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Symmetry analysis for Whitham-Broer-Kaup equations

Zhiyong Zhang [a], *Xuelin Yong* [b] *and Yufu Chen* [c]

[a] *School of Mathematical Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, P. R. China*
E-mail: zhiyong-2008@163.com

[b] *Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, P. R. China*
E-mail: yongxuelin2003@yahoo.com.cn

[c] *School of Mathematical Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, P. R. China*
E-mail: yfchen@gucas.ac.cn

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Abstract

We investigate a further group analysis of Whitham-Broer-Kaup(for short WBK) equations. An optimal system of one-dimensional subalgebras is derived and used to construct reduced equations and similarity solutions. Moreover, a special case of WBK equations is linearized and some new solutions are obtained. At last, conservation laws are also analyzed by means of scaling symmetry.

1 Introduction

Since Sophus Lie (1842-1899) introduced the notion of continuous transformation group, now known as Lie group, the theory of Lie group and Lie algebra have been evolved into one of the most explosive development of mathematics and physics throughout the past century. One of the main applications of Lie theory of symmetry group for differential equations is the construction of similarity(group invariant) solutions. Given any subgroup of the symmetry group, one can write down the equations for the similarity solution with respect to this subgroup. This reduced system is of fewer variables and easier to solve generally [1, 2, 3, 4, 5]. But a Lie group (or Lie algebra) usually contains infinitely many subgroups (or subalgebras) of the same dimension, it is not usually feasible to list all possible similarity solutions. Hence, one needs an effective, systematic means of classifying these solutions, leading to a "basis set" of similarity solutions from which every other such solution can be derived. This leads the notion of optimal system of symmetry subgroup introduced and some examples can be found in [1, 6, 7]. Simultaneously, constructing point transformation mapping the nonlinear partial differential equations(PDEs) which necessarily admit an infinite set of point (contact) symmetries to linear PDEs is also an important application of symmetry group theorem. Then, the solutions of nonlinear PDEs can be obtained from the linearized equations through the transformation. S. Anco *et.al* [8], G. Bluman and S. Kumei [9, 10] have completed some theoretic studies and performed some examples in this field.

Another important application is to construct conservation laws by the known symmetries [11, 12, 13, 14]. W. Hereman *et al.* [14] advocate a more direct approach by building the candidate density as a linear combination (with constant coefficients) of terms that are uniform in rank with respect to the scaling symmetry of the PDEs and can be implemented in most computer algebra systems such as Mathematica, Maple, and Reduce. Ü. Götaş and W.Hereman also applied this method to find higher-order symmetries for nonlinear evolution and lattice equations [15].

In this paper, we investigate nonlinear WBK equations in shallow water obtained by Whitham, Broer and Kaup [16, 17, 18]

$$\begin{aligned} u_t &= uu_x + v_x + \beta u_{xx}, \\ v_t &= vu_x + uv_x + \beta v_{xx} + \alpha u_{xxx}, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ is the field of horizontal velocity, $v = v(x,t)$ is the height that deviate from equilibrium position of liquid, α, β are constants that represent different diffusion power. If $\alpha = 0, \beta \neq 0$, Eq.(1.1) are classical long-wave equations that describe shallow water wave with diffusion [16]. If $\alpha = 1, \beta = 0$, Eq.(1.1) are modified Boussinesq equations [19]. If $\alpha = 0, \beta = 0$, Eq.(1.1) are one-dimensional shallow water equations on a flat bottom [20].

In the last decades, there have been several methods proposed to study Eq.(1.1) , which include inverse transformation formula [16, 19], homogeneous balance method [21], improved sine-cosine method and the Wu elimination method [22], Backlund transformation [23], hyperbolic function method [24] and nonclassical symmetries method [20]. In this paper, we present Lie point symmetries analysis and derive an optimal system of one-dimensional subalgebras, some reductions and similarity solutions are constructed. Furthermore, a special case for $\alpha = 0, \beta = 0$ is linearized by its admitted symmetry and some conservation laws are obtained by scaling symmetry.

The outline of the present paper is as follows. In Section 2, we investigate Lie point symmetries of Eq.(1.1). An optimal system of one-dimensional subalgebras is derived. Some symmetry reductions and similarity solutions are also obtained. In Section 3, a special case of WBK equations is linearized and some new solutions are derived. Conservation laws are considered in Section 4. Finally, we conclude this paper in Section 5.

2 Symmetry analysis

In this section, we first analyze Lie point symmetries of Eq.(1.1) and derive an optimal system of one-dimensional subalgebras, then similarity solutions and reduced equations are constructed.

