On the Weibull-X family of distributions

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In this paper, the Weibull-X family is proposed and some of its properties are discussed. A member of the Weibull-X family, the Weibull-logistic distribution, is defined and studied. Various properties of the Weibull-logistic distribution are obtained. The distribution is found to be unimodal and the shape can be symmetric, right skewed or left skewed. The structural analysis of the distribution in this paper includes limiting behavior, mode, quantiles, moments, skewness, kurtosis, Shannon's entropy and order statistics. The method of maximum likelihood estimation is proposed for estimating the model parameters. A real data set is used to illustrate the application of the Weibull-logistic distribution.

Keywords: Weibull-logistic distribution; T-X families of distributions; Shannon's entropy; reliability parameter; distribution of the sample extremums.

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1. Introduction

Continuous distributions are used in many applications. This attracts researchers to generate more distributions. Several methods to generate univariate continuous distributions have appeared in the literature. For good references, one can refer to Azzalini (1985), Johnson et al. (1994a), Eugene et al. (2002), Jones (2004) and Cordeiro and de Castro (2011). Recently, Alzaatreh, Lee and Famoye (2013) developed a new method to generate family of continuous distributions and called it the T-X family of distributions. Alzaatreh et al. (2012) used the T-X families of distributions to generate discrete distributions. For a recent review of methods for generating univariate continuous distributions, one can refer to Lee et al. (2013). In this paper we first study some general properties of the T-X family when T is a Weibull random variable. Then we study in detail a member of the Weibull-X family, the Weibull-logistic distribution.

Let $F(x)$ be the cumulative distribution function (c.d.f) of any random variable X and $r(t)$ be the probability density function (p.d.f) of a random variable T defined on $(-\infty,\infty)$. The CDF of the $T-X$ family of distributions defined by Alzaatreh, et al. (2013) is given by

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt. \tag{1.1}$$

When X is any continuous random variable, the probability density function of the T-X family is

$$g(x) = \frac{f(x)}{1-F(x)} r(-\log(1-F(x))) = h_f(x) r(H_f(x)), \tag{1.2}$$

where $h_f(x)$ and $H_f(x)$ are the hazard and cumulative hazard functions associated with f(x). If a random variable T follows the Weibull distribution, $r(t) = (c/\gamma)(t/\gamma)^{c-1}e^{-(t/\gamma)^c}, t \geq 0$, then from definition (1.2) we get,

$$g(x) = \frac{cf(x)}{\gamma(1-F(x))}\left(\frac{-\log(1-F(x))}{\gamma}\right)^{c-1} \exp\{-(-(\log(1-F(x)))/\gamma)^c\}, \tag{1.3}$$

The c.d.f of the Weibull-X family in (1.3) is given

by

$$G(x) = 1 - \exp\left\{-\left(-\frac{\log(1-F(x))}{\gamma}\right)^c\right\}, x \in \text{Supp}F. \tag{1.4}$$

A new generalization of the Pareto distribution, the Weibull-Pareto distribution, was proposed using the Weibull-X family in (1.3) [see Alzaatreh et al., 2013]. The remainder of this paper is organized in the following way: In section 2, we study some general properties of the Weibull-X family. In section 3, a member of the Weibull-X family, the Weibull-logistic distribution, is defined and various properties of the distribution including the limiting behavior, transformation, mode and Shannon's entropy are discussed. In section 4, the moment generating function, the moments and the mean deviations from the mean and the median are studied. Section 5 deals with the moment generating function of the r-th order statistic, the limiting distribution of the sample minimum and the sample maximum for a random sample of size n drawn from the Weibull-logistic distribution. In section 6, the maximum likelihood estimation is proposed to estimate the Weibull-logistic distribution parameters. Application of the Weibull-logistic distribution to a real data set is provided in section 7.

2. The Weibull-X family of distributions

In this section, we discuss some general properties of the Weibull-X family defined in (1.3).

Lemma 2.1. *If a random variable Y follows the Weibull distribution with parameters c and γ, then the random variable $X = F^{-1}(1-e^{-Y})$ follows the Weibull-X distribution.*

Proof. The result follows by using the transformation technique. $\square$

Lemma 2.1 gives an easy way of simulating a random sample from the Weibull-X distribution by first simulating random sample, Y, from Weibull distribution with parameters c and γ , then computing $X = F^{-1}(1-e^{-Y})$, which follows the Weibull-X distribution.

Lemma 2.2. *If Let $Q(\lambda), 0 < \lambda < 1$, denote the quantile function for the Weibull-X family. Then $Q(\lambda)$ can be written as*

$$Q(\lambda) = F^{-1}\left(1-\exp(\gamma(-\log(1-\lambda))^{1/c})\right). \tag{2.1}$$

Proof. The result follows immediately from (1.4). □

In some cases, one can face a situation where the third and fourth moments of the Weibull-X distribution are not in closed form. This makes the expression for the skewness and kurtosis measures in terms of moments are analytically intractable. However, Lemma 2.2 indicates that the quantile function of Weibull-X family is in closed form. Consequently, one can define the measure of skewness and kurtosis based on the quantile function. The Galton' skewness S defined by Galton (1883) and the Moors' kurtosis K defined by Moors (1988) are given by

$$S = \frac{Q(6/8)-2Q(4/8)+Q(2/4)}{Q(6/8)-Q(2/8)}. \tag{2.2}$$

$$K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}. \tag{2.3}$$

When the distribution is symmetric, $S = 0$ and when the distribution is right (or left) skew, $S > 0$ (or $S < 0$). As K increases the tail of the distribution becomes heavier.

