e'})_E \in R^*((x_e)_E)$, by R is symmetric, we have $(x_e)_E \in R^*((x'_{e'})_E)$. Thus $\mathscr{G}(E) = \emptyset$, a contradiction. Hence $\mathscr{G}(E) \subseteq \mathscr{L}(E)$. By Theorem 3.11, $g_E \,\widetilde{\subset}\, l_E$.

Therefore, $g_E \,\widetilde{\subset}\, \underline{apr}_P(\overline{apr}_P(g_E))$.

(2) $\Rightarrow$ (1). Let $(x_e)_E, (x'_{e'})_E \in \mathscr{F}(E)$ with $(x'_{e'})_E \in R^*((x_e)_E)$. Denote

$$g_E = (x_e)_E,\ h_E = \overline{apr}_P(g_E) \text{ and } l_E = \underline{apr}_P(h_E).$$

By Lemma 5.3 and Lemma 5.5, $\mathscr{L}_E = P_h(f_E,R)$ and

$$\mathscr{H}(E) = \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}.$$

Then $g_E \,\widetilde{\subset}\, l_E \,\widetilde{\subset}\, h_E$. This implies $(x_e)_E \in \mathscr{L}(E) \subseteq \mathscr{H}(E)$. So $(x_e)_E \in P^g(f_E,R)$. Thus $R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Since $(x'_{e'})_E \in R^*((x_e)_E)$, $(x'_{e'})_E \in \mathscr{G}(E)$. Then $(x'_{e'})_E = (x_e)_E$. Hence $(x'_{e'})_E \in \mathscr{H}(E)$. By Lemma 5.5, $(x_e)_E \in R^*((x'_{e'})_E)$.

Therefore, R is symmetric. □

Lemma 5.11 *Let R be reflexive on $f_E \in S(U,E)$ and let $g_E, h_E \in S_f(U,E)$.*

(1) *If $h_E = \underline{apr}_P(g_E)$ and R is transitive, then $P_g(f_E,R) = P_h(f_E,R) \subseteq P^h(f_E,R) \subseteq P^g(f_E,R)$;*

(2) *If $h_E = \overline{apr}_P(g_E)$ and R is Euclidean, then $P_g(f_E,R) \subseteq P^g(f_E,R) \subseteq P_h(f_E,R) \subseteq P^h(f_E,R)$.*

Proof. (1) Let $h_E = \underline{apr}_P(g_E)$. Since $\underline{apr}_P(g_E) \,\widetilde{\subset}\, g_E$, $h_E \,\widetilde{\subset}\, g_E$. By Lemma 4.10, $P_g(f_E,R) \supseteq P_h(f_E,R) \subseteq P^h(f_E,R) \subseteq P^g(f_E,R)$. It suffices to show that $P_g(f_E,R) \subseteq P_h(f_E,R)$.

Suppose $P_g(f_E,R) - P_h(f_E,R) \neq \emptyset$. Pick $(x_e)_E \in P_g(f_E,R) - P_h(f_E,R)$. Then $R^*((x_e)_E) \subseteq \mathscr{G}(E)$ and $R^*((x_e)_E) \nsubseteq \mathscr{H}(E)$, and so $R^*((x_e)_E) - \mathscr{H}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) - \mathscr{H}(E)$. Since R is transitive, $R^*((x'_{e'})_E) \subseteq R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Thus $(x'_{e'})_E \in P_g(f_E,R)$. By Lemma 5.3, $(x'_{e'})_E \in \mathscr{H}(E)$. But $(x'_{e'})_E \notin \mathscr{H}(E)$, a contradiction. Therefore, $P_g(f_E,R) \subseteq P_h(f_E,R)$.

(2) Let $h_E = \overline{apr}_P(g_E)$. Since $g_E \,\widetilde{\subset}\, \overline{apr}_P(g_E)$, $g_E \,\widetilde{\subset}\, h_E$. By Lemma 4.10, $P_g(f_E,R) \subseteq P^g(f_E,R)$ and $P_h(f_E,R) \subseteq P^h(f_E,R)$. It suffices to show that $P^g(f_E,R) \subseteq P_h(f_E,R)$.