2.1 Lie point symmetries

We consider a one-parameter Lie group of local transformations with an infinitesimal operator of the form

$$X = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v, \tag{2.1}$$

which leaves Eq.(1.1) invariant. Using the characteristic set algorithm for differential polynomial systems and its implemented program to the determining equations [26, 27, 28], we have two cases to discuss.
Case 1: $(\alpha, \beta) \neq (0,0)$.
In this case, we obtain

$$\xi = -\frac{1}{2}c_1 x - c_2 t + c_4, \tau = -c_1 t + c_3, \eta = \frac{1}{2}c_1 u + c_2, \phi = c_1 v, \tag{2.2}$$

and the corresponding infinitesimal operators are

$$X_1 = -\frac{1}{2}x\partial_x - t\partial_t + \frac{1}{2}u\partial_u + v\partial_v, X_2 = -t\partial_x + \partial_u, X_3 = \partial_x, X_4 = \partial_t. \tag{2.3}$$

Case 2: $(\alpha, \beta) = (0,0)$.
Here, we get

$$\begin{aligned} &\xi = c_3 x + (\frac{3}{4}c_1 u^2 - \frac{3}{2}c_1 v - c_4)t + g(u,v), \eta = \frac{1}{2}(\frac{1}{2}c_1 u^2 + c_2 u) + c_1 v + c_4, \\ &\tau = -\frac{1}{2}c_1 x + \frac{1}{2}(2c_3 - 3c_1 u - c_2)t + f(u,v), \phi = (c_1 u + c_2)v. \end{aligned} \tag{2.4}$$

The Lie algebra of infinitesimal symmetries of the WBK equations is spanned by the four finite-dimensional subalgebras

$$\begin{aligned} &X_5 = x\partial_x + t\partial_t, X_6 = \partial_u - t\partial_x, X_7 = -\tfrac{1}{2}t\partial_t + \tfrac{1}{2}u\partial_u + v\partial_v, \\ &X_8 = (\tfrac{3}{4}u^2 - \tfrac{3}{2}v)t\partial_x - (\tfrac{1}{2}x + \tfrac{3}{2}ut)\partial_t + (\tfrac{1}{4}u^2 + v)\partial_u + uv\partial_v. \end{aligned} \tag{2.5}$$

and one infinite-dimensional subalgebra

$$X_9 = g(u,v)\partial_x + f(u,v)\partial_t. \tag{2.6}$$

where f, g satisfy $g_v + uf_v - f_u = 0, g_u - vf_v + uf_u = 0$.

2.2 Optimal system of one-dimensional subalgebras

In this subsection we give an optimal system of one-dimensional subalgebras [29] for Eq.(1.1). Due to the complexity of calculations, we first present an optimal system for $X_5 \sim X_8$, then for $X_1 \sim X_4$ with the similar method.

2.2.1 Optimal system of $X_5 \sim X_8$.

We want to classify its one-dimensional subalgebras up to the adjoint representation for $X_5 \sim X_8$. Table 1 shows the Lie brackets of $X_5 \sim X_8$.

Table 1: Lie brackets of $X_5 \sim X_8$

	X_5	X_6	X_7	X_8
X_5	0	0	0	0
X_6	0	0	$\frac{1}{2}X_6$	$X_7 - \frac{1}{2}X_5$
X_7	0	$-\frac{1}{2}X_6$	0	$\frac{1}{2}X_8$
X_8	0	$\frac{1}{2}X_5 - X_7$	$-\frac{1}{2}X_8$	0

Each X_i of (2.5) generates an adjoint representation $Ad(exp(\epsilon X_i))X_j$ defined by [7]

$$Ad(exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \cdots . \tag{2.7}$$

From the commutator Table 1, we obtain the adjoint representation generated by $X_5 \sim X_8$ in Table 2, with the (i, j) entry indicating $Ad(exp(\epsilon X_i))X_j$.

Table 2: Adjoint representation generated by $X_5 \sim X_8$

Ad	X_5	X_6	X_7	X_8
X_5	X_5	X_6	X_7	X_8
X_6	X_5	X_6	$X_7 - \frac{\epsilon}{2}X_6$	$X_8 - \epsilon(X_7 - \frac{1}{2}X_5) + \frac{\epsilon^2}{4}X_6$
X_7	X_5	$e^{\frac{\epsilon}{2}}X_6$	X_7	$e^{-\frac{\epsilon}{2}}X_8$
X_8	X_5	$X_6 + \epsilon(X_7 - \frac{1}{2}X_5) + \frac{\epsilon^2}{4}X_8$	$X_7 + \frac{\epsilon}{2}X_8$	X_8

Hence they can be used to classify similar one-dimensional subalgebras. However, before proceeding with the classification scheme we need to identify invariants of the full adjoint action. These invariants place restrictions on how far we can expect to simplify a given arbitrary element spanned by $X_5 \sim X_8$

$$X = a_5X_5 + a_6X_6 + a_7X_7 + a_8X_8. \tag{2.8}$$

The adjoint representation group is generated (via Lie equations) by the Lie algebra $X_5 \sim X_8$ spanned by the following symmetries (see [1], vol. 2)

$$\triangle_i = c_{ij}^k e^j \frac{\partial}{\partial e^k}, i = 5, \cdots, 8, \tag{2.9}$$

where c_{ij}^k are the structure constants in Table 1. Explicitly we have

$$\begin{aligned}
&\triangle_5 = 0,\\
&\triangle_6 = \frac{1}{2}a_7\frac{\partial}{\partial a_6} + a_8(\frac{\partial}{\partial a_7} - \frac{1}{2}\frac{\partial}{\partial a_5}),\\
&\triangle_7 = -\frac{1}{2}a_6\frac{\partial}{\partial a_6} + \frac{1}{2}a_8\frac{\partial}{\partial a_8},\\
&\triangle_8 = \frac{1}{2}a_7\frac{\partial}{\partial a_8} - a_6(\frac{\partial}{\partial a_7} - \frac{1}{2}\frac{\partial}{\partial a_5}).
\end{aligned} \tag{2.10}$$