Lemma 2.3. *The rth non-central moments of the Weibull-X family is given by*

$$E(X^r) = \frac{c}{\gamma}\int_0^\infty [F^{-1}(1-e^{-u})]^r u^{c-1}e^{-u}du. \tag{2.4}$$

Proof. The result follows immediately from Lemma 2.1. □

The Shannon's entropy (Shannon, 1948) plays an important role in information theory and it is used as a measure of uncertainty. Shannon's entropy for a random variable X with p.d.f $g(x)$ is defined as $E[-\log(g(X))]$. According to Alzaatreh, et al. (2013), the Shannon's entropy for the Weibull-X family can be written as

$$-E\left[\log\left(f(F^{-1}(1-e^{-T}))\right)\right] - \gamma\Gamma(1+1/c) + \xi(1-1/c) - \log(c/\gamma) + 1. \tag{2.5}$$

where $F(.)$ and $f(.)$ are the c.d.f and p.d.f of the Transformer family respectively, and T follows Weibull with parameters c and γ, and ξ is the Euler's constant.

The hazard function associated with the Weibull-X family in (1.3) is

$$h_f(x) = \frac{g(x)}{1-G(x)} = \frac{cf(x)}{\gamma(1-F(x))}\left(-\frac{\log(1-F(x))}{\gamma}\right)^{c-1}. \tag{2.6}$$

Theorem 2.1.

(i) If $c \geq 1$ and $f(x)$ possess the increasing failure rate (IFR) property, then $g(x)$ in (1.3) possess the IFR property.
(ii) If $c < 1$ and $f(x)$ possess the decreasing failure rate (DFR) property, then $g(x)$ in (1.3) possess the DFR property.

Proof. Note that the hazard function of the Weibull-X family can be written as $h_g(x) = c\gamma^{-c}h_f(x)\left(H_f(x)\right)^{c-1}$. This implies

$$h'_g(x) = c\gamma^{-c}\left(H_f(x)\right)^{c-2}\left\{(c-1)h_f^2(x) + H_f(x)h'_f(x)\right\}. \tag{2.7}$$

If $f(x)$ possess the IFR property, then $h'_f(x) > 0$. This implies that $h'_g(x) > 0$ whenever $c \geq 1$. Part (ii) can be shown similarly and hence the proof. □

The reliability parameter R is defined as $R = P(X > Y)$, where X and Y are independent random variables. Numerous applications of the reliability parameter have appeared in the literature such as the area of classical stress-strength model and the break down of a system having two components. Other applications of the reliability parameter can be found in Hall (1984) and Weerahandi and Johnson (1992). If X and Y are two continuous and independent random variables with the c.d.fs $F_1(x)$ and $F_2(y)$ and their p.d.fs $f_1(x)$ and $f_2(y)$ respectively. Then the reliability parameter R can be written as

$$R = P(X > Y) = \int_{-\infty}^{\infty} F_2(t)f_1(t)dt.$$

Lemma 2.4. *Suppose Y and W are two independent random variables which follow the Weibull-X distributions with parameters (c_1,γ_1) and (c_2,γ_2) respectively. Then*

$$R = 1 - \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\gamma_1}{\gamma_2}\right)^{jc_2}\Gamma\left(j\frac{c_2}{c_1}+1\right). \tag{2.8}$$

Proof. From (1.3) and (1.4) and by using the substitution $u = -\log(1-F(t))$, we get

$$P(X > Y) = 1 - I, \tag{2.9}$$

where

$$\begin{aligned} I &= \frac{c_1}{\gamma_1}\int_0^{\infty}(u/\gamma_1)^{c_1-1}\exp\{-(u/\gamma_1)^{c_1} - (u/\gamma_2)^{c_2}\}du \\ &= \frac{c_1}{\gamma_1^{c}1}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\gamma_2^{jc_2}}\int_0^{\infty}e^{-(u/\gamma_1)^{c_1}}u^{c_1+jc_2-1}du \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\gamma_1}{\gamma_2}\right)^{jc_2}\Gamma(j\frac{c_2}{c_1}+1). \end{aligned} \tag{2.10}$$

The result follows immediately by substituting (2.10) in (2.9). □

In the next section, we define and study some properties of a member of the Weibull-X family, the Weibull-logistic distribution.

