Concomitants of Dual Generalized Order Statistics from Bivariate Burr III Distribution

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In this paper probability density function of single concomitant and joint probability density function of two concomitants of dual generalized order statistics from bivariate Burr III distribution are obtained and expressions for single and product moments are derived. Further, results are deduced for the order statistics and lower record values.

Keywords: Dual generalized order statistics; order statistics; lower record values; concomitants; bivariate Burr III distribution.

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1. Introduction

The probability density function (pdf) of bivariate Burr III distribution [Rodriguez, 1982] is given as

$$f(x,y) = c\,(c+1)\,\alpha_1 k_1\, x^{-(k_1+1)}\alpha_2 k_2\, y^{-(k_2+1)}\,(1+\alpha_1\, x^{-k_1}+\alpha_2\, y^{-k_2})^{-(c+2)},$$
$$x,y>0, \quad k_1,\ k_2, c,\ \alpha_1,\ \alpha_2\ >0 \tag{1.1}$$

and the corresponding distribution function (df) is

$$F(x,y) = (1+\alpha_1\, x^{-k_1}+\alpha_2\, y^{-k_2})^{-c}, \;\; x,y>0, \quad k_1,\ k_2, c,\ \alpha_1,\ \alpha_2\ >0. \tag{1.2}$$

The conditional pdf of Y given X is

$$f(y|x) = \frac{(c+1)\,\alpha_2 k_2\, y^{-(k_2+1)}(1+\alpha_1\, x^{-k_1})^{(c+1)}}{(1+\alpha_1\, x^{-k_1}+\alpha_2\, y^{-k_2})^{(c+2)}}, \;\; y>0. \tag{1.3}$$

The marginal pdf of X is

$$f(x) = \frac{c\,\alpha_1\,k_1\,x^{-(k_1+1)}}{\{1+\alpha_1\,x^{-k_1}\}^{(c+1)}}, \;\; x>0. \tag{1.4}$$

The marginal df of X is

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$$F(x) = (1 + \alpha_1 \, x^{-k_1})^{-c}, \; x > 0. \tag{1.5}$$

Order statistics, record values and several other model of ordered random variables can be viewed as special case of generalized order statistics (gos)[Kamps, 1995]. Burkschat (2003) introduced the concept of dual generalized order statistics $(dgos)$ to enable a common approach to descending ordered *rv's* like reverse order statistics and lower record values.

Let X be a continuous random variables with $df\ F(x)$ and the $pdf\ f(x),\ x \in (\alpha, \beta)$. Further, Let $n \in N,\ k \geq 1,\ \tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + (n-r) + M_r \geq 1$ for all $r \in 1, 2, ..., n-1$. Then $X_d(r, n, \tilde{m}, k)\ \ r = 1, 2, ..., n$ are called $dgos$ if their pdf is given by [Burkschat *et al.* (2003)]

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} \left[F(x_i)\right]^{m_i} f(x_i)\Big) \left[F(x_n)\right]^{k-1} f(x_n) \tag{1.6}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

If $m_i = 0,\ i = 1, 2, ..., n,\ k = 1$, then $X_d(r, n, m, k)$ reduces to the $(n - r + 1)$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and at $m = -1$, $X_d(r, n, m, k)$ reduces to r th, $k-$ lower record values. For more details on order statistics and record values, one may refer to David and Nagaraja (2003) and Ahsanullah (2004) respectively.

In view of (1.6), the pdf of $X_d(r, n, m, k)$ is

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.7}$$

and joint pdf of $X_d(r, n, m, k)$ and $X_d(s, n, m, k),\ \ 1 \leq r < s \leq n$ is

$$f_{r,s,n,m,k}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\ [F(x)]^m f(x)\ g_m^{r-1}\big(F(x)\big)$$

$$\times [h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y),\ x > y \tag{1.8}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1} & ,\ m \neq -1 \\ -\log x & ,\ m = -1 \end{cases}$$

and $\qquad g_m(x) = h_m(x) - h_m(1),\ x \in (0, 1).$

Let $(X_i, Y_i), i = 1, 2, ..., n$, be the n pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the $X-$ variates in descending order as

$X(1,n,m,k) \geq X(2,n,m,k) \geq ... \geq X(n,n,m,k)$ then $Y-$ variates paired (not necessarily in descending order) with these dual generalized ordered statistics are called the concomitants of dual generalized order statistics and are denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, ..., Y_{[n,n,m,k]}$. The pdf of $Y_{[r,n,m,k]}$, the $r-th$ concomitant of dual generalized order statistics is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f(y|x)\ f_{r,n,m,k}(x)\ dx \tag{1.9}$$

and the joint *pdf* of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$, $1 \leq r < s \leq n$ is

$$g_{[r,s,n,m,k]}(y_1,y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f(y_1|x_1)\ f(y_2|x_2)\ f_{r,s,n,m,k}(x_1,x_2)\ dx_2\ dx_1. \tag{1.10}$$