If a function $\rho(a_5, a_6, a_7, a_8)$ is an invariant of the full adjoint action, then the symmetries (2.10) yield

$$\triangle_i(\rho) = 0, i = 5, \cdots, 8. \tag{2.11}$$

Eq.(2.11) can be reduced to

$$a_6\frac{\partial\rho}{\partial a_6} - a_8\frac{\partial\rho}{\partial a_8} = 0, \quad a_7\frac{\partial\rho}{\partial a_6} + a_8(2\frac{\partial\rho}{\partial a_7} - \frac{\partial\rho}{\partial a_5}) = 0. \tag{2.12}$$

After direct computations, we get solutions of Eq.(2.12)

$$\rho = f(2a_5 + a_7, a_7^2 - 4a_6a_8). \tag{2.13}$$

In particular,

$$\eta_1(X) = a_7^2 - 4a_6a_8, \quad \eta_2(X) = 2a_5 + a_7 \tag{2.14}$$

are two invariants of the full adjoint action given in Table 2.

The invariants η_1 and η_2 provide us a key condition to simplify X by the action of adjoint maps. For example, If $\eta_1 \neq 0$, then we cannot simultaneously make a_6, a_7 and a_8 zero through adjoint maps. Similarly, if $\eta_2 \neq 0$, we cannot simultaneously make a_5 and a_7 zero through adjoint maps.

Hence, to begin the classification process, we investigate the coefficients a_6, a_7 and a_8. If X is presented in (2.8), then

$$\widehat{X} = \widehat{a_5}X_5 + \widehat{a_6}X_6 + \widehat{a_7}X_7 + \widehat{a_8}X_8 = Ad(exp(\beta X_8)) \circ Ad(exp(\alpha X_6))X \tag{2.15}$$

has coefficients

$$\widehat{a_5} = a_5, \quad \widehat{a_6} = a_6 - \frac{\alpha}{2}a_7 + \frac{\alpha^2}{4}a_8, \tag{2.16}$$

$$\widehat{a_7} = \beta a_6 + (1 - \frac{\alpha\beta}{2})a_7 + (\frac{\alpha^2\beta}{4} - \alpha)a_8, \tag{2.17}$$

$$\widehat{a_8} = \frac{\beta^2}{4}a_6 + (\frac{\beta}{2} - \frac{\alpha\beta^2}{8})a_7 + (\frac{\alpha^2\beta^2}{16} - \frac{\alpha\beta}{2} + 1)a_8. \tag{2.18}$$

In order to proceed, the following three cases about $\eta_1(X)$ should be considered.

Case 1: $\eta_1(X) > 0$.

In this case, we choose $\alpha = \dfrac{a_7 + \sqrt{\eta_1}}{a_8}, \beta = \dfrac{2a_8}{\alpha a_8 - a_7}$, then $\widehat{a_6} = \widehat{a_8} = 0$, $\widehat{a_7} = \sqrt{\eta_1} \neq 0$, X is equivalent to

$$\widehat{X} = \widehat{a_5}X_5 + \widehat{a_7}X_7. \tag{2.19}$$

No further simplification of $\widehat{X}$ is possible through the application of adjoint maps. After scaling $\widehat{X}$, we obtain every one dimensional subalgebra generated by X with $\eta_1 > 0$ is equivalent to the subalgebra spanned by

$$X_5 + aX_7, a \in R(\neq 0). \tag{2.20}$$

Case 2: $\eta_1(X) < 0 (\Rightarrow a_6a_8 > 0)$.

Here, we cannot make two of the coefficients $\widehat{a_6}, \widehat{a_7}$ and $\widehat{a_8}$ vanish simultaneously, but one of them can be annihilated. If X is the form as (2.8), then

$$\widehat{X} = \widehat{a_5}X_5 + \widehat{a_6}X_6 + \widehat{a_7}X_7 + \widehat{a_8}X_8 = Ad(exp(\alpha X_6))X \tag{2.21}$$

has coefficients

$$\begin{aligned}
\widehat{a_6} &= a_6 - \frac{\alpha}{2}a_7 + \frac{\alpha^2}{4}a_8,\\
\widehat{a_7} &= a_7 - \alpha a_8,\\
\widehat{a_8} &= a_8.
\end{aligned} \tag{2.22}$$

Set $\alpha = \dfrac{a_7}{a_8}$, then $\widehat{a_7} = 0, \widehat{a_6} = -\dfrac{\eta_1}{4a_8} \neq 0$, so X can be reduced to

$$\widehat{X} = \frac{\eta_2(X)}{2}X_5 - \frac{\eta_1(X)}{4a_8}X_6 + a_8X_8. \tag{2.23}$$

No further simplification of $\widehat{X}$ is possible through the application of adjoint maps, but we can simplify the coefficients of X_5, X_6, X_8.