Suppose $P^g(f_E,R) - P_h(f_E,R) \neq \emptyset$. Pick $(x_e)_E \in P^g(f_E,R) - P_h(f_E,R)$. Then $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$ and $R^*((x_e)_E) \nsubseteq \mathscr{H}(E)$. Pick $(x''_{e''})_E \in R^*((x_e)_E) \cap \mathscr{G}(E)$ and $(x'_{e'})_E \in R^*((x_e)_E) - \mathscr{H}(E)$. Then $(x'_{e'})_E \notin \mathscr{H}(E)$. Since R is Euclidean, $(x''_{e''})_E \in R^*((x'_{e'})_E)$. That implies $R^*((x'_{e'})_E) \cap \mathscr{G}(E) \neq \emptyset$. Thus $(x'_{e'})_E \in P^g(f_E,R)$. By Lemma 5.3, $(x'_{e'})_E \in \mathscr{H}(E)$, a contradiction. Therefore, $P^g(f_E,R) \subseteq P_h(f_E,R)$. □

Proposition 5.12 *Let R be reflexive on $f_E \in S(U,E)$. Then the following are equivalent.*

(1) *R is transitive;*

(2) $\forall\, g_E \in S_f(U,E)$,
$\underline{apr}_P(g_E) \,\widetilde{\subset}\, \underline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, \overline{apr}_P(\overline{apr}_P(g_E)) \,\widetilde{\subset}\, \overline{apr}_P(g_E)$.

Proof. (1) $\Rightarrow$ (2). Let $g_E \in S_f(U,E)$. Denote

$$h_E = \underline{apr}_P(g_E),\ k_E = \overline{apr}_P(g_E) \text{ and } l_E = \overline{apr}_P(k_E).$$

We will prove $h_E \,\widetilde{\subset}\, \underline{apr}_P(h_E)$. We can suppose $P_g(f_E,R) \neq \emptyset$. $\forall\, (x_e)_E \in P_g(f_E,R)$, by Lemma 5.11, $P_g(f_E,R) = P_h(f_E,R)$, then $(x_e)_E \in P_h(f_E,R)$. Hence $\underline{apr}_P(g_E) \,\widetilde{\subset}\, \underline{apr}_P(\underline{apr}_P(g_E))$. By Proposition 5.7,
$\underline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, \overline{apr}_P(\overline{apr}_P(g_E))$.

Suppose that $\mathscr{L}(E) - \mathscr{K}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{L}(E) - \mathscr{K}(E)$. Then $(x_e)_E \in \mathscr{L}(E)$. By Lemma 5.3, $(x_e)_E \in P^k(f_E,R)$. This implies $R^*((x_e)_E) \cap \mathscr{K}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) \cap \mathscr{K}(E)$. Then $(x'_{e'})_E \in \mathscr{K}(E)$. By Lemma 5.3, $(x_e)_E \in P^g(f_E,R)$. This implies $R^*((x'_{e'})_E \cap \mathscr{G}(E) \neq \emptyset$. Thus $\mathscr{G}(E) \neq \emptyset$. Since $(x_e)_E \notin \mathscr{K}(E)$, by Lemma 5.3, we have $(x_e)_E \notin P^g(f_E,R)$. So $R^*((x_e)_E) \cap \mathscr{G}(E) = \emptyset$. since R is reflexive, $(x_e)_E \in R^*((x_e)_E)$. Thus $\mathscr{G}(E) = \emptyset$, a contradiction. Hence $\mathscr{L}(E) \subseteq \mathscr{K}(E)$. By Theorem 3.11, $l_E \,\widetilde{\subset}\, k_E$.

Therefore, $\overline{apr}_P(\overline{apr}_P(g_E)) \,\widetilde{\subset}\, \overline{apr}_P(g_E)$.

(2) $\Rightarrow$ (1). Let $(x_e)_E, (x'_{e'})_E, (x''_{e''})_E \in \mathscr{F}(E)$ with $(x_e)_E \in R^*((x'_{e'})_E)$ and $(x'_{e'})_E \in R^*((x''_{e''})_E)$. Denote

$$g_E = (x_e)_E,\ h_E = \overline{apr}_P(g_E) \text{ and } l_E = \overline{apr}_P(h_E).$$

By Lemma 5.3 and Lemma 5.5, $\mathscr{L}_E = P^h(f_E,R)$ and

$$\mathscr{H}(E) = \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}.$$

$(x_e)_E \in R^*((x'_{e'})_E)$ implies $(x_e)_E \in \mathscr{H}(E)$. Note that $(x'_{e'})_E \in R^*((x''_{e''})_E)$. Then $R^*((x''_{e''})_E) \cap \mathscr{H}(E) \neq \emptyset$. So $(x''_{e''})_E \in P^h(f_E,R) = \mathscr{L}(E)$.