3. Weibull-logistic distribution and its properties

The simplicity of the logistic distribution and its importance as a growth curve have attracted many researchers to study this distribution. However, the limitation of the shape of the logistic distribution merits further investigation to various other different types of generalized logistic distribution to have more flexibility in modeling skewed data. In this section, we used the Weibull- X family to generate a member of its family namely, the Weibull-logistic distribution. If X follows a logistic distribution with parameters θ with the c.d.f $F(x) = 1-(1+e^{x/\theta})^{-1}$, $x \in \mathbb{R}$, then (1.3) reduces to

$$g(x) = \frac{ce^{x/\theta}}{\gamma\theta(1+e^{x/\theta})}\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{c-1}\exp\left\{-\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{c}\right\}, x \in \mathbb{R}, \qquad c,\gamma,\theta > 0. \tag{3.1}$$

Note that when $c = \gamma = 1$, the p.d.f in (3.1) reduces to the logistic distribution. When $c = 1$, the p.d.f in (3.1) reduces to the type I logistic distribution. From (3.1), the c.d.f of the Weibull-logistic distribution is given by

$$G(x) = 1-\exp\left\{-\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{c}\right\}, x \in \mathbb{R}. \tag{3.2}$$

A random variable X with the p.d.f $g(x)$ in (3.1) is said to follow the Weibull-logistic distribution with parameters c, γ and θ and will be denoted by WL(c,γ,θ). We provide below some characterizations of the WL(c,γ,θ) distribution which establishes the relation between Weibull-logistic, Weibull, Weibull-Pareto, exponential, and uniform distributions.

Lemma 3.1.

(i) *If a random variable X follows a Weibull distribution with parameters c and γ, then* $Y = \theta\log(e^{X}-1)$ *follows the WL(c,γ,θ) .*

(ii) *If a random variable X follows the Weibull-Pareto distribution (Alzaatreh, et al., 2013) with parameters c,* $1/\gamma$*, and* 1 *then* $Y = \theta\log(X-1)$ *follows the WL(c,γ,θ).*

(iii) *If a random variable X follows an exponential distribution with mean=* 1*, then* $Y = \theta\log(e^{\gamma X^{1/c}}-1)$ *follows the WL(c,γ,θ).*

(iv) *If a random variable X follows a standard uniform distribution, then* $Y = \theta\log(\exp(\gamma(-\log(1-X))^{1/c})-1)$ *follows the WL(c,γ,θ).*

From (2.6), the hazard function associated with the Weibull -logistic distribution is

$$h_g(x) = \frac{ce^{x/\theta}}{\gamma\theta(1+e^{x/\theta})}\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{c-1}, x \in \mathbb{R}. \tag{3.3}$$

The limiting behaviors of the WL(c,γ,θ) p.d.f and the hazard function are given in the following Lemma.

Lemma 3.2. *The limit of the Weibull logistic density function as $x \to \pm\infty$ is 0. Also the limit of the hazard function as $x \to -\infty$ is 0 and the limit as $x \to \infty$ is given by*

$$\lim_{x\to\infty} h_g(x) = \begin{cases} 0, & c < 1 \\ \frac{1}{\gamma\theta}, & c = 1. \\ \infty, & c > 1 \end{cases} \tag{3.4}$$

Theorem 3.1. *The Weibull-logistic distribution is unimodal and the mode is the solution of the equation $k(x) = 0$, where*

$$k(x) = e^{-x/\theta}\log(1+e^{x/\theta}) - c\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^c + c - 1. \tag{3.5}$$

Proof. The derivative of $g(x)$ in (3.1) can be written as

$$\frac{\partial}{\partial x}g(x) = c(A'(x))^2(A(x))^{c-2}e^{-(A(x))^c}k(x), \tag{3.6}$$

where $A(x) = (\log(1+e^{x/\theta}))/\gamma$. The critical values of $g(x)$ is the solution of $k(x) = 0$. Now to show that $k(x) = 0$ has only one solution, from (3.5), we can write

$$k'(x) = \gamma e^{-x/\theta}A'(x) - (\gamma/\theta)e^{-x/\theta}A(x) - c^2A'(x)(A(x))^{c-1}. \tag{3.7}$$

From (3.7), it is obvious that the last term in the expression of $k'(x)$ is negative. Next, consider the term

$$\gamma e^{-x/\theta}A'(x) - (\gamma/\theta)e^{-x/\theta}A(x) = \frac{1}{\theta(1+e^{x/\theta})} - \frac{\log(1+e^{x/\theta})}{\theta e^{x/\theta}}. \tag{3.8}$$

By using the inequality $\log(1+x) > x/(1+x)$, for $x > -1$ and $x \neq 0$, (Abramowitz and Stegun, 1972) we can write $\frac{\log(1+e^{x/\theta})}{e^{x/\theta}} > \frac{1}{(1+e^{x/\theta})}$. So the expression in (3.8) is negative. This implies that $k'(x) \leq 0$ and hence $g(x)$ has at most one maximum. Now, Lemma 3.2 and using the fact that $g(x) \geq 0$ imply that $g(x)$ must have one maximum. □

Lemma 3.3. *The quantile function for the WL(c, γ, θ) is given by*

$$Q(p) = \theta\log\left(\exp(\gamma(-\log(1-p))^{1/c}) - 1\right). \tag{3.9}$$

Proof. The result follows immediately from Lemma 2.2. □

In Figures 1 and 2, various graphs of $g(x)$ and $h_g(x)$ are provided for different parameter values. The plots indicate that the Weibull-logistic distribution can be symmetric, right-skewed or left-skewed. The Weibull-logistic hazard function can be an increasing failure rate, upside down or bathtub shapes. Also, when $c > 1$, the hazard function is an increasing function and this agrees with Theorem 1.
Next theorem defines expression for the Shannon's entropy for the Weibull-logistic distribution.