Acting on $\widehat{X}$ by $Ad(exp(\beta X_7))$ leads to

$$Ad(exp(\beta X_7))\widehat{X} = \frac{\eta_2(X)}{2}X_5 - \frac{\eta_1(X)}{4a_8}e^{\beta/2}X_6 + a_8e^{-\beta/2}X_8. \tag{2.24}$$

Due to $a_6a_8 > 0$ implies $a_8 > 0, a_6 > 0$ or $a_8 < 0, a_6 < 0$, so we first consider $a_8 > 0, a_6 > 0$.

SubCase 2.1: $\eta_2(X) > 0$.

Set $\beta = -2\ln(\dfrac{\eta_2}{2a_8})$,

$$\widehat{\widehat{X}} = \frac{\eta_2(X)}{2}(X_5 + X_8) - \frac{\eta_1(X)}{2\eta_2(X)}X_6. \tag{2.25}$$

After scaling $\widehat{\widehat{X}}$, every one-dimensional subalgebra generated by X with $\eta_1(X) < 0, \eta_2(X) > 0$ is equivalent to the subalgebra spanned by

$$X_5 + X_8 + aX_6, \quad a \in R(\neq 0). \tag{2.26}$$

SubCase 2.2: $\eta_2(X) < 0$.

Assume $\beta = 2\ln(\dfrac{2a_8\eta_2}{\eta_1})$,

$$\widehat{\widehat{X}} = \frac{\eta_2(X)}{2}(X_5 - X_6) + \frac{\eta_1(X)}{2\eta_2(X)}X_8. \tag{2.27}$$

After scaling $\widehat{\widehat{X}}$, every one-dimensional subalgebra generated by X with $\eta_1(X) < 0, \eta_2(X) < 0$ is equivalent to the subalgebra spanned by

$$X_5 - X_6 + aX_8, \quad a \in R(\neq 0). \tag{2.28}$$

SubCase 2.3: $\eta_2(X) = 0$.

Choose $\beta = \ln(-\dfrac{4a_8^2}{\eta_1})$, then

$$\widehat{\widehat{X}} = \frac{\sqrt{-\eta_1(X)}}{2}(X_8 + X_6). \tag{2.29}$$

After scaling $\widehat{\widehat{X}}$, every one-dimensional subalgebra generated by X with $\eta_1(X) < 0, \eta_2(X) = 0$ is equivalent to the subalgebra spanned by

$$X_8 + X_6. \tag{2.30}$$

Remarks: The case of $a_8 < 0$ is similar to $a_8 > 0$ except that X is simplified to (2.26) when $\eta_2(X) < 0$ and to (2.28) when $\eta_2(X) > 0$ with the same values of β respectively.

Case 3: $\eta_1(X) = 0$.

There are four subcases to consider here depending upon which of the coefficients a_6, a_7 and a_8 vanish.

(i): $a_6 = a_7 = a_8 = 0$, then $X = a_5X_5$, which can be scaled to X_5;

(ii): $a_6 = a_7 = 0, a_8 \neq 0$, then $X = a_5X_5 + a_8X_8$, which can be scaled to $X_5 + aX_8 (a \neq 0)$;

(iii): $a_7 = a_8 = 0, a_6 \neq 0$, then $X = a_5X_5 + a_6X_6$, which can be scaled to $X_5 + aX_6 (a \neq 0)$;

(iv): $a_6 \neq 0, a_7 \neq 0, a_8 \neq 0$, then $a_6a_8 > 0$, so we calculate it as in ***Case 2*** with additional conditions $\eta_1(X) = 0$. When $\eta_2(X) = 0$, then $X = a_8X_8$, which can be scaled to X_8; When $\eta_2(X) \neq 0$, then $X = e^{-\beta/2}a_8X_8 + \frac{\eta_2(X)}{2}X_5$, which can be scaled to $X_8 \pm X_5$ with appropriate value β.

Therefore, in this case, after scaling $\widehat{\widehat{X}}$ we have every one-dimensional subalgebra generated by X with $\eta_1(X) = 0$ is equivalent to the subalgebra spanned by

$$X_5, X_8, X_5 + aX_8, X_5 + aX_6, \quad a \in R(\neq 0). \tag{2.31}$$

In summary, an optimal system of one-dimensional subalgebras of the WBK algebra for $X_5 \sim X_8$ is generated by the elements in Table 3.