Since $\overline{apr}_P(\overline{apr}_P((x_e)_E)) \,\widetilde{\subset}\, \overline{apr}_P((x_e)_E)$, $\mathscr{L}(E) \subseteq \mathscr{H}(E)$. Thus $(x''_{e''})_E \in \mathscr{H}(E)$. By Lemma 5.5, $(x_e)_E \in R^*((x''_{e''})_E)$.

Therefore, R is transitive. □

Proposition 5.13 *Let R be reflexive on $f_E \in S(U,E)$. Then the following are equivalent.*

(1) *R is Euclidean;*

(2) $\forall\, g_E \in S_f(U,E)$,
$\overline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, \underline{apr}_P(g_E) \,\widetilde{\subset}\, \overline{apr}_P(g_E) \,\widetilde{\subset}\, \underline{apr}_P(\overline{apr}_P(g_E))$.

Proof. (1) $\Rightarrow$ (2). Let $g_E \in S_f(U,E)$. Denote

$$k_E = \underline{apr}_P(g_E),\ l_E = \overline{apr}_P(k_E) \text{ and } h_E = \overline{apr}_P(g_E).$$

Suppose $\mathscr{L}(E) - \mathscr{K}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{L}(E) - \mathscr{K}(E)$. Then $(x_e)_E \notin \mathscr{K}(E)$ and $(x_e)_E \in \mathscr{L}(E)$. By Lemma 5.3, $(x_e)_E \in P^k(f_E,R)$. This implies $R^*((x_e)_E) \cap \mathscr{K}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) \cap \mathscr{K}(E)$. Then $(x'_{e'})_E \in \mathscr{K}(E)$. By Lemma 5.3, $(x_e)_E \in P_g(f_E,R)$. This implies $R^*((x'_{e'})_E \subseteq \mathscr{G}(E)$. Since R is Euclidean, $R^*((x_e)_E) \subseteq R^*((x'_{e'})_E)$. Thus $R^*((x_e)_E) \subseteq \mathscr{G}(E)$. Since $(x_e)_E \notin \mathscr{K}(E)$, by Lemma 5.3, $(x_e)_E \notin P_g(f_E,R)$. So $R^*((x_e)_E) \nsubseteq \mathscr{G}(E)$, a contradiction. Hence $\mathscr{L}(E) \subseteq \mathscr{K}(E)$. By Theorem 3.11, $l_E \,\widetilde{\subset}\, k_E$.

Therefore, $\overline{apr}_P(\underline{apr}_P(g_E)) \,\widetilde{\subset}\, \underline{apr}_P(g_E)$.

By Proposition 5.7, $\underline{apr}_P(g_E) \,\widetilde{\subset}\, \overline{apr}_P(g_E)$.

We will prove $h_E \,\widetilde{\subset}\, \underline{apr}_P(h_E)$. Suppose $P^g(f_E,R) \neq \emptyset$. $\forall\, (x_e)_E \in P^g(f_E,R)$, by Lemma 5.11, $P^g(f_E,R) \subseteq P_h(f_E,R)$. Then $(x_e)_E \in P_h(f_E,R)$. Hence $\overline{apr}_P(g_E) \,\widetilde{\subset}\, \underline{apr}_P(\overline{apr}_P(g_E))$.

(2) $\Rightarrow$ (1) Let $(x_e)_E, (x'_{e'})_E, (x''_{e''})_E \in \mathscr{F}(E)$ with $(x_e)_E \in R^*((x''_{e''})_E)$ and $(x'_{e'})_E \in R^*((x''_{e''})_E)$. Denote

$$g_E = (x_e)_E,\ h_E = \overline{apr}_P(g_E) \text{ and } l_E = \underline{apr}_P(h_E).$$

By Lemma 5.3 and Lemma 5.5, $\mathscr{L}_E = P_h(f_E,R)$ and

$$\mathscr{H}(E) = \{(y_\varepsilon)_E \in \mathscr{F}(E) : (x_e)_E \in R^*((y_\varepsilon)_E)\}.$$

$(x_e)_E \in R^*((x''_{e''})_E)$ implies $(x''_{e''})_E \in \mathscr{H}(E)$. Since $\overline{apr}_P(g_E) \,\widetilde{\subset}\, \underline{apr}_P(\overline{apr}_P(g_E)))$, $\mathscr{H}(E) \subseteq \mathscr{L}(E)$. So $(x''_{e''})_E \in \mathscr{L}(E) = P_h(f_E,R)$. Thus $R^*((x''_{e''})_E) \subseteq \mathscr{H}(E)$. Note that $(x'_{e'})_E \in R^*((x''_{e''})_E)$. Then $(x'_{e'})_E \in \mathscr{H}(E)$. By Lemma 5.5, $(x_e)_E \in R^*((x'_{e'})_E)$.