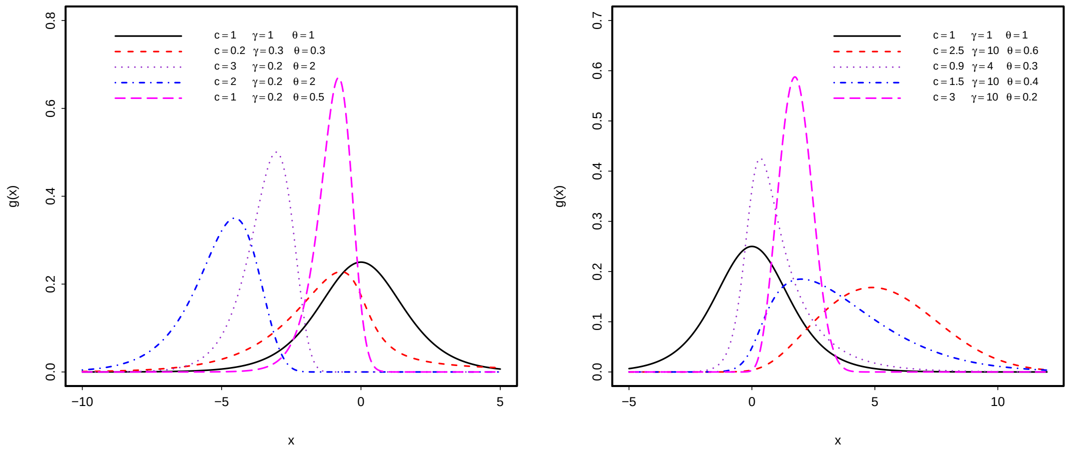

Fig. 1. Graphs of the WL p.d.f for various choices of c, γ, and θ.

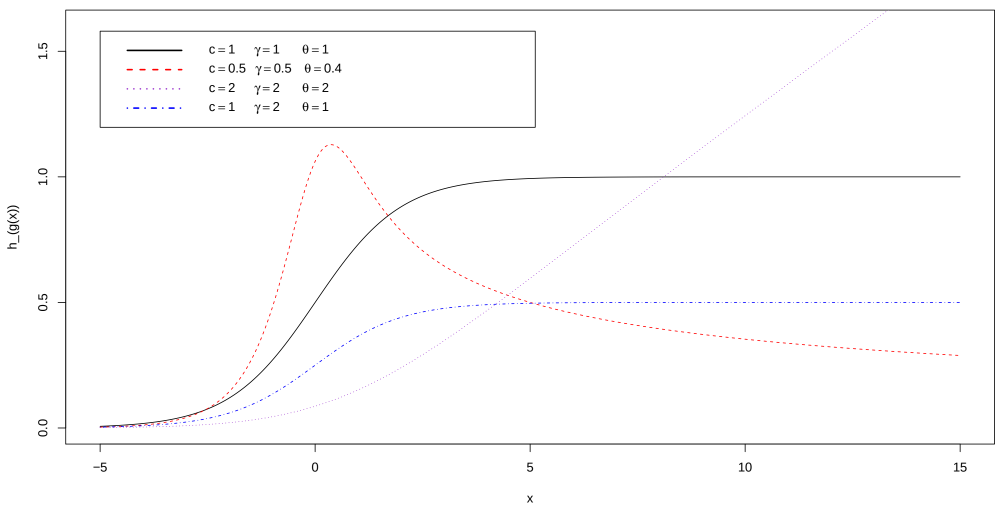

Fig. 2. Graphs of the WL hazard function for various choices of c, γ, and θ.

Theorem 3.2. *The Shannon's entropy for the random variable X which follows a $WL(c,\gamma,\theta)$ is given by*

$$\eta_x = c\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^j\Gamma(c(j+1))}{\gamma^{c(j+1)}k^{c(j+1)+1}} + \xi(1-1/c) - \log(c/\gamma) + \log\theta + 1. \tag{3.10}$$

Proof. In our case, $F(X) = 1-(1+e^{X/\theta})^{-1}, X \geq 0$. So that

$$\log\left\{f(F^{-1}(1-e^{T}))\right\} = \log(e^{T}-1) - 2T - \log\theta = \log(1-e^{-T}) - T - \log\theta, \tag{3.11}$$

where $T \sim$ Weibull(c,γ). Now, consider $E(\log(1-e^{-T}))$. Using the Taylor's series expansion of $\log(1-e^{-T})$, one can get

$$\begin{aligned} E(\log(1-e^{-T})) &= -\sum_{k=1}^{\infty} k^{-1} E(e^{-kT}) \\ &= -\frac{c}{\gamma}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(-1)^j k^{-1}\int_0^{\infty}\left(\frac{t}{\gamma}\right)^{c(j+1)-1} e^{-kt}dt \\ &= -c\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^j}{\gamma^{c(j+1)}k^{c(j+1)+1}}\Gamma(c(j+1)), \end{aligned} \tag{3.12}$$

also,

$$E(T) = \gamma\Gamma(1+1/c). \tag{3.13}$$

The result in (3.10) follows by substituting (3.12) and (3.13) in (2.5). □

4. Moments and mean deviations

The moment generating function for the WL(c,γ,θ) can be written as

$$M_X(t) = E(e^{tX}) = \frac{c}{\gamma\theta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}\frac{e^{x(t+1/\theta)}}{1+e^{x/\theta}}\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{kc+c-1} dx. \tag{4.1}$$