Table 3: Optimal system for $X_5 \sim X_8$

1	$X_5 + aX_7$	$\eta_1(X) > 0$		$a \in R(\neq 0)$
$2a$	$X_5 + X_8 - aX_6$	$\eta_1(X) < 0$	$a_8 > 0, \eta_2(X) > 0; a_8 < 0, \eta_2(X) < 0$	$a \in R(\neq 0)$
$2b$	$X_5 - X_6 + aX_8$	$\eta_1(X) < 0$	$a_8 > 0, \eta_2(X) < 0; a_8 < 0, \eta_2(X) > 0$	$a \in R(\neq 0)$
$2c$	$X_8 + X_6$	$\eta_1(X) < 0$	$a_8 > 0, \eta_2(X) = 0; a_8 < 0, \eta_2(X) = 0$	
$3a$	X_5	$\eta_1(X) = 0$	$a_6 = a_7 = a_8 = 0$	
$3b$	$X_5 + aX_8$	$\eta_1(X) = 0$	$a_6 = a_7 = 0, a_8 \neq 0$	$a \in R(\neq 0)$
$3c$	$X_5 + aX_6$	$\eta_1(X) = 0$	$a_7 = a_8 = 0, a_6 \neq 0$	$a \in R(\neq 0)$
$3d$	X_8	$\eta_1(X) = 0$	$\eta_2(X) = 0, a_6 \neq 0, a_7 \neq 0, a_8 \neq 0$	
$3e$	$X_8 \pm X_5$	$\eta_1(X) = 0$	$\eta_2(X) \neq 0, a_6 \neq 0, a_7 \neq 0, a_8 \neq 0$	

2.2.2 Optimal system of $X_1 \sim X_4$.

Similar to the discussion about $X_5 \sim X_8$, we give an optimal system of one-dimensional subalgebras for $X_1 \sim X_4$. Table 4 gives the Lie brackets of $X_1 \sim X_4$.

Table 4: Lie brackets of $X_1 \sim X_4$

	X_1	X_2	X_3	X_4
X_1	0	$-\frac{1}{2}X_2$	$\frac{1}{2}X_3$	X_4
X_2	$\frac{1}{2}X_2$	0	0	X_3
X_3	$-\frac{1}{2}X_3$	0	0	0
X_4	$-X_4$	$-X_3$	0	0

In Table 5, all the adjoint representations of $X_1 \sim X_4$ are presented, with the (i, j) entry indicating $Ad(exp(\epsilon X_i))X_j$ defined as (2.7).

Table 5: Adjoint representation generated by $X_1 \sim X_4$

	X_1	X_2	X_3	X_4
X_1	X_1	$e^{\frac{\epsilon}{2}}X_2$	$e^{\frac{-\epsilon}{2}}X_3$	$e^{-\epsilon}X_4$
X_2	$X_1 - \frac{\epsilon}{2}X_2$	X_2	X_3	$X_4 - \epsilon X_3$
X_3	$X_1 + \frac{\epsilon}{2}X_3$	X_2	X_3	X_4
X_4	$X_1 + \epsilon X_4$	$X_2 + \epsilon X_3$	X_3	X_4

They are used to identify one-dimensional subalgebras for $X_1 \sim X_4$. For a given arbitrary element

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{2.32}$$

after similar computations, we find an optimal system of one-dimensional subalgebras spanned by

$$\begin{aligned}
&(a) : X_1 = -\frac{1}{2}x\partial_x - t\partial_t + \frac{1}{2}u\partial_u + v\partial_v,\\
&(b_1) : X_2 + X_4 = -t\partial_x + \partial_u + \partial_t,\\
&(b_2) : X_2 = -t\partial_x + \partial_u,\\
&(c) : X_3 = \partial_x,\\
&(d) : X_4 = \partial_t.
\end{aligned} \tag{2.33}$$

The list is slightly reduced by the discrete symmetry $(x, t, u) \mapsto (-x, -t, u)$, not in the connected component of the identity of the full symmetry group, which maps $X_2 - X_4$ to $X_2 + X_4$.

2.3 Symmetry reductions and similarity solutions

One of the main purpose for calculating symmetry is to use them for obtaining symmetry reductions and hopefully similarity solutions. The goal of this subsection is to apply the symmetries calculated in the previous subsection to obtain symmetry reductions and exact solutions whenever it is possible.

Case 1. X_1. Solving the characteristic equations for the similarity variables [7, 2], one has

$$y = \frac{x^2}{t}, W = xu, T = tv. \tag{2.34}$$

Substituting these variables into Eq.(1.1), one finally converts it into ordinary differential equations

$$\begin{aligned}
&8\alpha y W_{yyy} + (\frac{6\alpha}{y} + 2T)W_y + 4\beta y T_{yy} + (2\beta + 2W + y)T_y + T - \frac{TW}{y} - \frac{6\alpha}{y^2}W = 0,\\
&\beta y^2 W_{yy} + (y^2 + 2yW - 2\beta y)W_y + 2y^2 T_y - W^2 + 2\beta W = 0.
\end{aligned} \tag{2.35}$$

Eq.(2.35) are highly nonlinear and should be solved numerically for given boundary conditions. However, it is much easier to solve this system numerically than the original partial differential equations.

In what follows, we omit the tedious computations and just present the final results. The similarity variables are listed in bracket.