2. Probability Density Function of $Y_{[r,n,m,k]}$

For the bivariate Burr III distribution as given in (1.1), the pdf of $g_{[r,n,m,k]}(y)$ in view of (1.3), (1.4), (1.5), (1.7) and (1.9) is given as

$$g_{[r,n,m,k]}(y) = \frac{\alpha_2 k_2\ C_{r-1}}{(r-1)!(m+1)^{r-1}}\ c\ (c+1)\ y^{-(1+k_2)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\times \int_0^{\infty} \frac{\alpha_1 k_1\ x^{-(1+k_1)}}{(1+\alpha_1 x^{-k_1}+\alpha_2 y^{-k_2})^{c+2}} \frac{1}{(1+\alpha_1 x^{-k_1})^{c(\gamma_{r-i}-1)}}\ dx. \tag{2.1}$$

Let $t = \alpha_1 x^{-k_1}$, then the R.H.S. of (2.1) reduces to

$$= \frac{\alpha_2 k_2\ C_{r-1}}{(r-1)!(m+1)^{r-1}}\ c\ (c+1)\ y^{-(1+k_2)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\times \int_0^{\infty} (1+t+\alpha_2 y^{-k_2})^{-(c+2)} (1+t)^{-c(\gamma_{r-i}-1)}\ dt. \tag{2.2}$$

Since

$$\int_0^{\infty} x^{v-1}\ (a+x)^{-\mu}\ (x+y)^{-\rho}\ dx = \frac{\Gamma v\ \Gamma(\mu-v+\rho)}{\Gamma(\mu+\rho)\ a^{\mu}}\ y^{(v-\rho)}{}_2F_1 \left[\begin{matrix} \mu, & v \\ & ;\ 1-\frac{y}{a} \\ \mu+\rho & \end{matrix} \right] \tag{2.3}$$

$$[\ |arg\ a| < \Pi,\ Re\ v > 0,\ \ |arg\ y| < \Pi,\ \ Re\ \rho > Re\ (v-\mu)],$$

where

$${}_2F_1 \left[\begin{matrix} a,\ b & \\ & ;\ x \\ c & \end{matrix} \right] = \sum_{p=0}^{\infty} \frac{(a)_p (b)_p\ x^p}{(c)_p\ p!} \tag{2.4}$$

is the Gauss hypergeometric series

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma\lambda},\ \lambda \neq 0,-1,-2,..., \qquad [\text{Prudnikov } et.\ al, 1986] \tag{2.5}$$

is the Pochhammer symbol .

Therefore (2.2), becomes

$$g_{[r,n,m,k]}\,(y) = \frac{\alpha_2 k_2\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\; c\,(c+1)\; y^{-(1+k_2)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\times \frac{(1+\alpha_2 y^{-k_2})^{-(c+1)}}{(c\gamma_{r-i}+1)}\; {}_2F_1\left[\begin{matrix}(c\,\gamma_{r-i}-c)\,,\,1 \\ \\ (c\,\gamma_{r-i}+2)\end{matrix}\;;-\alpha_2 y^{-k_2}\right]. \tag{2.6}$$

3. Moment of $Y_{[r,n,m,k]}$

For the bivariate Burr III distribution as given in (1.1), the $a-th$ moment of $Y_{[r,n,m,k]}$ is given as,

$$E\left(Y^{(a)}_{[r,n,m,k]}\right) = \int y^a g_{[r,n,m,k]}\,(y)\,dy \tag{3.1}$$

$$= \frac{\alpha_2 k_2\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\; c\,(c+1)\; \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \int_0^\infty y^a\; y^{-(1+k_2)} (1+\alpha_2 y^{-k_2})^{-(c+1)}\; {}_2F_1\left[\begin{matrix}(c\,\gamma_{r-i}-c)\,,\,1 \\ \\ (c\,\gamma_{r-i}+2)\end{matrix}\;;-\alpha_2 y^{-k_2}\right] dy \tag{3.2}$$

$$= \frac{\alpha_2 k_2\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\; c\,(c+1)\; \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p}{(c\gamma_{r-i}+2)_p\; p!} \int_0^\infty y^a\; y^{-(1+k_2)} (1+\alpha_2 y^{-k_2})^{-(c+1)}\; (-\alpha_2 y^{-k_2})^p\; dy. \tag{3.3}$$

Now letting $t = 1+\alpha_2 y^{-k_2}$ in (3.3), we get

$$= \frac{(\alpha_2)^{\frac{a}{k_2}}\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\; c\,(c+1)\; \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p\,(-1)^p}{(c\gamma_{r-i}+2)_p\; p!} \int_1^\infty t^{-(c+1)}\; (t-1)^{(p\,-\frac{a}{k_2})}\, dt. \tag{3.4}$$

Since

$$\int_y^\infty x^{-\lambda}\; (x-y)^{\mu-1}\, dx = \frac{\Gamma(\lambda-\mu)\,\Gamma\mu}{\Gamma\lambda}\; y^{(\mu-\lambda)},\;\; 0 < Re\;\mu < Re < \lambda \tag{3.5}$$

[*c.f.* Erdélyi *et al.*, 1954].

