The Additive Weibull-Geometric Distribution: Theory and Applications

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In this paper, we introduce a new class of lifetime distributions which is called the additive Weibull geometric (AWG) distribution. This distribution obtained by compounding the additive Weibull and geometric distributions. The new distribution has a number of well-known lifetime special sub-models such as modified Weibull geometric, Weibull geometric, exponential geometric, among several others. Some structural properties of the proposed new distribution are discussed. We propose the method of maximum likelihood for estimating the model parameters and obtain the observed information matrix. A real data set is used to illustrate the importance and flexibility of the new distribution.

Keywords: Additive Weibull distribution, Geometric distribution, Moments, Maximum likelihood.

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1. Introduction

The study of life length of organisms, structures, materials, etc., is very important in the biological and engineering sciences. A substantial part of such study is devoted to modeling the lifetime data by a failure distribution. The exponential, Rayleigh, and Weibull distributions are the most commonly used distributions in reliability and life testing. These distributions have several desirable properties and nice physical interpretations. Unfortunately, however, the exponential and Rayleigh distributions have constant and increasing failure rates, respectively. The Weibull distribution generalizes both these distributions which may have increasing, constant, or decreasing failure rates. The modified Weibull distribution (Lai et al. (2003)) attracts some interest among researchers and practitioners because of its ability in modeling bathtub shaped failure rate data. Also, Sarhan and Zaindin (2009) introduced the modified Weibull distribution. Xie and Lai

(1995) introduced a four-parameter distribution called the additive Weibull distribution based on the simple idea of combining the hazard rates of two Weibull distributions: one has a decreasing hazard rate and the other one has an increasing hazard rate.

The Weibull distribution, having exponential and Rayleigh as special cases, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub-shaped and the unimodal failure rates which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. An example of bathtub-shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

In this article, we propose a new lifetime distribution, referred to as the additive Weibull geometric (AWG) distribution which contains as special sub-models and study some of its properties. Several distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions. Adamidis and Loukas (1998) introduced a two-parameter exponential-geometric (EG) distribution by compounding an exponential distribution with a geometric distribution. In the same way, the exponential Poisson (EP) and exponential logarithmic (EL) distributions were introduced and studied by Kus (2007) and Tahmasbi and Rezaei (2008), respectively. Recently, Chahkandi and Ganjali (2009) proposed the exponential power series (EPS) family of distributions, which contains as special cases these distributions. Barreto-Souza et al. (2010) introduced the Weibull-geometric (WG). Mahmoudi and Shiran (2012) presented exponentiated Weibull-Geometric distribution and its Applications. Nadarajah et al. (2014) introduced A new four-parameter lifetime distribution (generalized linear failure rate geometric distribution). Wang and Elbatal, (2014) proposed modified Weibull geometric distribution. (Adamidis et al. (2005) proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the EEG distribution can be monotone decreasing, increasing or constant.

The paper is organized as follows. In Section 2, we define the *AWG* distribution, its probability density function (*pdf*), cumulative distribution function (*cdf*) and special cases of the *AWG* distribution. In Section 3, some properties of the new distribution are given. Residual life and reversed residual functions of the *AWG* distribution, Bonferroni and Lorenz Curves and mean deviations are discussed in Section 4. In section 5. we demonstrate the maximum likelihood estimates of the unknown parameters. Finally, In section 6 we present a data analysis to illustrate the usefulness of the proposed distribution.

2. A New Class

Our class can be derived as follows. Suppose that a company has *N* systems functioning independently and producing a certain product at a given time, where *N* is a random variable, which is often determined by economy, customers demand, etc. The reason for considering *N* as a random variable comes from a practical viewpoint in which failure (of a device for example) often occurs due to the present of an unknown number of initial defects in the system. In this paper, we focus on the case in which *N* is discrete random variable following a geometric distribution (truncated at zero) with the probability mass function given by

$$P(n; p) = P(N = n) = (1 - p)p^{n-1}, for\ n \in N\ and\ p\ \in (0,1). \tag{1}$$

Note that *N* can also be taken to follow other discrete distributions, such as binomial, Poisson, logarithmic, etc, whereas they need to be truncated zero because one must have $N \geq 1$. Now, let $X_1, X_2, \ldots, X_N$ be *N* independent and identically distributed (iid) random variables from additive Weibull distribution with cumulative distribution function(cdf) is given by

$$F(x) = 1 - e^{-(\theta x^{\alpha} + \beta x^{\lambda})}, \ x \geq 0 \tag{2}$$

where $\theta > 0, \beta > 0$ and $\alpha > \lambda > 0$,or $\lambda > \alpha > 0$ which gives identifiability to the model. When $\alpha > 0$ the hazard rate is increasing and when $0 < \lambda < 1$ hazard rate is decreasing. The corresponding probability density function is

$$f(x, \theta, \alpha, \beta, \lambda) = \left(\theta \alpha \, x^{\alpha-1} + \beta \lambda x^{\lambda-1}\right) e^{-(\theta x^{\alpha} + \beta x^{\lambda})}, \tag{3}$$