Case 2. $X_2 + X_4$. Reduced equations are ($y = x + \frac{t^2}{2}, W = u - t, T = v$)

$$\begin{aligned}
&WW_y + T_y + \beta W_{yy} - 1 = 0,\\
&(TW)_y + \beta T_{yy} + \alpha W_{yyy} = 0.
\end{aligned} \tag{2.36}$$

The first equation of (2.36) may be integrated once to give

$$T = -\beta W_y + y - \frac{1}{2}W^2 + c_1, \tag{2.37}$$

which makes the second equation of (2.36) become

$$\frac{-W(y)^3}{2} + W(y)\left(y + c_1 - 2\beta W'(y)\right) + \left(\alpha - \beta^2\right) W''(y) = c_2, \tag{2.38}$$

where c_1, c_2 are arbitrary constants.

Case 3. X_2. We find solutions

$$u = \frac{1}{t}(c_1 - x), v = \frac{c_2}{t} \tag{2.39}$$

with two arbitrary constants c_1, c_2.

Case 4. $X_3 + cX_4$. It is the traveling wave case, the reduced equations are ($y = x - ct, W = u, T = v$)

$$\begin{aligned}
&WW_y + T_y + \beta W_{yy} + cW_y = 0,\\
&WT_y + TW_y + \beta T_{yy} + \alpha W_{yyy} + cT_y = 0,
\end{aligned} \tag{2.40}$$

which is the researching object by many researchers [21, 22, 24].

Case 5. X_5. We get the reduced equations ($y = \frac{x}{t}, W = u, T = v$)

$$\begin{aligned}
&(W^2 + 2T)_y + 2yW_y = 0,\\
&WT_y + TW_y + yT_y = 0.
\end{aligned} \tag{2.41}$$

Case 6. X_7. Eq.(1.1) can be reduced to $(y = x, W = tu, T = t^2v)$

$$\begin{aligned} &W + WW_y + T_y = 0,\\ &2T + WT_y + TW_y = 0, \end{aligned} \tag{2.42}$$

whose solutions are $T(y) = -\int W(y)dy - \frac{1}{2}W(y)^2$ and $W(y)$ satisfy

$$W' = -1 \pm \frac{\sqrt{W^8 - c_1 W^4}}{c_1 - W^4}, \quad c_1 > 0. \tag{2.43}$$

Case 7. $X_5 + aX_6$. Eq.(1.1) can be reduced to $(y = \frac{x}{t} + a\ln t, W = \frac{u}{a} - \ln t, T = v)$

$$\begin{aligned} &a(1 + aW_y - aWW_y - yW_y) - T_y = 0,\\ &a(WT_y + TW_y - T_y) + yT_y = 0. \end{aligned} \tag{2.44}$$

Case 8. $X_5 + aX_7$. Eq.(1.1) can be reduced to $(y = \frac{x^{1-a/2}}{t}, W = \frac{u}{x^{a/2}}, T = \frac{v}{x^a})$

$$\begin{aligned} &\frac{a}{2}W^2 + (1 - \frac{a}{2})y(WW_y + T_y) + aT + y^2W_y = 0,\\ &(1 - \frac{a}{2})y(WT_y + TW_y) + \frac{3a}{2}TW + y^2T_y = 0. \end{aligned} \tag{2.45}$$

3 Linearization of WBK equations

In section **2**, we reduce Eq.(1.1) by an optimal system of one-dimensional subalgebras, but reduced equations and similarity solutions for the cases containing X_8 are not obtained, so we search solutions by means of linearization. For $\alpha = \beta = 0$, Eq.(1.1) become

$$\begin{aligned} &u_t = uu_x + v_x,\\ &v_t = vu_x + uv_x, \end{aligned} \tag{3.1}$$

which are one-dimensional shallow water equations on a flat bottom [20].

Now, we review an important theorem on invertible linearization mappings of nonlinear PDEs to linear PDEs through admitted symmetry [8, 9, 10].

Theorem 1. *Let $R\{x, u\}$ denote a given k th-order nonlinear system of M PDEs with n independent variables $x = (x_1, \cdots, x_n)$ and m dependent variables $u = (u_1, \cdots, u_m)$ and $S\{z, w\}$ denote a k th-order linear target system of M PDEs with n independent variables $z = (z_1, \cdots, z_n)$ and m dependent variables $w = (w_1, \cdots, w_m)$.*

Suppose a given nonlinear system $R\{x, u\}$ of PDEs admits infinitesimal point symmetries