Therefore, R is Euclidean. □

6. Soft topologies induced by s-relations on special soft sets

In this section, we investigate soft topologies induced by a reflexive s-relation on a special soft set and give their structure.

6.1. s-relations on $\widetilde{U}$

Proposition 6.1 *Let R be a s-relation on $\widetilde{U}$. Then $\forall\, g_E \in S(U,E)$, we have*
(1) $\underline{apr}_P(g_E) = \widetilde{U} - \overline{apr}_P(\widetilde{U} - g_E)$;
(2) $\overline{apr}_P(g_E) = \widetilde{U} - \underline{apr}_P(\widetilde{U} - g_E)$.

Proof. (1) Denote $h_E = \widetilde{U} - g_E$, $q_E = \overline{apr}_P(h_E)$, $l_E = \widetilde{U} - q_E$ and $k_E = \underline{apr}_P(g_E)$.

To prove $l_E \widetilde{\subset} k_E$, by Theorem 3.11, it suffices to show $\mathscr{L}(E) \subseteq \mathscr{K}(E)$.

Suppose $\mathscr{L}(E) - \mathscr{K}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{L}(E) - \mathscr{K}(E)$. Then $(x_e)_E \notin \mathscr{K}(E)$. By Lemma 5.3, $(x_e)_E \notin P_g(f_E,R)$. Thus $R^*((x_e)_E) \not\subseteq \mathscr{G}(E)$. Pick $(x'_{e'})_E \in R^*((x_e)_E) - \mathscr{G}(E)$. Then $(x'_{e'})_E \notin \mathscr{G}(E)$. This implies $x' \notin g(e')$. So $x' \in U - g(e') = h(e')$.

Since $(x_e)_E \in \mathscr{L}(E)$, $(x_e)_E \in \mathscr{U}(E) - \mathscr{Q}(E)$. Then $(x_e)_E \notin \mathscr{Q}(E)$. By Lemma 5.3, $(x_e)_E \notin P^h(f_E,R)$. So $R^*((x_e)_E) \cap \mathscr{H}(E) = \emptyset$. Since $(x'_{e'})_E \in R^*((x_e)_E)$, $(x'_{e'})_E \notin \mathscr{H}(E)$. This implies $x' \notin h(e')$, a contradiction. Hence $\mathscr{L}(E) \subseteq \mathscr{K}(E)$.

Therefore, $\widetilde{U} - \overline{apr}_P(\widetilde{U} - g_E) \widetilde{\subset} \underline{apr}_P(g_E)$.

Conversely, to prove $k_E \widetilde{\subset} l_E = \widetilde{U} - q_E$, by Remark 2.11, it suffices to show $k_E \widetilde{\cap} q_E = \widetilde{\emptyset}$.

Suppose $w_E = k_E \widetilde{\cap} q_E \neq \widetilde{\emptyset}$. By Proposition 3.5, $\mathscr{W}(E) = \mathscr{K}(E) \cap \mathscr{Q}(E)$. Pick $(x_e)_E \in \mathscr{W}(E)$. Then $(x_e)_E \in \mathscr{K}(E)$ and $(x_e)_E \in \mathscr{Q}(E)$. By Lemma 5.3, $(x_e)_E \in P_g(f_E,R)$ and $(x_e)_E \in P^h(f_E,R)$. Thus $R^*((x_e)_E) \subseteq \mathscr{G}(E)$ and $R^*((x_e)_E) \cap \mathscr{H}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) \cap \mathscr{H}(E)$. Then $(x'_{e'})_E \in \mathscr{H}(E)$. Thus $x' \in h(e') = U - g(e')$ and so $x' \notin g(e')$. But $(x'_{e'})_E \in R^*((x_e)_E)$. This implies $(x'_{e'})_E \in \mathscr{G}(E)$. Thus $x' \in g(e')$, a contradiction. Hence $\underline{apr}_P(g_E) \widetilde{\subset} \widetilde{U} - \overline{apr}_P(\widetilde{U} - g_E)$.

Therefore, $\underline{apr}_P(g_E) = \widetilde{U} - \overline{apr}_P(\widetilde{U} - g_E)$.

(2) Denote $h_E = \widetilde{U} - g_E$, $q_E = \underline{apr}_P(h_E)$, $l_E = \widetilde{U} - q_E$ and $k_E = \overline{apr}_P(g_E)$.