Therefore R.H.S. of (3.4) becomes

$$= \frac{(\alpha_2)^{\frac{a}{k_2}}\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\, c\,(c+1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p\,(-1)^p}{(c\gamma_{r-i}+2)_p\; p!}\; \frac{\Gamma(c+\frac{a}{k_2}-p)\,\Gamma(p+1-\frac{a}{k_2})}{\Gamma(c+1)}. \tag{3.6}$$

Now on application of (2.5) and using

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n},\;\; n=1,2,3,\ldots,\; \lambda \neq 0,\; \pm 1,\; \pm 2,\ldots \;\text{[Srivastava and Karlsson, 1985]} \tag{3.7}$$

in (3.6) and simplifying, we get

$$E\left(Y^{(a)}_{[r,n,m,k]}\right) = \frac{(\alpha_2)^{\frac{a}{k_2}}\, C_{r-1}}{(r-1)!(m+1)^{r-1}}\, c\,(c+1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p\,(1)_p\,(1-\frac{a}{k_2})_p}{(c\gamma_{r-i}+2)_p\,(1-c-\frac{a}{k_2})_p\; p!}\; \frac{\Gamma(c+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}. \tag{3.8}$$

We have the generalized Gauss series or the generalized hypergeometric series.

$$ {}_pF_q \left[\begin{matrix} \alpha_1 \,\ldots\, \alpha_p & \\ & ,z \\ \beta_1 \,\ldots\, \beta_q & \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k} \frac{z^k}{k!}. \tag{3.9}$$

Thus applying (3.9) in (3.8), we have

$$E\left(Y^{(a)}_{[r,n,m,k]}\right) = \frac{(\alpha_2)^{\frac{a}{k_2}}\, C_{r-1} c\,(c+1)}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1} \frac{\Gamma(c+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}$$

$$\times\; {}_3F_2 \left[\begin{matrix} (c\gamma_{r-i}-c), & 1, & (1-\frac{a}{k_2}) & \\ & & & ,1 \\ (c\gamma_{r-i}+2), & (1-c-\frac{a}{k_2}) & & \end{matrix} \right]. \tag{3.10}$$

Noting that [prudnikov *et al*, 1986]

$$ {}_3F_2\left[\begin{matrix}-N, & 1, & a & \\ & & & ,1 \\ b, & a-m, & & \end{matrix}\right] $$

$$ = \frac{(b-1)(a-m-1)}{(b+N-1)(a-1)}\,{}_3F_2\left[\begin{matrix}-m, & 1, & 2-b & \\ & & & ,1 \\ 2-b-N, & 2-a, & & \end{matrix}\right]. \tag{3.11} $$

Now using (3.11) in (3.10), we can write

$$ = \frac{(\alpha_2)^{\frac{a}{k_2}}\ c\ C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\Gamma(c+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}\frac{(c+\frac{a}{k_2})}{(\frac{a}{k_2})} $$

$$ \times {}_3F_2\left[\begin{matrix}-c, & 1, & (-c\gamma_{r-i}) & \\ & & & ,1 \\ -c, & 1+\frac{a}{k_2}, & & \end{matrix}\right]. \tag{3.12} $$

Using relation (3.9) in (3.12), we get

$$ = \frac{(\alpha_2)^{\frac{a}{k_2}}\ c\ C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\Gamma(c+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}\frac{(c+\frac{a}{k_2})}{(\frac{a}{k_2})} $$

$$ \times {}_2F_1\left[\begin{matrix}1, & (-c\gamma_{r-i}) & \\ & & ;1 \\ (1+\frac{a}{k_2}) & & \end{matrix}\right]. \tag{3.13} $$

Noting that [Prudnikov *et al.*, 1986].

$$ {}_2F_1\left[\begin{matrix}a\ b & \\ & ;1 \\ c & \end{matrix}\right] = \frac{\Gamma\,(c)\,\Gamma\,(c-a-b)}{\Gamma\,(c-a)\,\Gamma\,(c-b)},\ \ Re\,(c-a-b)\ >\ 0,\ \ c\neq,\ \pm 1,\ \pm 2,\ ...., \tag{3.14} $$

Applying (3.14) in (3.13), we get

$$= \frac{(\alpha_2)^{\frac{a}{k_2}}\ C_{r-1}}{(r-1)!(m+1)^{r-1}} \ \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{(c+\frac{a}{k_2})\ \Gamma(c+\frac{a}{k_2})\ \Gamma(1-\frac{a}{k_2})}{c\ \Gamma(c)}\frac{1}{\gamma_{r-i}+\frac{a}{c\ k_2}} \tag{3.15}$$

$$= \frac{(\alpha_2)^{\frac{a}{k_2}}\ C_{r-1}}{(r-1)!(m+1)^{r}} \ \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{(c+\frac{a}{k_2})\ \Gamma(c+\frac{a}{k_2})\ \Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}$$

$$\times B\Big(\frac{k}{m+1}+(n-r)+i+\frac{a}{c\ k_2\ (m+1)}\ \ ,\ \ 1\Big). \tag{3.16}$$

Since,

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,\ \ c)=B(k,\ \ c+b) \tag{3.17}$$

where $B(a,b)$ is the complete beta function.