Here θ and β are scale parameters, and α and λ are shape parameters. The interpretation of model (2) is evident. Suppose a system composed of two interconnected independent series sub-systems that affect the system in a different way, each one having a Weibull distribution with proper parameters. The hazard time of the system follows (2), since it occurs when the first of the two sub-systems fails. Let $X_{(1)} = min(X_i)_{i=1}^{N}$,then the conditional cumulative distribution of $X_{(1)}|N = n$ is given by

$$G_{\{X^{(1)}|N=n\}}(x) = 1 - [1 - F(x)]^{n} = 1 - e^{-n(\theta x^{\alpha} + \beta x^{\lambda})},$$

the cumulative distribution function of $X_{(1)}$ is given by

$$F(x; \theta, \alpha, \beta, \lambda, p) = (1 - p) \sum_{n=1}^{\infty} p^{n-1} \left[1 - e^{-n(\theta x^{\alpha} + \beta x^{\lambda}),}\right]$$

$$= \frac{1 - e^{-(\theta x^{\alpha} + \beta x^{\lambda})}}{1 - pe^{-(\theta x^{\alpha} + \beta x^{\lambda})}}, \quad x \geq 0 \tag{4}$$

The corresponding probability density function (pdf) of $X_{(1)}$ is given by

$$f(x, \theta, \alpha, \beta, \lambda, p) = \frac{(1 - p)\left(\theta \alpha \, x^{\alpha-1} + \beta \lambda x^{\lambda-1}\right) e^{-(\theta x^{\alpha} + \beta x^{\lambda})}}{\left(1 - pe^{-(\theta x^{\alpha} + \beta x^{\lambda})}\right)^{2}}, \quad x \geq 0 \tag{5}$$

We shall refer to the distribution given by Equations (4) and (5) as the as the additive Weibull geometric (AWG) distribution. If a random variable X has the AWG distribution, then we write $X \frown AWG(\theta, \alpha, \beta, \lambda, p)$.

The survival, hazard (hf) and reverse hazard (rh) functions of the (AWG) distribution are given respectively by

$$\bar{F}(x; \theta, \alpha, \beta, \lambda, p) = \frac{(1 - p)e^{-(\theta x^{\alpha} + \beta x^{\lambda})}}{1 - pe^{-(\theta x^{\alpha} + \beta x^{\lambda})}},$$

$$h(x) = \frac{f(x)}{(1 - F(x))} = \frac{\left(\theta \alpha \, x^{\alpha-1} + \beta \lambda x^{\lambda-1}\right)}{1 - pe^{-(\theta x^{\alpha} + \beta x^{\lambda})}}, \tag{6}$$

and

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{(1 - p)\left(\theta \alpha \, x^{\alpha-1} + \beta \lambda x^{\lambda-1}\right) e^{-(\theta x^{\alpha} + \beta x^{\lambda})}}{\{1 - pe^{-(\theta x^{\alpha} + \beta x^{\lambda})}\}\{1 - e^{-(\theta x^{\alpha} + \beta x^{\lambda})}\}}. \tag{7}$$

respectively.

Figure 1 (a) and (b) provide some plots of the density curves for different values of the parameters $\theta, \alpha, \beta, \lambda$ and p
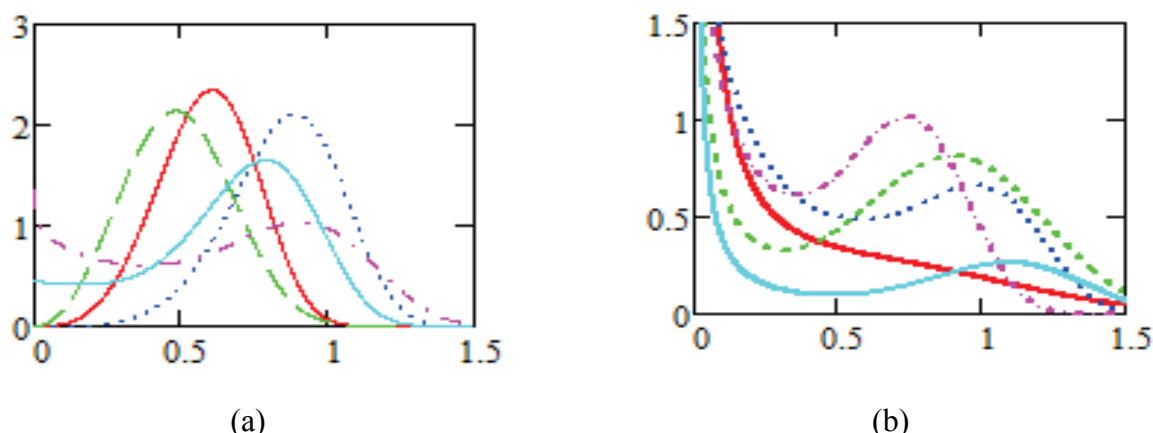
(a)	(b)

Fig.1. Plots of the AWG density function for some parameter values. (a) For different values of $\theta, \alpha, \beta, \lambda$ with $p = 0.4$.(b) For different values of $\theta, \alpha, \beta, \lambda$ with $p = 0.1$.

Figure 2 does the same for the associated hazard rate function, showing that it is quite flexible for modelling survival data.
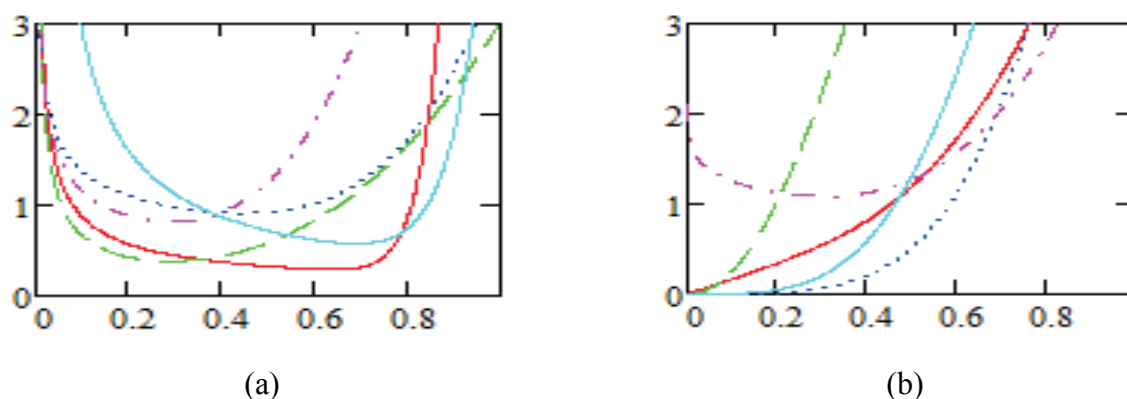

(a)	(b)

Fig. 2. (a) and (b) are plots of the hazard rate function for some parameter values. (a) For different values of $\theta, \alpha, \beta, \lambda$ with $p = 0.4$.(b) For different values of $\theta, \alpha, \beta, \lambda$ with $p = 0.1$.