To prove $l_E \widetilde{\subset} k_E$, by Theorem 3.11, it suffices to show $\mathscr{L}(E) \subseteq \mathscr{K}(E)$.

Suppose $\mathscr{L}(E) - \mathscr{K}(E) \neq \emptyset$. Pick $(x_e)_E \in \mathscr{L}(E) - \mathscr{K}(E)$. Then $(x_e)_E \notin \mathscr{K}(E)$. By Lemma 5.3, $(x_e)_E \notin P^g(f_E,R)$. Then $R^*((x_e)_E) \cap \mathscr{G}(E) = \emptyset$.

Since $(x_e)_E \in \mathscr{L}(E)$, $(x_e)_E \in \mathscr{U}(E) - \mathscr{Q}(E)$. Then $(x_e)_E \notin \mathscr{Q}(E)$. By Lemma 5.3, $(x_e)_E \notin P_h(f_E,R)$. Then $R^*((x_e)_E) \not\subseteq \mathscr{H}(E)$ and so $R^*((x_e)_E) - \mathscr{H}(E) \neq \emptyset$. Pick $(x'_{e'})_E \in R^*((x_e)_E) - \mathscr{H}(E)$. Then $(x'_{e'})_E \notin \mathscr{H}(E)$. Thus $x' \notin h(e') = g'(e') = U - g(e')$ and so $x' \in g(e')$.

Since $(x'_{e'})_E \in R^*((x_e)_E)$, $(x'_{e'})_E \notin \mathscr{G}(E)$. Thus $x' \notin g(e')$, a contradiction. Hence $\mathscr{L}(E) \subseteq \mathscr{K}(E)$.

Therefore, $\widetilde{U} - \underline{apr}_P(\widetilde{U} - g_E) \widetilde{\subset} \overline{apr}_P(g_E)$.

Conversely, to prove $k_E \widetilde{\subset} l_E = \widetilde{U} - q_E$, by Remark 2.11, it suffices to show $k_E \widetilde{\cap} q_E = \widetilde{\emptyset}$.

Suppose $w_E = k_E \widetilde{\cap} q_E \neq \widetilde{\emptyset}$. By Proposition 3.5, $\mathscr{W}(E) = \mathscr{K}(E) \cap \mathscr{Q}(E)$. Pick $(x_e)_E \in \mathscr{W}(E)$. Then $(x_e)_E \in \mathscr{K}(E)$ and $(x_e)_E \in \mathscr{Q}(E)$. By Lemma 5.3, $(x_e)_E \in P^g(f_E,R)$ and $(x_e)_E \in P_h(f_E,R)$. Thus $R^*((x_e)_E) \cap \mathscr{G}(E) \neq \emptyset$ and $R^*((x_e)_E) \subseteq \mathscr{H}(E)$. Pick $(x'_{e'})_E \in R^*((x_e)_E) \cap \mathscr{G}(E)$. Then $(x'_{e'})_E \in \mathscr{G}(E)$. Thus $x' \in g(e')$. But $(x'_{e'})_E \in R^*((x_e)_E)$. This implies $(x'_{e'})_E \in \mathscr{H}(E)$. Thus $x' \in h(e') = U - g(e')$ and so $x' \notin g(e')$, a contradiction. Hence $\overline{apr}_P(g_E) \widetilde{\subset} \widetilde{U} - \underline{apr}_P(\widetilde{U} - g_E)$.

Therefore, $\overline{apr}_P(g_E) = \widetilde{U} - \underline{apr}_P(\widetilde{U} - g_E)$. □

Corollary 6.2 *Let R be a s-relation on $\widetilde{U}$. Then the following are equivalent.*
(1) *R is serial;*
(2) $\forall\, g_E \in S(U,E)$, $\underline{apr}_P(g_E) \widetilde{\subset} \overline{apr}_P(g_E)$;
(3) $\underline{apr}_P(\widetilde{\emptyset}) = \widetilde{\emptyset}$;
(4) $\overline{apr}_P(\widetilde{U}) = \widetilde{U}$.

Proof. This follows from Proposition 5.7 and Proposition 6.1. □

Corollary 6.3 *Let R be a s-relation on $\widetilde{U}$. Then the following are equivalent.*
(1) *R is reflexive;*
(2) $\forall\, g_E \in S(U,E)$, $\underline{apr}_P(g_E) \widetilde{\subset} g_E$;
(3) $\forall\, g_E \in S(U,E)$, $g_E \widetilde{\subset} \overline{apr}_P(g_E)$.