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta_\tau \frac{\partial}{\partial u_\tau} \tag{3.2}$$

of the form

$$\xi_i = \alpha_{i\sigma} F^\sigma(x, u), \qquad \eta^\tau = \beta^\tau_\sigma F^\sigma(x, u), \tag{3.3}$$

involving an arbitrary solution $F(X)$ of a linear system

$$L[X]F = 0 \tag{3.4}$$

with specific independent variables $X = (X_1(x,u), \cdots, X_n(x,u))$. If the m first order linear homogeneous PDEs

$$\alpha_{i\sigma}(x,u)\frac{\partial \phi}{\partial x_i} + \beta_\sigma^\tau \frac{\partial \phi}{\partial u^\tau} = 0, \quad \sigma = 1, \cdots, m, \tag{3.5}$$

whose coefficients are formed from (3.3) have $\phi_1 = X_1(x,u), \cdots, \phi_n = X_n(x,u)$ as n functionally independent solutions, and if the m^2 first order linear inhomogeneous PDEs

$$\alpha_{i\sigma}(x,u)\frac{\partial \psi^\gamma}{\partial x_i} + \beta_\sigma^\tau \frac{\partial \psi^\gamma}{\partial u^\tau} = 0, \quad \gamma, \sigma = 1, \cdots, m, \tag{3.6}$$

(where δ_σ^γ is the Kronecker symbol) have a particular solution $\psi = (\psi^1(x,u), ..., \psi^m(x,u))$, then the mapping μ defined by

$$z_i = X_i(x,u), i = 1, \cdots, n, \qquad w^\sigma = \psi^\sigma(x,u), \quad \sigma = 1, \cdots, m \tag{3.7}$$

is invertible and transforms $R\{x,u\}$ to the linear system $S\{z,w\}$ of PDEs given by $L[z]w = g(z)$, for some inhomogeneous term $g(z)$.

Obviously, the linear target system $S\{z,w\}$ arises from the admitted infinitesimal point symmetries of the given nonlinear system (first procedure). Moreover, these admitted symmetries yield a specific mapping (second procedure). Then, the solutions of $R\{x,u\}$ can be obtained from the linear target system $S\{z,w\}$ through the invertible mapping. Next we apply the **Theorem 1** to linearize Eq.(3.1).

In subsection **2.1**, the nonlinear system (3.1) is found to admit an infinite set of point symmetries given by the infinitesimal generator

$$X = (\int f_u dv - uf)\frac{\partial}{\partial x} + f\frac{\partial}{\partial t} \tag{3.8}$$

where f satisfy $f_{uu} = 2f_v + vf_{vv}$. Therefore, we have $F_1 = \int f_u dv, F_2 = f$ with $\alpha_{11} = 1, \alpha_{12} = -u, \alpha_{21} = 0, \alpha_{22} = 1, \beta_\sigma^\tau = 0$ in (3.3). The associated homogeneous system (3.5) give $S_1 = u, S_2 = v$ as functionally independent solutions and the corresponding linear inhomogeneous system (3.6) has a particular solution $(\psi^1, \psi^2) = (x + ut, t)$. Then from (3.8) we have that $F = (F_1, F_2)$ satisfies the linear system

$$\frac{\partial F_1}{\partial S_2} = \frac{\partial F_2}{\partial S_1}, \qquad \frac{\partial F_1}{\partial S_1} = S_2\frac{\partial F_2}{\partial S_2} + F_2.$$

One obtains the invertible point transformation

$$z_1 = u, \ z_2 = v, \ w_1 = x + ut, \ w_2 = t \tag{3.9}$$

mapping the given nonlinear system (3.1) into the linear system

$$\frac{\partial w_1}{\partial z_2} = \frac{\partial w_2}{\partial z_1}, \qquad \frac{\partial w_1}{\partial z_1} = z_2\frac{\partial w_2}{\partial z_2} + w_2. \tag{3.10}$$

Therefore, we can get the solutions of Eq.(3.1) through transformation (3.9) if the solutions of the linearized Eq.(3.10) are known. According to different solutions of Eq.(3.10), Eq.(3.1) have the following solutions.

Case I. We have solutions in the form

$$u = t + c_1, v = \frac{1}{2}(t^2 + 2x - 2c_1 t + c_1^2 - 2c_2), \tag{3.11}$$

where c_1, c_2 are arbitrary constants.

Case II. Solutions are

$$u = \frac{2\,5^{\frac{1}{3}}\,t + 2^{\frac{1}{3}}\left(-15\,x + \sqrt{5}\,\sqrt{-4\,t^3 + 45\,x^2}\right)^{\frac{2}{3}}}{10^{\frac{2}{3}}\left(-15\,x + \sqrt{5}\,\sqrt{-4\,t^3 + 45\,x^2}\right)^{\frac{1}{3}}}, \tag{3.12}$$

$$v = \frac{3\,t}{5} - \frac{\left(\frac{2}{5}\right)^{\frac{2}{3}}t^2}{\left(-15\,x + \sqrt{5}\,\sqrt{-4\,t^3 + 45\,x^2}\right)^{\frac{2}{3}}} - \frac{\left(-15\,x + \sqrt{5}\,\sqrt{-4\,t^3 + 45\,x^2}\right)^{\frac{2}{3}}}{5\,2^{\frac{2}{3}}\,5^{\frac{1}{3}}}.$$