It is known, not many lifetime distributions exhibit bathtub hazard rates. The AWG model shows flexibility in accommodating all forms of the hazard rate function as seen from Figure (2) (by changing its parameter values) seems to be an important distribution that can be used.

2.1. *Special Cases of the AWG Distribution*

The exponentiated modified Weibull- geometric is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the exponentiated modified Weibull-geometric distribution is explained in the following. If X is a random variable with cdf (4), then we have the following cases.

Special Cases:

1- If $= 1$, then (4) reduces to the modified Weibull geometric which introduced by (Wang and Elbatal (2014)).

2- If $\theta = 0$ we get the Weibull geometric distribution which introduced by (Souza et al. (2012)).

3- $\alpha = 1$ and $\lambda = 2$ we get the linear failure rate geometric distribution.

4- For $\alpha = 1$ and $\beta = 0$ we get the exponential geometric distribution.

5- If $\theta = 0$ and $\lambda = 2$, we get the Rayleigh geometric distribution.

6- For $p \downarrow 0$ and $\alpha = 1$ we get the modified Weibull distribution.

7- For $p \downarrow 0$ and $\theta = 0$ we get the Weibull distribution.

8- For $\downarrow 0$, $\alpha = 1$ and $\lambda = 2$ we get the linear failure rate distribution.

9- For $\downarrow 0$, $\alpha = 1$ and $\beta = 0$ we get the exponential distribution.

10- For $p \downarrow 0$, $\theta = 0$ and $= 2$, we get the Rayleigh distribution.

Note that, using the fact that if $|z| < 1$ and $k > 0$; we have the series representation

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k + j)}{\Gamma(k)j!} z^j, \tag{8}$$

using (8) in equation (5) yields

$$f(x; \theta, \alpha, \beta, \lambda, p) = (1 - p)\left(\theta\alpha\, x^{\alpha-1} + \beta\lambda x^{\lambda-1}\right) \sum_{j=0}^{\infty} p^j(j + 1)\, e^{-(j+1)(\theta x^\alpha + \beta x^\lambda)} \tag{9}$$

but the series expansion of $e^{-(j+1)(\beta x^\lambda)}$ is given by

$$e^{-(j+1)(\beta x^\lambda)} = \sum_{k=0}^{\infty} \frac{\left[-\beta(j + 1)x^\lambda\right]^k}{k!}, \tag{10}$$

substituting from (10) into (9), we get

$$f(x; \theta, \alpha, \beta, \lambda, p) = (1 - p) \sum_{j,K=0}^{\infty} \frac{p^j(-\beta)^k(j + 1)^{k+1}x^{\lambda k}}{k!} e^{-(j+1)(\theta x^\alpha + \beta x^\lambda)}$$

$$\times \left(\theta\alpha\, x^{\alpha+\lambda k-1} + \beta\lambda x^{\lambda(k+1)-1}\right)e^{-(j+1)(\theta x^\alpha)}$$

$$= \omega_{(j,k)}\left(\theta\alpha\, x^{\alpha+\lambda k-1} + \beta\lambda x^{\lambda(k+1)-1}\right)e^{-(j+1)(\theta x^\alpha)}, \tag{11}$$

where

$$\omega_{(j,k)} = (1 - p) \sum_{j,K=0}^{\infty} \frac{p^j(-\beta)^k(j + 1)^{k+1}}{k!}.$$

3. Statistical Properties

In this section we discuss the statistical properties of the additive Weibull geometric distribution, in particular, moment, moment generating function and conditional moment.

3.1. *Moments*

In this subsection we discuss the r_{th} moment for $AWG(\theta, \alpha, \beta, \lambda, p)$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 1.

If X has $AWG\,(\phi, x)$, $\phi = (\theta, \alpha, \beta, \lambda, p)$ then the r_{th} moment of X is given by the following

$$\acute{\mu}_r(x) = \omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{r+\lambda k}{\alpha}+1\right)}{[(j+1)\theta]^{\frac{r+\lambda k+1}{\alpha}}} + \frac{\beta\lambda\Gamma\left(\frac{r+\lambda(k+1)}{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{r+\lambda(k+1)}{\alpha}}} \right] \qquad (12)$$

Proof:

Let X be a random variable with density function (11). The r_{th} ordinary moment of the *AWG* distribution is given by

$$\acute{\mu}_r(x) = E(X^r) = \int_0^\infty x^r f(x,\phi)dx$$

$$= \int_0^\infty \omega_{(j,k)} \left(\theta\alpha\, x^{r+\alpha+\lambda k-1} + \beta\lambda x^{r+\lambda(k+1)-1}\right) e^{-(j+1)(\theta x^\alpha)}\, dx$$

$$= \omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{r+\lambda k}{\alpha}+1\right)}{[(j+1)\theta]^{\frac{r+\lambda k+1}{\alpha}}} + \frac{\beta\lambda\Gamma\left(\frac{r+\lambda(k+1)}{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{r+\lambda(k+1)}{\alpha}}} \right]. \qquad (13)$$

Which completes the proof.