Proof. This follows from Proposition 5.8 and Proposition 6.1. □

Corollary 6.4 *Let R be a s-relation on $\widetilde{U}$. Then the following are equivalent.*
(1) *R is symmetric;*
(2) $\forall\, g_E \in S_f(U,E)$, $g_E \widetilde{\subset} \underline{apr}_P(\overline{apr}_P(g_E))$;
(3) $\forall\, g_E \in S_f(U,E)$, $\overline{apr}_P(\underline{apr}_P(g_E)) \widetilde{\subset} g_E$.

Proof. This follows from Proposition 5.10 and Proposition 6.1. □

Corollary 6.5 *Let R be a s-relation on $\widetilde{U}$. Then the following are equivalent.*

(1) *R is transitive;*

(2) $\underline{apr}_P(g_E) \tilde{\subset} \underline{apr}_P(\underline{apr}_P(g_E)) \ \forall \ g_E \in S_f(U,E)$;

(3) $\overline{apr}_P(\overline{apr}_P(g_E)) \tilde{\subset} \overline{apr}_P(g_E) \ \forall \ g_E \in S_f(U,E)$.

Proof. This follows from Proposition 5.12 and Proposition 6.1. □

Corollary 6.6 *Let R be a s-relation on $\widetilde{U}$. Then the following are equivalent.*

(1) *R is Euclidean;*

(2) $\forall \, g_E \in S_f(U,E)$, $\overline{apr}_P(g_E) \tilde{\subset} \underline{apr}_P(\overline{apr}_P(g_E))$;

(3) $\forall \, g_E \in S_f(U,E)$, $\overline{apr}_P(\underline{apr}_P(g_E)) \tilde{\subset} \underline{apr}_P(g_E)$.

Proof. This follows from Proposition 5.13 and Proposition 6.1. □

6.2. Soft topologies induced by relations on $\widetilde{U}$

Theorem 6.7 *Let R be reflexive on $\widetilde{U}$. Then $\tau_R = \{g_E \in S(U,E) : \underline{apr}_P(g_E) = g_E\}$ is a soft topology over U.*

Proof. (1) By Proposition 5.9, $\widetilde{\emptyset}, \widetilde{U} \in \tau_R$.

(2) Let g_E, $h_E \in \tau_R$. Since $g_E = \underline{apr}_P(g_E)$ and $h_E = \underline{apr}_P(h_E)$, by Proposition 5.6, $g_E \tilde{\cap} h_E = \underline{apr}_P(g_E) \tilde{\cap} \underline{apr}_P(h_E) = \underline{apr}_P(g_E \tilde{\cap} h_E)$.

(3) Let $(g_\alpha)_E \in \tau_R \ \forall \ \alpha \in \bigwedge$, we will show that $\tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\} = \underline{apr}_P(\tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\})$. Since R is reflexive, by Proposition 5.8, $\underline{apr}_P(\tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\}) \tilde{\subset} \tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\}$.

Conversely, since $(g_\alpha)_E = \underline{apr}_P((g_\alpha)_E)$, by Proposition 5.6, we have $\tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\} = \tilde{\cup} \{\underline{apr}_P((g_\alpha)_E) : \alpha \in \bigwedge\} \tilde{\subset} \underline{apr}_P(\tilde{\cup} \{(g_\alpha)_E : \alpha \in \bigwedge\})$.

Therefore, $\tau_R = \{g_E \in S_f(U,E) : \underline{apr}_P(g_E) = g_E\}$ is a soft topology on f_E. □

Definition 6.8 *Let R be reflexive on $\widetilde{U}$. Then τ_R is called the soft topology induced by R on $\widetilde{U}$.*

The following Theorem 6.9 gives the structure of the soft topology induced by a reflexive *s*-relation on $\widetilde{U}$.

Theorem 6.9 *Let R be reflexive on $\widetilde{U}$ and τ_R the soft topology induced by R on U. Then*

(1) *a)* $\tau_R = \{\underline{apr}_P(g_E) : g_E \in S(U,E)\}$ *whenever R is transitive.*

b) $\{\overline{apr}_P(g_E) : g_E \in S(U,E)\} \subseteq \tau_R$ *whenever R is Euclidean.*

(2) *$\underline{apr}_P$ is a soft interior operator of τ_R.*

(3) *$\overline{apr}_P$ is a soft closure operator of τ_R.*

Proof. (1) *a)* Let $g_E \in S(U,E)$. By Corollary 6.5, $\underline{apr}_P(\underline{apr}_P(g_E)) = \underline{apr}_P(g_E)$. This implies $\underline{apr}_P(g_E) \in \tau_R$. Thus $\tau_R \supseteq \{\underline{apr}_P(g_E) : g_E \in S(U,E)\}$. Hence $\tau_R = \{\underline{apr}_P(g_E) : g_E \in S(U,E)\}$.

b) By Corollary 6.6, $\{\overline{apr}_P(g_E) : g_E \in S(U,E)\} \subseteq \tau_R$.