Therefore,

$$E\Big(y^{(a)}_{[r,n,m,k]}\Big)=\frac{(\alpha_2)^{\frac{a}{k_2}}\ C_{r-1}}{(m+1)^r}\ \frac{\Gamma(c+1+\frac{a}{k_2})\ \ \Gamma(1-\frac{a}{k_2})}{c\ \Gamma(c)}\frac{\Gamma(\frac{k+(n-r)(m+1)+\frac{a}{c\ k_2}}{m+1})}{\Gamma(\frac{k+n(m+1)+\frac{a}{c\ k_2}}{m+1})} \tag{3.18}$$

$$E\Big(y^{(a)}_{[r,n,m,k]}\Big)=(\alpha_2)^{\frac{a}{k_2}}\ \frac{\Gamma(c+1+\frac{a}{k_2})\ \ \Gamma(1-\frac{a}{k_2})}{c\ \Gamma(c)}\frac{1}{\prod_{i=1}^{r}(1+\frac{a}{c\ k_2\ \gamma_i})}. \tag{3.19}$$

Remark 3.1 : If $m=0,\ k=1$ in (3.11), we get single moment of concomitants of order statistics from bivariate Burr III distribution as;

$$E\Big(y^{(a)}_{[n-r+1:n]}\Big)=\frac{n!}{(n-r)!}\ \ (\alpha_2)^{\frac{a}{k_2}}\frac{\Gamma(c+1+\frac{a}{k_2})\ \ \Gamma(1-\frac{a}{k_2})}{c\ \Gamma(c)}\frac{\Gamma(n-r+1+\frac{a}{c\ k_2})}{\Gamma(n+1+\frac{a}{c\ k_2})}.$$

If we replace $n-r+1$ by r , then

$$E\Big(y^{(a)}_{[r:n]}\Big)=\frac{n!}{(r-1)!}\ \ (\alpha_2)^{\frac{a}{k_2}}\frac{\Gamma(c+1+\frac{a}{k_2})\ \ \Gamma(1-\frac{a}{k_2})}{c\ \Gamma(c)}\frac{\Gamma(r+\frac{a}{c\ k_2})}{\Gamma(n+1+\frac{a}{c\ k_2})}$$

and at $r\ =\ n$, we get

$$E\left(y_{[n:n]}^{(a)}\right) = \frac{n}{(n\,c + \frac{a}{k_2})}\ (\alpha_2)^{\frac{a}{k_2}} \frac{\Gamma(c+1+\frac{a}{k_2})\ \Gamma(1-\frac{a}{k_2})}{\Gamma(c)}. \tag{3.20}$$

It may be noted that for the order statistics, the k^{th} moments of $Y_{[r:n]}$ is

$$\mu_{[r:n]}^{k}(y) = \sum_{i=r}^{n} (-1)^{i-r} \binom{i-1}{n-r} \binom{n}{i} \mu_{[i:i]}^{k}(y). \tag{3.21}$$

Therefore,

$$E\left(y_{[r:n]}^{(a)}\right) = \sum_{i=r}^{n} (-1)^{i-r} \binom{i-1}{r-1} \binom{n}{i} \frac{i}{(i\,c + \frac{a}{k_2})}\ (\alpha_2)^{\frac{a}{k_2}} \frac{\Gamma(c+1+\frac{a}{k_2})\ \Gamma(1-\frac{a}{k_2})}{\Gamma(c)} \tag{3.22}$$

as obtained by (Begum and Khan,1998).

Remark 3.2 : Set $m = -1$ in (3.12), to get moment of $k-th$ lower record value from bivariate Burr III distribution as;

$$E\left(y_{[r,n,-1,k]}^{(a)}\right) = (\alpha_2)^{\frac{a}{k_2}}\ \frac{\Gamma(c+1+\frac{a}{k_2})\ \Gamma(1-\frac{a}{k_2})}{c\,\Gamma(c)} \frac{1}{(1+\frac{a}{c\,k\,k_2})^{r}}. \tag{3.23}$$

4. Joint Probability Density Function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For the bivariate Burr III distribution as given in (1.1), the joint pdf of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ in view of (1.3), (1.4), (1.5), (1.8) and (1.10) is given as

$$g_{[r,s,n,m,k]}(y_1,y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\ c^2\ (c+1)^2\ (\alpha_1 k_1)\ (\alpha_2 k_2)^2$$

$$\times y_1^{-(1+k_2)}\ y_2^{-(1+k_2)} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$

$$\times \int_0^{\infty} \frac{x_1^{-(1+k_1)}}{(1+\alpha_1\ x_1^{-k_1})^{c(s-r+i-j)(m+1)-c}} \frac{1}{(1+\alpha_1\ x_1^{-k_1}+\alpha_2\ y_1^{-k_2})^{c+2}}\ I(x_1,y_2)\ dx_1 \tag{4.1}$$

where,

$$I(x_1,y_2) = \int_0^{x_1} \frac{\alpha_1 k_1\ x_2^{-(1+k_1)}}{(1+\alpha_1\ x_2^{-k_1})^{c(\gamma_{s-j}-1)}} \frac{1}{(1+\alpha_1\ x_2^{-k_1}+\alpha_2\ y_2^{-k_2})^{c+2}}\ dx_2. \tag{4.2}$$

Setting $t_2 = (1+\alpha_1\ x_2^{-k_1})$, then the R.H.S. of (4.2) reduces to

$$I(x_1,y_2) = (\lambda)^{-\beta} \int_{1+\alpha_1 x_2^{-k_1}}^{\infty} t_2^{-\alpha} \, (1+\frac{t_2}{\lambda})^{-\beta} \, dt_2 \tag{4.3}$$

where $\alpha = (c\gamma_{s-j} - c)$, $\beta = (c+2)$ and $\lambda = \alpha_2 \, y_2^{-k_2}$

Note that [Prudnikov *et al.*, 1986].