On using the substitution $u = \log(1+e^{x/\theta})$, (4.1) can be written as

$$M_X(t) = c\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+k}\frac{(t\theta)^j}{j!k!\gamma^{c(k+1)}}\int_0^{\infty} e^{-u(j-t\theta)}u^{kc+c-1}du.$$

On writing $(e^u-1)^{t\theta} = e^{ut\theta}(1-e^{-u})^{t\theta}$ and using the generalized binomial expression of $(1-e^{-u})^{t\theta}$, we get

$$M_X(t) = c\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}(t\theta)^{(j)}\Gamma(kc+c)}{j!k!\gamma^{c(k+1)}(j-t\theta)^{kc+c}}, \tag{4.2}$$

provided $t < \frac{1}{\theta}$ and $(t\theta)^{(j)} = t\theta(t\theta-1)\cdots(t\theta-j+1)$.
Next, we derive the expression for the rth order raw moment for the Weibull-logistic distribution. For any $r \in N$, using Lemma 2.3, we have

$$\begin{aligned} E(X^r) &= \theta^r E\left[\left(Y+\log(1+e^{-Y})\right)^r\right] \\ &= \theta^r\sum_{j=0}^{r}\binom{r}{j}E\left(Y^{r-j}(\log(1-e^{-Y}))^j\right) \end{aligned} \tag{4.3}$$

where $Y \sim \text{Weibull}(c,\gamma)$. By using the generalized expression of $(\log(1-e^{-Y}))^j = \sum_{k_1,k_2,\cdots,k_j=1}^{\infty}(-1)^j\frac{e^{-S_jY}}{a_j}$, where $S_j = k_1+k_2+\cdot+k_j$, and $a_j = k_1k_2\cdots k_j$, (4.3) reduces to

$$\begin{aligned}
E(X^r) &= \theta^r\sum_{j=0}^{r}\sum_{k_1,k_2,\cdots,k_j=1}^{\infty}(-1)^j\binom{r}{j}a_j^{-1}E(Y^{r-j}e^{-S_jY}) \\
&= c\theta^r\sum_{j=0}^{r}\sum_{k_1,k_2,\cdots,k_j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{r}{j}\frac{\Gamma(ic+c+r-j)}{i!\gamma^{c(i+1)}a_jS_j^{ic+c+r-j}}.
\end{aligned}$$

The deviation from the mean and the deviation from the median are used to measure the dispersion and the spread in a population from the center. If we denote the median by M, then the mean deviation from the mean, $D(\mu)$, and the mean deviation from the median, $D(M)$, can be written as

$$D(\mu) = E|X-\mu| = 2\mu G(\mu) - 2\int_{-\infty}^{\mu}xg(x)dx. \tag{4.4}$$

$$D(M) = E|X-M| = \mu - 2\int_{-\infty}^{M}xg(x)dx. \tag{4.5}$$

Now, consider

$$\begin{aligned}
I_m &= \int_{-\infty}^{m}xg(x)dx \\
&= \frac{c}{\gamma\theta}\int_{-\infty}^{m}\frac{xe^{x/\theta}}{1+e^{x/\theta}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{cj+c-1}dx. \qquad (4.6)
\end{aligned}$$

Using the substitution $u = (\log(1+e^{x/\theta}))/\gamma$ in (4.6), we get

$$\begin{aligned}
I_m &= c\theta\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_0^{(\log(1+e^{m/\theta}))/\gamma}\log(e^{u\gamma}-1)u^{cj+c-1}du \\
&= c\theta\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left\{\int_0^{(\log(1+e^{m/\theta}))/\gamma}\gamma u^{cj+j}du + \int_0^{\log(1+e^{m/\theta})}u^{cj+c-1}\log(1-e^{-u\gamma})du\right\}.
\end{aligned}$$

On using the Taylor series expansion of $\log(1-e^{-u\gamma})$, we get

$$I_m = c\theta\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\gamma^{c(j+1)}}\left\{\frac{(\log(1+e^{m/\theta}))^{c(j+1)+1}}{c(j+1)+1} - \sum_{k=1}^{\infty}\frac{\Gamma(c(j+1),k\log(1+e^{m\theta}))}{k^{c(j+1)+1}}\right\}, \tag{4.7}$$

where $\Gamma(\alpha,x)$ is the incomplete gamma function defined as $\Gamma(\alpha,x) = \int_0^x u^{\alpha-1}e^{-u}du$.
By using equations (4.4) and (4.5), the mean deviation from the mean and the mean deviation from the median are, respectively, given by

$$\begin{aligned}
D(\mu) &= 2\mu\left(1-\exp(-((\log(1+e^{\mu/\theta}))/\gamma)^c)\right) - 2I_\mu. \\
D(M) &= \mu - 2I_M.
\end{aligned}$$

Since the expression of the third and fourth moments of the $\text{WL}(c,\gamma,\theta)$ are not in closed form and the quantile function of $\text{WL}(c,\gamma,\theta)$ is in closed form, the Galton' skewness S and the Moors' kurtosis K are used to investigate the effect of the two shape parameters c and γ on the Weibull logistic distribution. Equations (6) and (7) are used to obtain the Galton' skewness and the Moors' kurtosis where the quantile function is defined in Lemma 3.3. Figure 4 displays the Galtons' skewness and Moors' kurtosis for the WL(c,γ,θ) when $\theta = 1$. From Figure 3, the WL(c,γ,θ) distribution can be left skewed, right skewed or symmetric. For fixed γ, the skewness and the kurtosis increase as c decreases.