(2) It suffices to show $\underline{apr}_P(g_E) = int(g_E)$ for any $g_E \in S(U,E)$.

By (1), $\underline{apr}_P(g_E) \in \tau_R$. By Corollary 6.3, $\underline{apr}_P(g_E) \tilde{\subset} g_E$. Thus $\underline{apr}_P(g_E) \tilde{\subset} int(g_E)$.

Conversely, suppose $h_E \in \tau_R$ and $h_E \tilde{\subset} g_E$, by Proposition 5.6, $h_E = \underline{apr}_P(h_E) \tilde{\subset} \underline{apr}_P(g_E)$. By Remark 2.5,

$$int(g_E) = \tilde{\cup} \{h_E : h_E \in \tau_R \text{ and } h_E \tilde{\subset} g_E\} \tilde{\subset} \underline{apr}_P(g_E).$$

Thus $\underline{apr}_P(g_E) = int(g_E)$.

(3) By Proposition 2.17 and Proposition 6.1,

$\overline{apr}_P(g_E) = \widetilde{U} - \underline{apr}_P(\widetilde{U} - g_E) = \widetilde{U} - int(\widetilde{U} - g_E) = cl(g_E)$. □

Theorem 6.10 *Let R be reflexive and transitive on $\widetilde{U}$ and τ_R the soft topology induced by R on $\widetilde{U}$. Then $\forall \, g_E \in S(U,E)$, $g_E \in \tau_R \Leftrightarrow g_E \in \tau'_R$.*

Proof. Necessity. Let $g_E \in \tau_R$. Then $\underline{apr}_P(g_E) = g_E$. By Proposition 6.1 and Remark 2.9, $\overline{apr}_P(g'_E) = \widetilde{U} - \underline{apr}_P((g'_E)') = \widetilde{U} - \underline{apr}_P(g_E) = \widetilde{U} - g_E = g'_E$. By Theorem 6.9, $g'_E = \overline{apr}_P(g'_E) \in \tau_R$. Thus $g_E \in \tau'_R$.

Sufficiency. Let $g_E \in \tau'_R$. Then $g'_E \in \tau_R$ and $\underline{apr}_P(g'_E) = g'_E$. By Proposition 6.1 and Remark 2.9, $\overline{apr}_P(g_E) = \widetilde{U} - \underline{apr}_P(g'_E) = g_E$. By Theorem 6.5, $g_E = \overline{apr}_P(g_E) \in \tau_R$. □

Definition 6.11 *Let τ be a topology on U. τ is called a pseudo-discrete topology on U, if $A \subseteq U$ is open in U if and only if A is closed in U.*

Theorem 6.12 *Let R be reflexive and transitive on $\widetilde{U}$. Then τ_R is a pseudo-discrete topology over U.*

Proof. This holds by Theorem 6.7 and Theorem 6.10. □

7. Conclusions

In this paper, the fact that soft sets can be translated into soft point sets has been proved. Thus, we may expediently handled soft set like ordinary sets. We have proposed s-relations on soft sets. By means of soft points and s-relations, a pair of soft rough approximate operations has been defined. Serial, reflexive, symmetric, transitive and Euclidean s-relations have been characterized by using soft rough approximate operations. In addition, we have investigated soft topologies induced by a reflexive s-relation on a special soft set and given their structure. In the future, we will investigate the axiomatization of the proposed soft rough approximate operations and consider some concrete applications of our proposed notions.