Case III. We get solutions

$$v = t - u^2, \tag{3.13}$$

where u satisfy

$$7\,u^6 + 12\,t\,u^3 + 36\,u^2\,x - 18\,t^2 = 0. \tag{3.14}$$

Case IV. We arrive at

$$u = \frac{\mathrm{ProductLog}[-t^2 e^{-x}]}{t}, v = -\frac{\mathrm{ProductLog}[-t^2 e^{-x}]}{t^2}, \tag{3.15}$$

where ProductLog$[z]$ gives the principal solution for equation $z = we^w$, and

$$u = -\frac{x}{t}, v = -\frac{1}{t}. \tag{3.16}$$

Case V. We find

$$u = -\frac{x}{t}, v = 0. \tag{3.17}$$

and

$$u = \pm\sqrt{4\,v + t^2\,v^2}, \tag{3.18}$$

where v satisfies

$$-2\ln\frac{1}{\sqrt{v}}(vt \pm \sqrt{4\,v + t^2\,v^2}) = x \pm \sqrt{4\,v + t^2\,v^2}t. \tag{3.19}$$

Case VI. Solutions u, v satisfy the following equations

$$\begin{aligned} &e^u\,\left(c_1\mathrm{BesselI}(0, 2\sqrt{v}) + 2c_2\mathrm{BesselK}(0, 2\sqrt{v})\right) = x + ut, \\ &e^u\,\left(c_1\;\mathrm{BesselI}\;(1, 2\sqrt{v}) - 2c_2\mathrm{BesselK}(1, 2\sqrt{v})\right) = t\sqrt{v}, \end{aligned} \tag{3.20}$$

where BesselI$[n, z]$ gives the modified Bessel function of the first kind $I_n(z)$, BesselK$[n, z]$ gives the modified Bessel function of the second kind $K_n(z)$, they both satisfy the differential equation $z^2y'' + zy' - (z^2 + n^2)y = 0$.

4 Conservation laws

In this section, we use the scaling symmetry to obtain the polynomial form conservation laws of Eq.(1.1).

In reference [14], there is an direct method to construct conservation laws. First, build a candidate density as a linear combination (with undetermined coefficients) of "building blocks" that are homogeneous under the scaling symmetry of the PDEs. If no such symmetry exists, one is constructed by introducing weighted parameters. Next, use the Euler operator (variational derivative) to derive a linear algebraic system for the undetermined coefficients. After the system is analyzed and solved, use the homotopy operator to compute the flux(for details in [14]).

Eq.(1.1) is invariant under the scaling symmetry with infinitesimal operator X_1

$$(x, t, u, v) \longrightarrow (\lambda^{-1}x, \lambda^{-2}t, \lambda^{-1}u, \lambda^{-2}v), \tag{4.1}$$

where λ is an arbitrary scaling parameter. Introducing the *weight*, W, of a variable as the exponent of λ that multiplies the variable, if we set $W(x) = -1$ or $W(\partial/\partial x) = 1$, then $W(u) = 1, W(v) = 2, W(\partial/\partial t) = 2$. The *rank* of a monomial equals the sum of all of its weights. An expression (or equation) is *uniform* in rank if its monomial terms have equal rank. Observe that the first and second equation of (1.1) are uniform of ranks 3 and 4, respectively.

By virtue of the method in [14], after tedious and complicated calculations, we obtain three density-flux pairs

$$\begin{aligned}
&\rho^{(1)} = -v, J^{(1)} = uv + \alpha u_{xx} + \beta v_x. \\
&\rho^{(2)} = -u_x, J^{(2)} = uu_x + \beta u_{xx} + v_x. \\
&\rho^{(3)} = -v^2 - u^2 v + \alpha u_x^2, \\
&J^{(3)} = 2uv^2 + u^3 v + \frac{1}{3}(2\alpha + 3)u^2 u_{xx} + 2(1 - 2\alpha)uu_x^2 - 2\alpha u_x v_x + 3\alpha uv_{xx} (\beta = 0).
\end{aligned} \tag{4.2}$$

Obviously, the above densities are uniform in ranks 2 and 4. Both $\rho^{(1)}$ and $\rho^{(2)}$ are of rank 2 and $\rho^{(3)}$ is of rank 4. The corresponding fluxes are also uniform in rank with ranks 3 and 5. Both $J^{(1)}$ and $J^{(2)}$ are of rank 3 and $J^{(3)}$ is of rank 5.

5 Conclusion

We have performed Lie symmetry analysis for the WBK equations and derived an optimal system of one-dimensional subalgebras. Some exact solutions and symmetry reductions with respect to the optimal system are constructed. Furthermore, a special case of WBK equations is linearized through its admitted infinite set of point symmetries and some new solutions are obtained. Polynomial form conservation laws are also analyzed by means of scaling symmetry.

In particular, the above conservation laws can be used for construction of potential systems, potential symmetries and potential conservation laws. For example, from $\rho^{(1)}$

and $J^{(1)}$, we can construct potential equations

$$\begin{aligned} &w_x = v, \\ &w_t = uv + \alpha u_{xx} + \beta v_x, \\ &u_t = uu_x + v_x + \beta u_{xx}, \end{aligned} \tag{5.1}$$

for potential symmetries and potential conservation laws. It would be interesting to investigate them in our future work.

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