The central moments μ_r and cumulants κ_r of the *AWG* distribution can be determined from expression (12) as $\mu_r = \sum_{m=0}^r (-1)^m \acute{\mu}_1 \acute{\mu}_{r-m}$ and $\kappa_r = \acute{\mu}_r - \sum_{m=1}^{r-1} \binom{r-1}{m-1} k_m \acute{\mu}_{r-m}$, respectively, Additionally, the skewness and kurtosis can be calculated from the third and fourth standardized cumulants in the forms $SK = ((\kappa_3)/(\sqrt{(\kappa_2{}^3)}))$ and $KU = ((\kappa_4)/(\kappa_2{}^2))$, respectively.

Theorem 2.

The moment generating function of AWG distribution is given by

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{r+\lambda k}{\alpha}+1\right)}{[(j+1)\theta]^{\frac{r+\lambda k}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{r+\lambda(k+1)}{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{r+\lambda(k+1)}{\alpha}}} \right] \qquad (14)$$

Proof.

We start with the well-known definition of the moment generating function given by $M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{AWG}(x,\phi)$, since $\sum_{r=0}^\infty \frac{t^r}{r!} x^r f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(x^r)$ by replacing $E(x^r)$. Hence using (12) the *MGF* of *AWG* distribution is given by

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{r+\lambda k}{\alpha}+1\right)}{[(j+1)\theta]^{\frac{r+\lambda k}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{r+\lambda(k+1)}{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{r+\lambda(k+1)}{\alpha}}} \right].$$

which completes the proof.

Similarly, the characteristic function of the *AWG* distribution becomes $\varphi_X(t) = M_X(it)$ where $i = \sqrt{(-1)}$ is the unit imaginary number.

3.2. Conditional Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. For lifetime models, it is also of interest to find the conditional moments and the mean residual lifetime function. The conditional moments for AWG distribution is given by

$$v_s = E(X|X > t) - t = \int_t^\infty x^s f_{AWG}(x, \phi) dx$$

$$= \omega_{(j,k)} \int_t^\infty \left(\theta\alpha\, x^{s+\alpha+\lambda k-1} + \beta\lambda x^{s+\lambda(k+1)-1} \right) e^{-(j+1)(\theta x^\alpha)} dx$$

$$= \omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{s+\lambda k}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{s+\lambda(k+1)+1}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{s+\lambda(k+1)}{\alpha}}} \right]. \tag{14}$$

Where $\Gamma(s, t) = \int_t^\infty x^{s-1} e^{-x} dx$ is the upper incomplete gamma function. The mean residual lifetime function is given by

$$v_s = E(X|X > t) - t = \omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{\lambda k}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)}{\alpha}}} \right] - t. \tag{15}$$

The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate (FR) function. Lifetimes can exhibit $IMRL$ (increasing MRL) or $DMRL$ (decreasing MRL). MRL functions that first decreases (increases) and then increases (decreases) are usually called bathtub (upside-down bathtub) shaped, $BMRL$ ($UMRL$). Many authors such as Ghitany (1998), Mi (1995), Park (1985) and Tang et al. (1999) have been studied the relationship between the behaviors of the MRL and FR functions of a distribution.

4. Numbering and spacing

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X - t | X > t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely (Gupta and Gupta, 1983). Therefore, we obtain the rth-order moment of the residual lifetime can be obtained via the general formula

$$\mu_r(t) = E((X - t)^r | X > t) = \frac{1}{\bar{F}(t)} \int_t^\infty (x - t)^r f(x, \phi) dx, r \geq 1.$$

Applying the binomial expansion of $(x - t)^r$ into the above formula, we get

$$\mu_r(t) = \frac{1}{\bar{F}(t)} \sum_{d=0}^r (-t)^d \binom{r}{d} \int_t^\infty x^{r-d} f(x) dx$$

$$= \frac{\omega_{(j,k)}}{\bar{F}(t)} \sum_{d=0}^{r} (-t)^d \binom{r}{d} \int_{t}^{\infty} \left(\theta\alpha\, x^{r+\alpha+\lambda k-d-1} + \beta\lambda x^{r+\lambda(k+1)-d-1} \right) e^{-(j+1)(\theta x^\alpha)}\, dx$$

$$= \frac{\omega_{(j,k)}}{\bar{F}(t)} \sum_{d=0}^{r} (-t)^d \binom{r}{d} \left[\frac{\theta\Gamma\left(\frac{\lambda k}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)}{\alpha}}} \right] \quad (16)$$

The mean residual life (MRL) of the AWG distribution is given by

$$\mu(t) = \frac{\omega_{(j,k)}}{\bar{F}(t)} \left[\frac{\theta\Gamma\left(\frac{\lambda k}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)}{\alpha}}} \right] - t$$

The variance of the residual life of the AWG distribution can be obtained easily by using $\mu_2(t)$ and $\mu(t)$.