$$(1+z)^{-a} = \sum_{p=0}^{\infty} \frac{(-1)^p (a)_p \, z^p}{p!}. \tag{4.4}$$

Now in view of (4.4), (4.3) after simplification yields

$$I(x_1,y_2) = (\lambda)^{-\beta} \sum_{l=0}^{\infty} (-1)^l \, \frac{(\beta)_l \, (\frac{1}{\lambda})^l}{l!} \, \frac{(1+\alpha_1 \, x_1^{-k_1})^{-(\alpha-l-1)}}{(\alpha-l-1)}. \tag{4.5}$$

Now putting the value of $I(x_1,y_2)$ from (4.5) in (4.1), we obtain

$$g_{[r,s,n,m,k]}(y_1,y_2) = \frac{c^2 \, (c+1)^2 \, (\alpha_1 k_1) \, (\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \, y_1^{-(1+k_2)} \, y_2^{-(1+k_2)}$$

$$\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} (\lambda)^{-\beta} \sum_{l=0}^{\infty} \frac{(-1)^l}{(\alpha-l-1)} \, \frac{(\beta)_l \, (\frac{1}{\lambda})^l}{l!}$$

$$\times \int_0^{\infty} \frac{x_1^{-(1+k_1)}}{(1+\alpha_1 \, x_1^{-k_1})^{c(s-r+i-j)(m+1)-c+(c\gamma_{s-j}-c)-l-1}} \frac{1}{(1+\alpha_1 \, x_1^{-k_1} + \alpha_2 \, y_1^{-k_2})^{c+2}} \, dx_1. \tag{4.6}$$

Setting $t_1 = (1+\alpha_1 \, x_1^{-k_1})$ in (4.6), and after simplification we get

$$g_{[r,s,n,m,k]}(y_1,y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \, c^2 \, (c+1)^2 \, (\alpha_2 k_2)^2$$

$$\times y_1^{-(1+k_2)} \, y_2^{-(1+k_2)} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$

$$\times (\lambda)^{-\beta} (\delta)^{-\beta} \sum_{l=0}^{\infty} \frac{(\beta)_l \, (-\frac{1}{\lambda})^l}{(1-\alpha+l) \, l!} \sum_{p=0}^{\infty} \frac{(\beta)_p \, (-\frac{1}{\delta})^p}{(2-\theta-\alpha+l+p) \, p!} \tag{4.7}$$

where $\delta = \alpha_2 \, y_1^{-k_2}$ and $\theta = c(s-r+i-j)(m+1)-c$.

By setting $d = 1-\alpha$ and $g = 2-\theta-\alpha$ in (4.7), we get

$$= \frac{c^2\ (c+1)^2\ \ (\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$

$$\times y_1^{-(1+k_2)}\ y_2^{-(1+k_2)} (\lambda)^{-\beta} (\delta)^{-\beta} \sum_{l=0}^{\infty} \frac{(\beta)_l\ (-\frac{1}{\lambda})^l}{(d+l)\ l!} \ \sum_{p=0}^{\infty} \frac{(\beta)_p\ (-\frac{1}{\delta})^p}{(g+p+l)\ p!}. \tag{4.8}$$

Now after putting the value of λ and δ in (4.8), we have

$$= \frac{c^2\ (c+1)^2\ \ (\alpha_2 k_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} y_1^{-(1+k_2)}$$

$$\times\ y_2^{-(1+k_2)} (\alpha_2 y_2^{-k_2})^{-\beta} (\alpha_2 y_1^{-k_2})^{-\beta} \sum_{l=0}^{\infty} \frac{(\beta)_l\ (\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{(d+l)\ l!} \ \sum_{p=0}^{\infty} \frac{(\beta)_p\ (\frac{-1}{\alpha_2 y_1^{-k_2}})^p}{(g+p+l)\ p!} \tag{4.9}$$

Noting that [Srivastava and Karlsson, 1985].

$$(\lambda+m) = \frac{\lambda\ (\lambda+1)_m}{(\lambda)_m} \tag{4.10}$$

$$(\lambda+m+n) = \frac{\lambda\ (\lambda+1)_{m+n}}{(\lambda)_{m+n}}. \tag{4.11}$$

Using relation (4.10) and (4.11) in (4.9), it becomes

$$= \frac{c^2\ (c+1)^2\ C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \frac{1}{dg}$$

$$\times \frac{(\alpha_2 k_2\ y_1^{-(1+k_2)})}{(\alpha_2\ y_1^{-k_2})^{\beta}} \frac{(\alpha_2\ k_2\ y_2^{-(1+k_2)})}{(\alpha_2\ y_2^{-k_2})^{\beta}} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \frac{(g)_{p+l}}{(g+1)_{p+l}} \frac{(\beta)_l\ (d)_l\ (\beta)_p}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_1^{-k_2}})^p}{p!} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}. \tag{4.12}$$

We have the Kampede Feriet's series [Srivastava and Karlsson, 1985]

$$F^{p:q;k}_{l:m;n} \left[\begin{matrix} (a_p)\ :(b_q)\ ;(c_k) \\ \\ (\alpha_l)\quad (\beta_m)\ \ (\gamma_n) \end{matrix} ;x,y \right]$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p}\ (a_j)_{r+s}\ \prod_{j=1}^{q}\ (b_j)_r\ \prod_{j=1}^{k}\ (c_j)_s}{\prod_{j=1}^{l}\ (\alpha_j)_{r+s}\ \prod_{j=1}^{m}\ (\beta_j)_r\ \prod_{j=1}^{n}\ (\gamma_j)_s} \frac{x^r}{r!}\ \ \frac{y^s}{s!}. \tag{4.13}$$