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References

1. A.Aygünoğlu, H.Aygün, "Introduction to fuzzy soft groups," *Comput. Math. Appl.*, **58**, 1279-1286 (2009).
2. H.Aktas, N.Cağman, "Soft sets and soft groups," *Inform. Sci.*, **177**, 2726-2735 (2007).
3. M.I.Ali, F.Feng, X.Liu, W.K.Min, M.Shabir, "On some new operations in soft set theory," *Comput. Math. Appl.*, **57**, 1547-1553 (2009).
4. M.I.Ali, M.Shabir, M.Naz, "Algebraic structures of soft sets associated with new operations," *Comput. Math. Appl.*, **61**, 2647-2654 (2011).
5. K.V.Babitha, J.J.Sunil, "Soft set relations and functions," *Comput. Math. Appl.*, **60**, 1840-1849 (2010).
6. F.Feng, C.Li, B.Davvaz, M.I.Ali, "Soft sets combined with fuzzy sets and rough sets: a tentative approach," *Soft Comput.*, **14**, 899-911 (2010).
7. F.Feng, X.Liu, V.Leoreanu-Fotea, Y.B.Jun, "Soft sets and soft rough sets," *Inform. Sci.*, **181**, 1125-1137 (2011).
aking," *J. Comput. Appl. Math.*, **234**, 10-20 (2010).
8. X.Ge, Z.Li, Y.Ge, "Topological spaces and soft sets," *J. Comput. Anal. Appl.*, **13**, 881-885 (2011).
9. S.Hussain, B.Ahmad, "Some properties of soft topological spaces," *Comput. Math. Appl.*, **62**, 4058-4067 (2011).
10. Y.Jiang, Y.Tang, Q.Chen, J.Wang, S.Tang, "Extending soft sets with description logics," *Comput. Math. Appl.*, **59** , 2087-2096 (2010).
11. Y.Liu, M.Luo, "Fuzzy Topology", World Scientific Publishing, Singapore, 1998.
12. Z.Li, G.Wen, Y.Han, "Decision making based on intuitionistic fuzzy soft sets and its algorithm," *J. Comput. Anal. Appl.*, **17**, 620-631 (2014).
13. Z.Li, T.Xie, "Roughness of fuzzy soft sets and related results," *Int. J. Comput. Intell. Syst.*, **8**, 278-296 (2015).
14. Z.Li, T.Xie, "The relationship among soft sets, soft rough sets and topologies," *Soft Comput.*, **18**, 717-728 (2014).
15. Z.Li, N.Xie, G.Wen, "Soft coverings and their parameter reductions," *Appl. Soft Comput.*, **31**, 48-60 (2015).
16. Z.Li, D.Zheng, J.Hao, "*L*-fuzzy soft sets based on complete Boolean lattices," *Comput. Math. Appl.*, **64**, 2558-2574 (2012).
17. G.Liu, W.Zhu, "The algebraic structures of generalized rough sets," *Inform. Sci.*, **178**, 4105-4113 (2008).
18. P.K.Maji, R.Biswas, A.R.Roy, "Soft set theory," *Comput. Math. Appl.*, **45**, 555-562 (2003).
19. P.K.Maji, R.Biswas, A.R.Roy, "Fuzzy soft sets," *J.*

Fuzzy Math., **9**, 589-602 (2001).
20. P.K.Maji, A.R.Roy, "An application of soft sets in a decision making problem," *Comput. Math. Appl.*, **44**, 1077-1083 (2002).
21. D.Molodtsov, "Soft set theory-first result," *Comput. Math. Appl.*, **37**, 19-31 (1999).
22. Z.Pawlak, "Rough sets: Theoretical sspects of reasoning about data", Kluwer Academic Publishers, Boston, 1991.
23. B.Pu, Y.Liu, "Fuzzy Topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence," *J. Math. Anal. Appl.*, **76**, 571-599 (1980).
24. A.R.Roy, P.K.Maji, "A fuzzy soft set theoretic approach to decision making problems," *J. Comput. Appl. Math.*, **2003**, 412-418 (2007).
25. M.Shabir, M.Naz, "On soft topological spaces," *Comput. Math. Appl.*, **61**, 1786-1799 (2011).
26. Y.Y.Yao, "Constructive and algebraic methods of the theory of rough sets," *Inform. Sci.*, **109**, 21-47 (1998).
27. L.A.Zadeh, "Fuzzy sets," *Inform. Control*, **8**, 338-353 (1965).
28. X.Zhang, J.Dai, Y.Yu, "On the union and intersection operations of rough sets based on various approximation spaces," *Inform. Sci.*, **292**, 214-229 (2015).
29. X.Zhang, D.Miao, C.Liu, M.Le, "Constructive methods of rough approximation operators and multigranulation rough sets," *Knowl-Based Syst.*, **91**, 114-125 (2016).
30. W.Zhang, W.Wu, J.Liang, D.Li, "Rough set theorey and methods", Chinese Scientific Publishers, Beijing, 2001.