On the other hand, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable $t - X | X \leq t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable may also be called the inactivity time (or time since failure); for more details, you may (see, Kundu and Nanda, (2010) and Nanda et al. (2003). Also, in reliability, the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely. the reversed failure (or reversed hazard) rate function is given by Equation (2.5). The rth -order moment of the reversed residual life can be obtained by the well-known formula

$$m_r(t) = E((t-X)^r | X \leq t) = \frac{1}{F(t)} \int_{0}^{t} (t-X)^r f(x, \phi)\, dx, r \geq 1.$$

Applying the binomial expansion of $(t - X)^r$ into the above formula gives

$$m_r(t) = \frac{\omega_{(i,j,k)}}{F(t)} \sum_{d=0}^{r} (-t)^d \binom{r}{d} \int_{0}^{t} \left(\theta\alpha\, x^{r+\alpha+\lambda k-d-1} + \beta\lambda x^{r+\lambda(k+1)-d-1} \right) e^{-(j+1)(\theta x^\alpha)}\, dx$$

$$= \frac{\omega_{(i,j,k)}}{F(t)} \sum_{d=0}^{r} (-t)^d \binom{r}{d} \left[\frac{\theta\zeta\left(\frac{r+\lambda k-d}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{r+\lambda k-d}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{r+\lambda(k+1)-d}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{r+\lambda(k+1)-d}{\alpha}}} \right] (17)$$

where $\zeta(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function. Thus, the mean of the reversed residual life of the AWG distribution is given by

$$m(t) = t - \frac{\omega_{(j,k)}}{F(t)} \left[\frac{\theta\Gamma\left(\frac{\lambda k}{\alpha}+1, (j+1)\theta t^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta t^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)}{\alpha}}} \right] \quad (18)$$

Using $m(t)$ and $m_2(t)$ one can obtain the variance and the coefficient of variation of the reversed residual life of the AWG distribution.

4.1. *Bonferroni and Lorenz Curves*

In this subsection we proposed the Bonferroni and Lorenz Curves. The Bonferroni and Lorenz curves (Bonferroni 1930) and the Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined by

$$
\begin{aligned}
B(p) &= \frac{1}{p\mu} \int_0^q x f(x) dx \\
&= \frac{\omega_{(j,k)}}{p\mu} \left[\frac{\theta \zeta \Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta q^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta \lambda \zeta \Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta q^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right]
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
L(p) &= \frac{1}{\mu} \int_0^q x f(x) dx \\
&= \frac{\omega_{(j,k)}}{\mu} \left[\frac{\theta \zeta \Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta q^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta \lambda \zeta \Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta q^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right]
\end{aligned}
\tag{20}
$$

4.2. *Mean deviation*

In statistics, mean deviation about the mean and mean deviation about the median measure the ammount of scatter in a population. For random variable X with pdf (x), distribution function $F(x)$, mean $\mu = E(X)$ and $M = Median\ (X)$, the mean deviation about the mean and mean deviation about the median, are defined by

$$
\begin{aligned}
\delta_1(x) &= \int_0^\infty |x - \mu| f(x) dx \\
&= 2\mu F(\mu) - 2\mu + 2 \int_\mu^\infty x f(x) dx.
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2(x) &= \int_0^\infty |x - M| f(x) dx \\
&= 2MF(M) - M - \mu + 2 \int_M^\infty x f(x) dx.
\end{aligned}
$$

respectively, if X is AWG random variable then

$$
\int_\mu^\infty x f(x) dx = \omega_{(j,k)} \left[\frac{\theta \Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta \mu^\alpha\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta \lambda \Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta \mu^\alpha\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right],
\tag{21}
$$

and

$$\int\limits_{M}^{\infty} xf(x)dx = \omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta M^{\alpha}\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta\mu^{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right], \qquad (22)$$

so that

$$\delta^{1}(x) = 2\mu F(\mu) - 2\mu + 2\omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta\mu^{\alpha}\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta\mu^{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right],$$

and

$$\delta_{2}(x) = -\mu + 2\omega_{(j,k)} \left[\frac{\theta\Gamma\left(\frac{\lambda k + 1}{\alpha} + 1, (j+1)\theta M^{\alpha}\right)}{[(j+1)\theta]^{\frac{\lambda k+1}{\alpha}+1}} + \frac{\beta\lambda\Gamma\left(\frac{\lambda(k+1)+1}{\alpha}, (j+1)\theta M^{\alpha}\right)}{\alpha[(j+1)\theta]^{\frac{\lambda(k+1)+1}{\alpha}}} \right].$$

5. Maximum Likelihood Estimation

Statistical inference can be carried out in three different ways: point estimation, interval estimation and hypothesis testing. Several approaches for parameter point estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The *MLEs* enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. Large sample theory for these estimates delivers simple approximations that work well in finite samples. Statisticians often seek to approximate quantities such as the density of a test statistic that depend on the sample size in order to obtain better approximate distributions. The resulting approximation for the *MLEs* in distribution theory is easily handled either analytically or numerically. Here, we determine the maximum likelihood estimates (*MLEs*) of the parameters of the *AWG* distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the *AWG* distribution given by (5). Let $\varphi = (\theta, \alpha, \beta, \lambda, p)^{T}$ be $p \times 1$ vector of parameters. The total log-likelihood function for φ is given by

$$L_n = L_n(\emptyset) = n\log(1 - p) + \sum_{i=1}^{n} \log\left(\theta\alpha\, x_i^{\alpha-1} + \beta\lambda\, x_i^{\lambda-1}\right) - \theta\sum_{i=1}^{n} x_i^{\alpha} - \beta\sum_{i=1}^{n} x_i^{\lambda}$$
$$- 2\sum_{i=1}^{n} \log\left(1 - pe^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}\right). \qquad (23)$$