Therefore after using (4.13), we finally get

$$g_{[r,s,n,m,k]}(y_1,y_2)=\frac{c^2\,(c+1)^2\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\,(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg}$$

$$\times\frac{(\alpha_2 k_2\,y_1^{-(1+k_2)})}{(\alpha_2\,y_1^{-k_2})^{\beta}}\frac{(\alpha_2 k_2\,y_2^{-(1+k_2)})}{(\alpha_2\,y_2^{-k_2})^{\beta}}F^{1:2;1}_{1:1;0}\left[\begin{matrix}(g); & (\beta); & (d);\,(\beta); & \\ & & ; & \frac{-1}{\alpha_2 y_2^{-k_2}},\frac{-1}{\alpha_2 y_1^{-k_2}}\\ (g+1);\,(d+1); & & & \end{matrix}\right]. \tag{4.14}$$

5. Product Moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For Burr type III dstribution, the product Moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given as

$$E\left(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]}\right)=\int_0^{\infty}\int_0^{\infty}y_1^a\,y_2^b\;g_{[r,s,n,m,k]}(y_1,y_2)\,dy_1\,dy_2. \tag{5.1}$$

In view of (4.14) and (5.1), we have

$$E\left(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]}\right)=A\int_0^{\infty}\int_0^{\infty}y_1^a\,y_2^b\frac{(\alpha_2 k_2\,y_1^{-(1+k_2)})}{(\alpha_2\,y_1^{-k_2})^{\beta}}\frac{(\alpha_2 k_2\,y_2^{-(1+k_2)})}{(\alpha_2\,y_2^{-k_2})^{\beta}}$$

$$\times F^{1:2;1}_{1:1;0}\left[\begin{matrix}(g)\,; & (\beta)\,; & (d);\,(\beta); & \\ & & ; & \frac{-1}{\alpha_2 y_2^{-k_2}},\frac{-1}{\alpha_2 y_1^{-k_2}}\\ (g+1)\,;\,(d+1)\,; & & & \end{matrix}\right]dy_1\,dy_2 \tag{5.2}$$

where,

$$A=\frac{c^2\,(c+1)^2\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\,(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg}. \tag{5.3}$$

Thus,

$$E\left(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]}\right)=A\int_0^{\infty}\int_0^{\infty}y_1^a\,y_2^b\frac{(\alpha_2 k_2\,y_1^{-(1+k_2)})}{(\alpha_2\,y_1^{-k_2})^{\beta}}\frac{(\alpha_2 k_2\,y_2^{-(1+k_2)})}{(\alpha_2\,y_2^{-k_2})^{\beta}}$$

$$\times\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(g)_{p+l}}{(g+1)_{p+l}}\frac{(\beta)_l\,(d)_l\,(\beta)_p}{(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_1^{-k_2}})^p}{p!}\frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}dy_1\,dy_2. \tag{5.4}$$

By Srivastava and Karlsson (1985), we have

$$(\lambda)_{m+n} = (\lambda)_m\ (\lambda+m)_n. \tag{5.5}$$

On applying (5.5) in (5.4), we get

$$= A \int_0^\infty y_2^b \frac{(\alpha_2 k_2\ y_2^{-(1+k_2)})}{(\alpha_2\ y_2^{-k_2})^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l\ (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}$$

$$\times \Big\{ \int_0^\infty y_1^a\ \frac{(\alpha_2 k_2\ y_1^{-(1+k_2)})}{(\alpha_2\ y_1^{-k_2})^\beta} \sum_{p=0}^\infty \frac{(\beta)_p\ (g+l)_p}{(g+1+l)_p} \frac{(\frac{-1}{\alpha_2 y_1^{-k_2}})^p}{p!} dy_1 \Big\}\ dy_2 \tag{5.6}$$

$$= A \int_0^\infty y_2^b \frac{(\alpha_2 k_2\ y_2^{-(1+k_2)})}{(\alpha_2\ y_2^{-k_2})^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l\ (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}$$

$$\times \int_0^\infty y_1^a\ \frac{(\alpha_2 k_2\ y_1^{-(1+k_2)})}{(\alpha_2\ y_1^{-k_2})^\beta}\, {}_2F_1 \left[\begin{matrix} (\beta); \quad (g+l) \\ \\ (g+l+1) \end{matrix} ;\ \frac{-1}{\alpha_2\ y_1^{-k_2}}{}_1 \right] dy_1\ dy_2. \tag{5.7}$$

Now letting $\alpha_2\ y_1^{-k_2} = z-1$ in (5.7), we have

$$= A\ (\alpha_2)^{\frac{a}{k_2}} \int_0^\infty y_2^b \frac{(\alpha_2 k_2\ y_2^{-(1+k_2)})}{(\alpha_2\ y_2^{-k_2})^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l\ (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}$$

$$\times \int_1^\infty (z-1)^{-(\beta+\frac{a}{k_2})}\, {}_2F_1 \left[\begin{matrix} (\beta); \quad (g+l) \\ \\ (g+l+1) \end{matrix} ;\ \frac{1}{1-z} \right] dz\ dy_2 \tag{5.8}$$