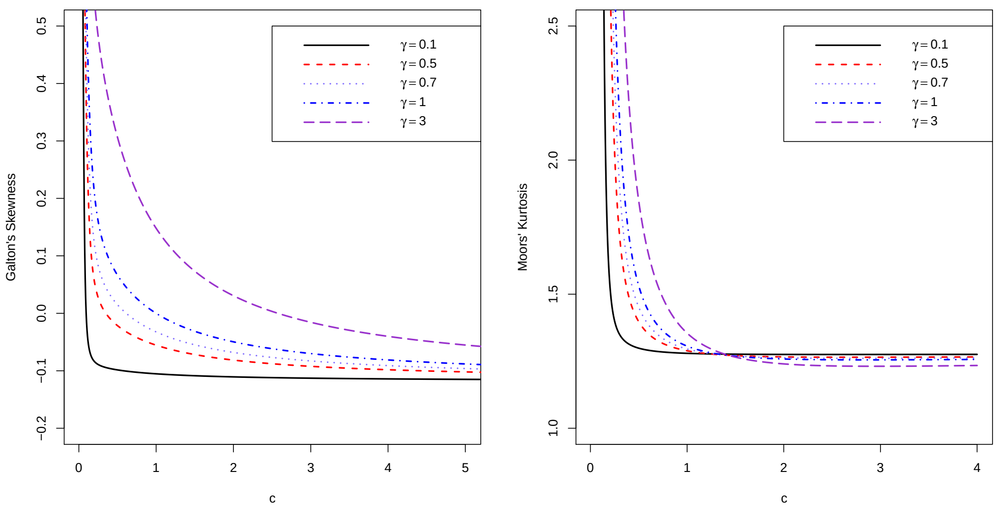

Fig. 3. Graphs of Skewness and Kurtosis for the WL p.d.f when $\theta = 1$.

5. Order statistics for the Weibull logistic distribution

In this section we will consider the expression for the general r-th order statistic and the large sample distribution of the sample minimum and the sample maximum when a random sample of size n are drawn from the WL(c,γ,θ) distribution. The density function of the r-th order statistic, $X_{r:n}$ for a random sample of size n drawn from (3.1), is given by

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!(n-r)!}(G(x))^{r-1}(1-G(x))^{n-r}g(x) \\ &= \frac{cn!e^{x/\theta}}{\gamma\theta(r-1)!(n-r)!(1+e^{x/\theta})}\left(1-\exp(-((\log(1+e^{x/\theta}))/\gamma)^{c})\right)^{r-1} \\ &\quad \times\left\{\exp\left(-(n-r+1)\left((\log(1+e^{x/\theta}))/\gamma\right)^{c}\right)\right\}\left((\log(1+e^{x/\theta}))/\gamma\right)^{c-1}, x\in\mathbb{R}. \quad (5.1) \end{aligned}$$

From (5.1), the distribution of the sample maximum $X_{n:n} = \max(X_1,X_2,\cdots,X_n)$, and the sample minimum $X_{1:n} = \min(X_1,X_2,\cdots,X_n)$ are

$$f_{X_{n:n}}(x) = \frac{cne^{x/\theta}}{\gamma\theta(1+e^{x/\theta})}\left((\log(1+e^{x/\theta}))/\gamma\right)^{c-1}\exp\left(-((\log(1+e^{x/\theta}))/\gamma)^c\right)$$
$$\times\left\{1-\exp(-((\log(1+e^{x/\theta}))/\gamma)^c)\right\}^{n-1}, x\in\mathbb{R}, \tag{5.2}$$

and

$$f_{X_{1:n}}(x) = \frac{cne^{x/\theta}}{\gamma\theta(1+e^{x/\theta})}\left(\frac{\log(1+e^{x/\theta})}{\gamma}\right)^{c-1}\left(\exp(-n((\log(1+e^{x/\theta}))/\gamma)^c)\right), x\in\mathbb{R}. \tag{5.3}$$

In order to find the large sample distribution of $X_{n:n}$, we will use the sufficient condition for weak convergence due to von Mises (1936) which is stated in the following theorem:

Theorem 5.1. *Let G be an absolutely continuous c.d.f and suppose $h_g(x)$ is nonzero and differentiable function. If*

$$\lim_{x\to G^{-1}(1)}\frac{d}{dx}\left(\frac{1}{h_g(x)}\right)=0,$$

then $G\in\mathscr{D}(G_1)$, where $G_1(x)=\exp(-\exp(x))$.