The log-likelihood can be maximized either directly by using the SAS program or R-language (2012) or by solving the nonlinear likelihood equations obtained by differentiating (23). The associated components of the score function
$U_n(\Phi) = [\frac{\partial L_n}{\partial\theta}, \frac{\partial L_n}{\partial\alpha}, \frac{\partial L_n}{\partial\beta}, \frac{\partial L_n}{\partial\lambda}, \frac{\partial L_n}{\partial p}]^{T}$ are

$$\frac{\partial L_n}{\partial\theta} = \sum_{i=1}^{n} \frac{\alpha\, x_i^{\alpha-1}}{\left(\theta\alpha\, x_i^{\alpha-1} + \beta\lambda\, x_i^{\lambda-1}\right)} - \sum_{i=1}^{n} x_i^{\alpha} - 2p\sum_{i=1}^{n} \frac{x_i^{\alpha} e^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}}{\left(1 - pe^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}\right)}, \qquad (24)$$

$$\frac{\partial L_n}{\partial\alpha} = \sum_{i=1}^{n} \frac{\theta\, x_i^{\alpha-1}(\alpha\ln(x_i)+1)}{\left(\theta\alpha\, x_i^{\alpha-1} + \beta\lambda\, x_i^{\lambda-1}\right)} - \theta\sum_{i=1}^{n} x_i^{\alpha}\ln(x_i) - 2p\sum_{i=1}^{n} \frac{\theta x_i^{\alpha}\ln(x_i)\, e^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}}{\left(1 - pe^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}\right)}, \qquad (25)$$

$$\frac{\partial L_n}{\partial\beta} = \sum_{i=1}^{n} \frac{\lambda\, x_i^{\lambda-1}}{\left(\theta\alpha\, x_i^{\alpha-1} + \beta\lambda\, x_i^{\lambda-1}\right)} - \sum_{i=1}^{n} x_i^{\lambda} - 2p\sum_{i=1}^{n} \frac{x_i^{\lambda} e^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}}{\left(1 - pe^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}\right)}, \qquad (26)$$

$$\frac{\partial L_n}{\partial\lambda} = \sum_{i=1}^{n} \frac{\beta\, x_i^{\lambda-1}(\alpha\ln(x_i)+1)}{\left(\theta\alpha\, x_i^{\alpha-1} + \beta\lambda\, x_i^{\lambda-1}\right)} - \beta\sum_{i=1}^{n} x_i^{\lambda}\ln(x_i) - 2p\sum_{i=1}^{n} \frac{\beta x_i^{\lambda}\ln(x_i)\, e^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}}{\left(1 - pe^{-(\theta x_i^{\alpha} + \beta x_i^{\lambda})}\right)}, \qquad (27)$$

and

$$\frac{\partial L_n}{\partial p} = \frac{-n}{1-p} + 2\sum_{i=1}^{n}\frac{e^{-(\theta x_i{}^\alpha + \beta x_i{}^\lambda)}}{\left(1 - pe^{-(\theta x_i{}^\alpha + \beta x_i{}^\lambda)}\right)}. \tag{28}$$

And The maximum likelihood estimation (MLE) of ϕ, say $\hat{\phi}$, is obtained by solving the nonlinear system $U_n(\phi) = 0$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton--Raphson type algorithm to obtain the estimate $\hat{\phi}$. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 5×5 observed information matrix is given by

$$I_n(\varphi) = - \begin{pmatrix} I_{\theta\theta} & I_{\theta\alpha} & I_{\theta\beta} & I_{\theta\lambda} & I_{\theta p} \\ I_{\alpha\theta} & I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} & I_{\alpha p} \\ I_{\beta\beta} & I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} & I_{\beta p} \\ I_{\lambda\theta} & I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} & I_{\lambda p} \\ I_{p\theta} & I_{p\alpha} & I_{p\beta} & I_{p\lambda} & I_{pp} \end{pmatrix}$$

whose elements are given in Appendix. Applying the usual large sample approximation, MLE of ϕ, i.e $\hat{\phi}$ can be treated as being approximately $N_5(\phi, J_n(\phi)^{-1})$, where $J_n(\phi) = E[I_n(\phi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\phi} - \phi)$ is $_5(0, J(\phi)^{-1})$, where $J(\phi) = lim\_\{n \to \infty\}n^{-1}I_n(\phi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\phi)$ is replaced by the average sample information matrix evaluated at $\hat{\phi}$, say $n^{-1}I_n(\hat{\phi})$. The estimated asymptotic multivariate normal $N_5(\phi, I_n(\hat{\phi})^{-1})$ distribution of ϕ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1 - \gamma)$ asymptotic confidence interval for each parameter ϕ_r is given by

$$ACI_r = (\phi_r - z_{\frac{\gamma}{2}}\sqrt{\hat{I}_{rr}}\hat{\phi}_r + z_{\frac{\gamma}{2}}\sqrt{\hat{I}_{rr}})$$

where z_γ is the upper 100γ the percentile of the standard normal distribution.

6. Applications

In this section, we use two real data sets to see how the new model works in practice. compare the fits of the AWG distribution with others models. In each case, the parameters are estimated by maximum likelihood as described in Section 5, using the R code.

6.1. *Data Set 1*

   The first data set represents failure times are reported in the book Weibull Models by Murthy" et al. (2004, page 297) as follows:
   0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911,2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924,1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089,2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278,1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224,3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663 .
   In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov), -2L, AIC (Akaike information criterion), AICC (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller KS, -2L, AIC and AICC values:

$$AIC = -2L + 2K,$$

$$AICC = -2L + \left(\frac{2Kn}{n - K - 1}\right),$$

and

$$BIC = -2L + KLog(n).$$

where L denotes the log-likelihood function evaluated at the maximum likelihood estimates, K is the number of parameters, and n is the sample size.

Also, for calculating the values of KS we use the sample estimates of $\theta, \alpha, \beta, \lambda$ and P. Table 1 shows the parameter estimation based on the maximum likelihood and gives the values of the criteria $AIC, AICC, BIC$, and KS test. The values in Table 1 indicate that the AWG distribution leads to a better fit over all the other models.