Since

$${}_2F_1 \left[\begin{matrix} a,\ b \\ \\ c \end{matrix} ;\ x \right] = (1-x)^{-a}\ 2F_1 \left[\begin{matrix} a,\ c-b \\ \\ c \end{matrix} ;\ -\frac{x}{1-x} \right]. \tag{5.9}$$

Therefore on using relation (5.9) in (5.10), we have

$$= A\ (\alpha_2)^{\frac{a}{k_2}} \int_0^\infty y_2^b \frac{(\alpha_2 k_2\ y_2^{-(1+k_2)})}{(\alpha_2\ y_2^{-k_2})^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l\ (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}$$

$$\times \sum_{p=0}^\infty \frac{(\beta)_p\ (1)_p}{(g+l+1)_p} \Big\{ \int_1^\infty (z-1)^{(1-\frac{a}{k_2})-1}\ z^{-(\beta+p)} dz \Big\}\ dy_2. \tag{5.10}$$

Now using relation (3.5) in (5.10), we get

$$= A\,(\alpha_2)^{\frac{a}{k_2}}\ \frac{g}{(g+l)}\ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}\ \int_0^\infty y_2^b \frac{(\alpha_2 k_2\, y_2^{-(1+k_2)})}{(\alpha_2\, y_2^{-k_2})^\beta}$$

$$\times \sum_{l=0}^\infty \frac{(\beta)_l\,(d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}\ {}_2F_1\left[\begin{matrix}(\beta-1+\frac{a}{k_2});\,1 & \\ & ;\,1\\ (g+l+1) & \end{matrix}\right] dy_2. \tag{5.11}$$

Now in view of (3.14), (5.11) becomes

$$= (\alpha_2)^{\frac{a}{k_2}}\, g\, A\ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}\ \left\{ \int_0^\infty y_2^b \frac{(\alpha_2 k_2\, y_2^{-(1+k_2)})}{(\alpha_2\, y_2^{-k_2})^\beta} dy_2 \right\}$$

$$\times \sum_{l=0}^\infty \frac{(\beta)_l\,(d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!} \frac{1}{(g+1-\beta-\frac{a}{k_2}+l)}. \tag{5.12}$$

Using relation (4.10) in (5.12), we get

$$= (\alpha_2)^{\frac{a}{k_2}}\ A\ \frac{g}{(g+1-\beta-\frac{a}{k_2})}\ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}\ \left\{ \int_0^\infty y_2^b \frac{(\alpha_2 k_2\, y_2^{-(1+k_2)})}{(\alpha_2\, y_2^{-k_2})^\beta}\ dy_2 \right\}$$

$$\times \sum_{l=0}^\infty \frac{(\beta)_l\,(d)_l}{(d+1)_l} \frac{(g+1-\beta-\frac{a}{k_2})_l}{(g+2-\beta-\frac{a}{k_2})_l} \frac{(\frac{-1}{\alpha_2 y_2^{-k_2}})^l}{l!}. \tag{5.13}$$

Further setting $z-1 = \alpha_2 y_2^{-k_2}$ in (5.13), it becomes

$$= (\alpha_2)^{\frac{(a+b)}{k_2}}\ A\ \frac{g}{(g+1-\beta-\frac{a}{k_2})}\ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \int_1^\infty (z-1)^{-(\beta+\frac{b}{k_2})}\,{}_3F_2\left[\begin{matrix}(d);\quad (g+1-\beta-\frac{a}{k_2});\,(\beta) & \\ & ;\ \frac{1}{1-z}\\ (d+1);\ (g+2-\beta-\frac{a}{k_2}) & \end{matrix}\right] dz. \tag{5.14}$$

Set $z-1 = \frac{1}{t}$ in (5.14), to get

$$= (\alpha_2)^{\frac{(a+b)}{k_2}}\ A\ \frac{g}{(g+1-\beta-\frac{a}{k_2})}\ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \int_0^\infty t^{(\beta+\frac{b}{k_2}-1)-1} {}_3F_2 \left[\begin{matrix} (d); \quad (g+1-\beta-\frac{a}{k_2}); \beta \\ ; \ -t \\ (d+1); \ (g+2-\beta-\frac{a}{k_2}) \end{matrix} \right] dz. \tag{5.15}$$

Note that [Prudnikov *et al.*, 1986]

$$\int_0^\infty x^{s-1} {}_3F_2 \left[\begin{matrix} (a_1), (a_2), (a_3) \\ ; \quad -x \\ (b_1), \ (b_2) \end{matrix} \right] dx$$

$$= \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(s)\,\Gamma(a_1-s)\,\Gamma(a_2-s)\,\Gamma(a_3-s)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(b_1-s)\,\Gamma(b_2-s)} \tag{5.16}$$

Now using relations (5.16) in (5.15), we get

$$= (\alpha_2)^{\frac{(a+b)}{k_2}} A \, \frac{g}{(g+2-2\beta-\frac{a}{k_2}-\frac{b}{k_2})\,(d+1-\beta-\frac{b}{k_2})} \ \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)}. \tag{5.17}$$