In our case $G^{-1}(1)=\infty$ and from (17), we have

$$\lim_{x\to\infty}\frac{d}{dx}\left(\frac{1}{h_g(x)}\right) = \lim_{x\to\infty}\left\{\frac{\gamma\theta}{c}(\log(1+e^{x/\theta}))^{-c}\left((1-c)(e^{-x/\theta}+1)-e^{-x/\theta}\log(1+e^{x/\theta})\right)\right\}$$
$$=0.$$

Hence the large sample distribution of $X_{n:n}$ is of extreme value type. The asymptotic distribution of the sample minima $X_{1:n}$ can be viewed as $G^*(x)$, where $G^*(x)=1-G_1(-x)$.
Next, we consider the moment generating function for any $X_{r:n}, 1\le r\le n$.
On using the substitution $u=\log(1+e^{y/\theta})$, we can write

$$M_{X_{r:n}}(t) = \frac{c}{\gamma}\int_0^\infty (e^u-1)^{t\theta}(u/\gamma)^{c-1}\left(\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\exp(-(n-r+j+1)(u/\gamma)^c)\right)$$
$$=\frac{c}{\gamma}\sum_{k=0}^{\infty}\sum_{j=0}^{r-1}\frac{(-1)^{j+k}(n-r+j+1)^k}{k!}\binom{r-1}{j}\int_0^\infty (e^u-1)^{t\theta}(u/\gamma)^{kc+c-1}du.$$

Following same technique as in (4.2), we get

$$M_{X_{r:n}}(t) = c\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{r-1}\frac{(-1)^{j+k+l}(n-r+j+1)^k(t\theta)^{(\ell)}}{k!\ell!\gamma^{kc+c}(\ell-t\theta)^{kc+c}}\binom{r-1}{j}\Gamma(kc+c),$$

provided $t<\frac{1}{\theta}$.

6. Maximum likelihood estimation

In this section we address the parameter estimation of the WL(c,γ,θ) under the classical set up. Let X_1, X_2, $\cdots$, X_n be a random sample of size n drawn from the density in (3.1). The log-likelihood function is given by

$$\ell = \log L(c,\gamma,\theta) = n\log c - nc\log\gamma - n\log\theta + \frac{n\bar{X}}{\theta} - \sum_{i=1}^{n}\log(1+e^{X_i/\theta})$$
$$+(c-1)\sum_{i=1}^{n}\log\left(\log(1+e^{X_i/\theta})\right) - \gamma^{-c}\sum_{i=1}^{n}\left(\log(1+e^{X_i/\theta})\right)^{c}. \tag{6.1}$$

The derivatives of (6.1) with respect to c, γ and θ are given by

$$\frac{\partial\ell}{\partial c} = \frac{n}{c} - n\log\gamma + \sum_{i=1}^{n}\log\left(\log(1+e^{X_i/\theta})\right) - \sum_{i=1}^{n}\left(\frac{\log(1+e^{X_i/\theta})}{\gamma}\right)^{c}\log\left(\frac{\log(1+e^{X_i/\theta})}{\gamma}\right). \tag{6.2}$$

$$\frac{\partial\ell}{\partial\gamma} = -\frac{nc}{\gamma} + c\gamma^{-c-1}\sum_{i=1}^{n}\left(\log(1+e^{X_i/\theta})\right)^{c}. \tag{6.3}$$

$$\frac{\partial\ell}{\partial\theta} = -\frac{n}{\theta} - \frac{n\bar{X}}{\theta^2} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{X_i e^{X_i/\theta}}{1+e^{X_i/\theta}}\left\{1 - \frac{c-1}{\log(1+e^{X_i/\theta})} + c\gamma^{-c}\left(\log(1+e^{X_i/\theta})\right)^{c-1}\right\}. \tag{6.4}$$

The MLE of $\hat{c}$, $\hat{\gamma}$ and $\hat{\theta}$ are obtained by setting (6.2), (6.3) and (6.4) to zero and solving them simultaneously.

The initial value for the parameter θ can be obtained by assuming that a random sample of size n, $(X_1, X_2, \cdots, X_n)$ are taken from the logistic distribution with parameter θ. Hence the initial value for θ (Johnson et al., 1994b, p. 117) is $\theta_0 = \sqrt{3}s_x/\pi$, where s_x is the sample standard deviation of X_i. Using Lemma 3.1(i), the initial values for the parameters c and γ can be obtained by considering $y_i = \theta_0\log(e^{x_i}-1)$, $i = 1,2,\cdots,n$ for a random sample of size n drawn from the Weibull distribution with parameters c and γ. Hence the initial values for the parameters c and γ (Johnson et al., 1994a, pp. 642-643) are $c_0 = \frac{\pi}{\sqrt{6}S_{\log y_i}}$ and $\gamma_0 = \exp(\bar{x}_{\log y_i} + \xi/c_0)$, where $S_{\log y_i}$ and $\bar{x}_{\log y_i}$ are the sample standard deviation and the sample mean for $\log y_i$, and ξ is the Euler gamma constant which approximately equals 0.57722.

A simulation study is done for evaluating the performance of the MLE, we consider several parameter choices for c, γ and θ. Two different sample sizes $n = 100$ and 300 are considered. For each parameter combination, we generate a random sample $y_1, y_2, \cdots, y_n$ from Weibull distribution with parameters c and γ. By using the transformation $x_i = log(exp(y_i)-1)$ [see Lemma 3.1(i)], we obtain a random sample, $x_1, x_2, \cdots, x_n$ which follow WL(c,γ,θ). The MLE for the parameters c,γ and θ are obtained for each WL(c,γ,θ) sample. This process is repeated 200 times in order to find the means and standard deviations of the parameter estimates. To conserve space, the bias (estimate - actual) and the standard deviation for some parameter choices are presented in Tables 1. The small values of bias and standard deviation indicate that the maximum likelihood estimation method performs quite well in estimating the model parameters.