Table1.MLEs the measures AIC, AICC and BIC, and KS test.

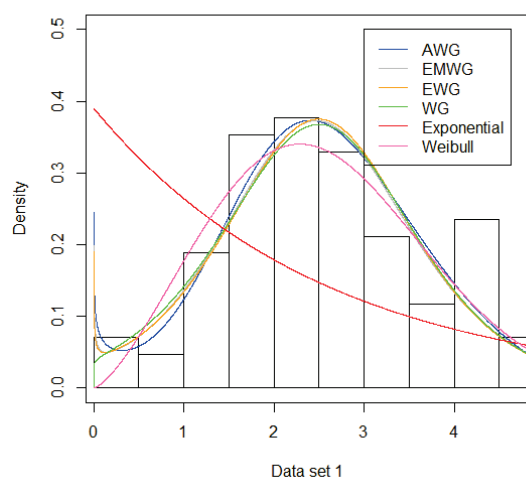| Model	Estimates	S E	$-logL$	AIC	AICC	BIC	KS
AWG	$\hat{\theta} = 0.011$	0.0001	127.11	264.22	264.98	264.98	0.07
	$\hat{\alpha} = 3.483$	1.0215					
	$\hat{\beta} = 0.019$	0.3341					
	$\hat{\lambda} = 0.718$	0.3383					
	$\hat{p} = 0.602$	0.0123					
EMWG	$\hat{\theta} = 18.15$	6.3254	129.25	268.50	269.26	269.26	0.08
	$\hat{\alpha} = 1.312$	0.3254					
	$\hat{\beta} = 0.110$	0.2446					
	$\hat{\lambda} = 3.374$	1.0225					
	$\hat{p} = 0.984$	0.0221					
EWG	$\beta = 0.593$	0.9991	129.28	266.56	267.06	276.33	0.82
	$\hat{\theta} = 0.975$	0.3938					
	$\hat{\lambda} = 1.545$	1.2363					
	$\hat{p} = 0.986$	0.0142					
WG	$\hat{\beta} = 1.074$	0.4168	129.37	264.75	265.04	272.07	0.81
	$\hat{\theta} = 1.232$	1.0650					
	$\hat{p} = 0.962$	0.060					
W	$\hat{\beta} = 2.393$	0.210	131.28	266.57	266.72	271.46	0.09
	$\hat{\theta} = 0.3486$	0.016					
E	$\hat{\theta} = 0.390$	0.0017	164.98	331.97	332.02	334.41	0.291

Fig. 3. Estimated densities of data set 1
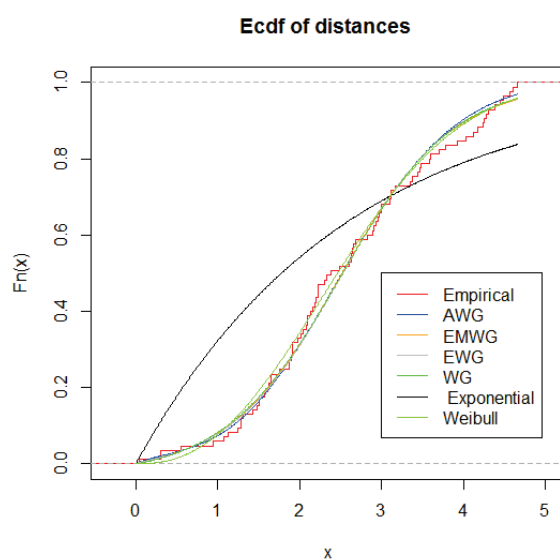
Ecdf of distances

Fig. 4. Empirical, fitted AWG, EMWG, MWG, WG,
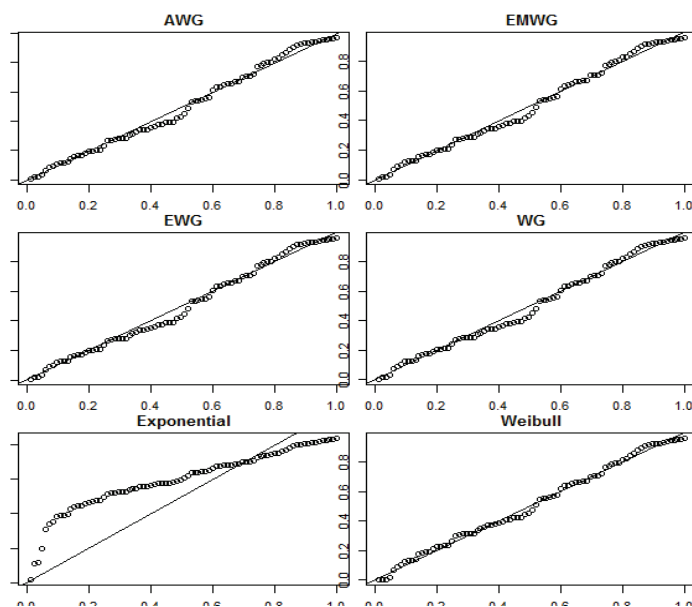Exponential and Weibull distributions of data set 1.

Fig. 5. Probability plots for the fits of the AWG, EMWG, MWG,
WG, Exponential and Weibull distributions of data set1.

6.2. *Data Set 2*

The second data set represents failure time of 50 items reported in Aarset [1].

Table 2. MLEs the measures AIC, AICC and BIC, and KS test to failure time data for the models.