Finally putting the values of $A,\ d,\ g$ and β in (5.17), we have

$$E\left(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}\right) = (\alpha_2)^{\frac{(a+b)}{k_2}} \frac{c^2\,(c+1)^2\,C_{s-1}}{(r-1)!\,(s-r-1)!(m+1)^{s-2}} \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$

$$\times \frac{1}{[c\{k+(n-s+j)(m+1)\}+\frac{b}{k_2}]} \ \frac{1}{[c\{k+(n-r+i)(m+1)\}+\frac{a}{k_2}+\frac{b}{k_2}]} \tag{5.18}$$

$$= (\alpha_2)^{\frac{(a+b)}{k_2}} \frac{c^2\,(c+1)^2\,C_{s-1}}{(r-1)!\,(s-r-1)!(m+1)^{s}} \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} B\Big(\frac{k}{m+1}+(n-r+i)+\frac{a+b}{ck_2(m+1)} \ , \ 1\Big)$$

$$\times \sum_{j=0}^{s-r-1} (-1)^j \binom{s-r-1}{j} B\Big(\frac{k}{m+1}+(n-s+j)+\frac{b}{ck_2(m+1)} \ , \ 1\Big). \tag{5.19}$$

On using relation (3.17) in (5.19), we get

$$= (\alpha_2)^{\frac{(a+b)}{k_2}} \frac{(c+1)^2\, C_{s-1}}{(r-1)!\,(s-r-1)!(m+1)^s} \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} B\Big(\frac{k}{m+1}+(n-r)+\frac{a+b}{ck_2(m+1)}\ ,\ r\Big)$$

$$\times B\Big(\frac{k}{m+1}+(n-s)+\frac{b}{ck_2(m+1)}\ ,\ s-r\Big), \tag{5.20}$$

which after simplification yields

$$E\Big(Y^{(a)}_{[r,n,m,k]}Y^{(b)}_{[s,n,m,k]}\Big) = (\alpha_2)^{\frac{(a+b)}{k_2}} (c+1)^2 \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} \frac{1}{\prod_{i=1}^{r}(1+\frac{a+b}{c\,k_2\gamma_i})\ \prod_{j=r+1}^{s}(1+\frac{b}{c\,k_2\gamma_i})}. \tag{5.21}$$

Remark 5.1 : By setting $m=0,\ k=1$ in (5.18), we get product moments of concomitant of order statistics from bivariate Burr III distribution as

$$E\Big(Y^{(a)}_{[n-s+1:n]}Y^{(b)}_{[n-r+1:n]}\Big) = (\alpha_2)^{\frac{(a+b)}{k_2}} \frac{c^2\,(c+1)^2\,n!}{(r-1)!\,(s-r-1)!(n-s)!} \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}.$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times \frac{1}{[c(n-s+1+j)+\frac{b}{k_2}]}\ \frac{1}{[c(n-r+1+i)+\frac{a}{k_2}+\frac{b}{k_2}]} \tag{5.22}$$

Replace $n-s+1$ by r and $n-r+1$ by s in (5.22), we get

$$E\Big(Y^{(a)}_{[r:n]}Y^{(b)}_{[s:n]}\Big) = (\alpha_2)^{\frac{(a+b)}{k_2}} \frac{c^2\,(c+1)^2\,n!}{(r-1)!\,(s-r-1)!(n-s)!} \frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)} \sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{n-s}{i}\binom{s-r-1}{j}$$

$$\times \frac{1}{[cr+cj+\frac{b}{k_2}]}\ \frac{1}{[cs+ic+\frac{a}{k_2}+\frac{b}{k_2}]}$$

$$E\left(Y^{(a)}_{[r:n]}Y^{(b)}_{[s:n]}\right) = \frac{n!}{(r-1)!}(\alpha_2)^{\frac{(a+b)}{k_2}} \frac{\Gamma(c+1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})\Gamma(c+1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(c+1)}$$

$$\times \frac{\Gamma(r+\frac{b}{ck_2})}{\Gamma(s+\frac{b}{ck_2})}\,\frac{\Gamma(s+\frac{a+b}{ck_2})}{\Gamma(n+1+\frac{a+b}{ck_2})}$$

as obtained by (Begum and Khan, 1998).

Remark 5.2 : If $m = -1$ in (5.21), we get product moment of concomitants of $k-th$ lower record values from bivariate Burr III distribution as

$$E\left(Y^{(a)}_{[r,n,-1,k]}Y^{(b)}_{[s,n,-1,k]}\right) = (\alpha_2)^{\frac{(a+b)}{k_2}}\,(c+1)^2\frac{\Gamma(\beta-1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(\beta)}$$

$$\times \frac{\Gamma(\beta-1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(\beta)}\,\frac{1}{(1+\frac{a+b}{c\,k\,k_2})^r\,(1+\frac{b}{c\,k\,k_2})^{s-r}}. \tag{5.23}$$

Putting the value of β in (5.22), we get

$$E\left(Y^{(a)}_{[r,n,-1,k]}Y^{(b)}_{[s,n,-1,k]}\right) = (\alpha_2)^{\frac{(a+b)}{k_2}}\,\frac{\Gamma(c+1+\frac{a}{k_2})\,\Gamma(1-\frac{a}{k_2})}{\Gamma(c+1)}$$

$$\times \frac{\Gamma(c+1+\frac{b}{k_2})\,\Gamma(1-\frac{b}{k_2})}{\Gamma(c+1)}\,\frac{1}{(1+\frac{a+b}{c\,k\,k_2})^r\,(1+\frac{b}{c\,k\,k_2})^{s-r}} \tag{5.24}$$

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