Table 1. Bias and standard error of the estimates under the maximum likelihood method.

Sample Size	Actual Values			Bias			Standard Deviation		
n	c	γ	θ	$\hat{c}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{c}$	$\hat{\gamma}$	$\hat{\theta}$
100	1	1	1	0.1697	−0.0295	0.1885	0.4445	0.1182	0.3758
	2.5	1.5	1	0.0735	0.0972	0.0283	0.6794	0.4227	0.3758
	3	2	2	−0.0639	0.2801	−0.1135	0.5060	0.6526	0.6227
	0.5	1	2	−0.0113	−0.0641	−0.0613	0.1560	0.2965	0.7352
300	1	1	1	0.0544	−0.0137	0.0674	0.1412	0.0686	0.1680
	2.5	1.5	1	0.0421	0.0194	0.0203	0.3731	0.1893	0.2140
	3	2	2	−0.0202	0.0966	−0.0324	0.3463	0.3511	0.4231
	0.5	1	2	0.0194	0.0073	0.1034	0.0456	0.1253	0.2357

7. Application

In this section, the Weibull-logistic distribution is applied to a data set from Smith and Naylor (1987) on the strengths of 1.5 cm glass fibres measured at the National Physical Laboratory in England. Recently, Nadarajah (2009) used Azzalini (1985) approach to propose a two-parameter skew logistic distribution. The Weibull-logistic distribution is fitted to the data set and compared the results with the logistic and the skew logistic distributions. The maximum likelihood estimates, the log-likelihood value, the AIC (Akaike Information Criterion), the Kolmogorov-Smirnov test statistic (*K-S*), and the p-value for the *K-S* statistic for the fitted distributions are reported in Tables 2. The results in Table 2 show that only the Weibull-logistic distribution gives an adequate fit while the logistic and skew logistic distributions do not give an adequate fit. After these results, we decided to add a location parameter to the skew logistic distribution. The results from Table 2 show that the location parameter improves the skew logistic distribution fits significantly. Figure 4 displays the empirical and fitted cumulative distribution functions for the strength of 1.5 glass fibres data. This Figure supports the results in Table 2.

Table 2. Parameter estimates for the glass fibre data.

Distribution	Logistic	Skew logistic	Skew logistic with location	Weibull-logistic
Parameter Estimates	$\hat{\theta} = 0.7425$	$\hat{\lambda} = 251.9920$ $\hat{\beta} = 0.7425$	$\hat{\lambda} = -2.8240$ $\hat{\beta} = 0.2210$ $\hat{m} = 1.4170$	$\hat{c} = 10.9867$ $\hat{\gamma} = 1.2397$ $\hat{\theta} = 1.3909$
Log likelihood	-75.4164	-43.5316	-2.2925	-2.0928
AIC	152.8328	91.0632	10.5850	10.1856
K-S K-S p-value	0.6544 0.0000	0.3833 0.0000	0.0516 0.9997	0.0417 1.0000

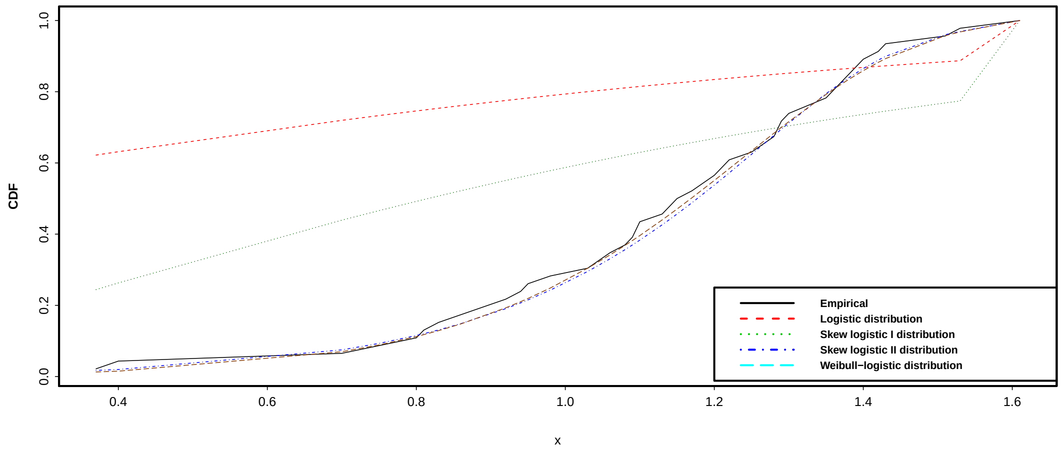

Fig. 4. C.d.f for fitted distributions of the strength of 1.5 glass fibres data

8. Concluding remarks

In this paper, we study some properties of the Weibull-X family of distributions and focuses the attention on studying a special case of the Weibull-X family, namely, the Weibull-logistic distribution. Various properties of the Weibull-logistic distribution are studied, including moments, hazard function, mode and Shannon entropy. It is observed that the distribution can be symmetric, positively skewed and negatively skewed. A data set on the strengths of 1.5 cm glass fibres was fitted to the Weibull-logistic distribution and compared with other distributions. The results show that the Weibull-logistic distribution gives almost a perfect fit to the data. Moreover, since the distribution function is in a closed form, it can also be used to model censored data. Further research is needed to investigate the applicability of this distribution to model censored data.

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