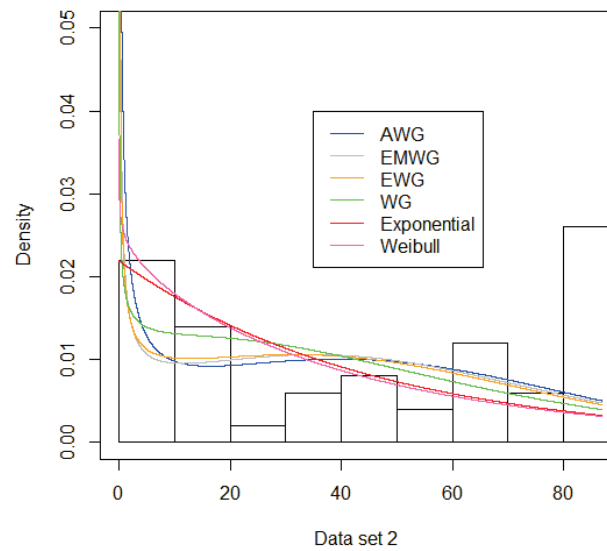| Model	Estimates	S E	−logL	AIC	AICC	BIC	KS
AWG	$\hat{\theta} = 0.0001$	0.0000	229.39	468.78	470.14	478.34	0.13
	$\hat{\alpha} = 2.1455$	1.0002					
	$\hat{\beta} = 0.0825$	0.0496					
	$\hat{\lambda} = 0.4178$	0.1087					
	$\hat{p} = 0.0099$	0.4644					
EMWG	$\hat{\theta} = 13.569$	0.8470	233.51	477.03	478.40	486.59	0.15
	$\hat{\alpha} = 0.0327$	0.0064					
	$\hat{\beta} = 0.0452$	0.0377					
	$\hat{\lambda} = 7.0182$	6.1686					
	$\hat{p} = 0.9991$	0.0003					
EWG	$\hat{\beta} = 1.0111$	0.6691	234.83	477.66	478.55	485.31	0.14
	$\hat{\theta} = 0.0259$	0.0038					
	$\hat{\lambda} = 1.0689$	0.2463					
	$\hat{p} = 0.0365$	0.0142					
WG	$\hat{\beta} = 0.699$	0.1582	237.70	481.41	481.93	487.15	0.16
	$\hat{\theta} = 0.1585$	0.1380					
	$\hat{p} = 0.8505$	0.1224					
W	$\hat{\beta} = 0.948$	0.1197	240.97	485.95	486.21	489.78	0.17
	$\hat{\theta} = 0.0222$	0.0034					
E	$\hat{\theta} = 0.0218$	0.0000	241.06	484.13	484.21	486.04	0.64

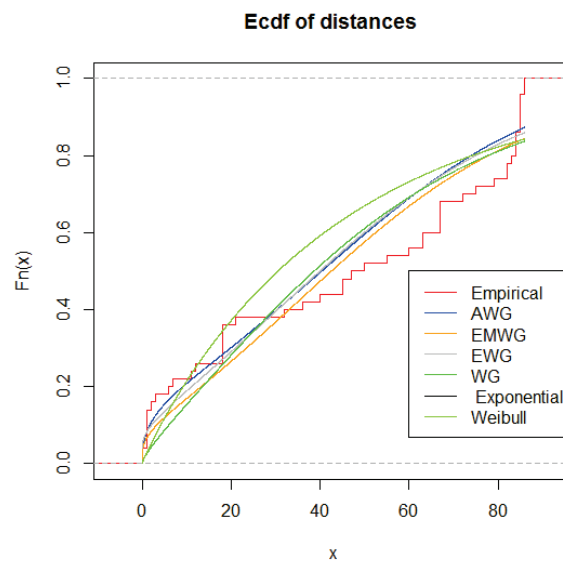Fig 6. Estimated densities of data set 2.

Fig 7. Empirical, fitted AWG, EMWG, MWG, WG,
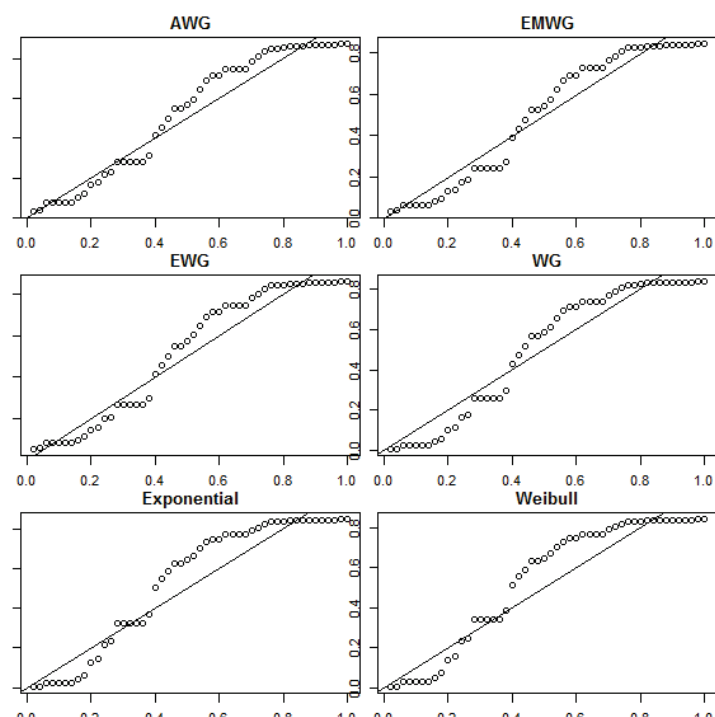Exponential and Weibull distributions of data set 2.

Fig 8. Probability plots for the fits of the AWG, EMWG, MWG, WG,

Exponential and Weibull distributions of data set2.

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