# Solutions of some second order ODEs by the extended Prelle-Singer method and symmetries

*A Ghose Choudhury* [a], *Partha Guha* [b] *and Barun Khanra* [c]

[a] *Department of Physics, Surendranath College,*
*24/2 Mahatma Gandhi Road, Calcutta-700009, India*
*E-mail: a_ghosechoudhury@rediffmail.com*

[b] *Max Planck Institute for Mathematics in the Sciences*
*Inselstrasse 22, D-04103 Leipzig, Germany and*
*S.N. Bose National Centre for Basic Sciences*
*JD Block, Sector III, Salt Lake, Kolkata - 700098, India*
*E-mail: partha.guha@mis.mpg.de*

[c] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street*
*Calcutta-700 004, India*
*E-mail: barunkhanra@rediffmail.com*

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**Abstract**

In this paper we compute first integrals of nonlinear ordinary differential equations using the extended Prelle-Singer method, as formulated by Chandrasekar *et al* in J. Math. Phys. 47 (2), 023508, (2006). We find a new first integral for the Painlevé-Gambier XXII equation. We also derive the first integrals of generalized two-dimensional Kepler system and the Liénard type oscillators.

## 1 Introduction

The problem of finding first integrals of ordinary differential equations (ODE) has a long and interesting history which may be traced to the seminal works of Darboux and Lie in the latter half of the nineteenth century. For an $n$th-order ODE, a first integral is an expression involving the independent variable, the dependent variable and its derivatives to order $n-1$. In fact, if $r$ such first integrals are known for an $n$th-order ODE, then the latter may be reduced to an $(n-r)$th-order ODE. The familiar case of a simple harmonic oscillator, is an example of a second-order ODE with a first integral given by the mechanical energy. The latter enables us to reduce the problem to solving a first-order ODE.

Dynamical systems, described by nonlinear oscillators, are a common occurrence in many areas of physics and the applied sciences. The main difficulty involved in solving

such ODEs is that they are often nonlinear, involve several degrees of freedom, and are usually coupled to one other in a non-trivial way. Moreover, these equations are generally non Hamiltonian in nature and describe the time evolution of physical processes which are usually dissipative in character.
The techniques involved in finding first integrals of systems of one or more ODEs generally make use of integrating factors, which are functions multiplying each of the ODEs to yield a first integral.

In 1878 Darboux showed that one can construct an integrating factor (and first integrals) of a planar polynomial differential system, if there exists a sufficient number of invariant algebraic curves (real or complex). On the other hand for first-order scalar ODEs, S. Lie devised a method for constructing an integrating factor from each admitted point symmetry. Then, after almost a century, a major breakthrough in the construction of an algorithm for solving first-order ordinary differential equations was put forward by Prelle and Singer [1] in 1983. The method is a semi algorithmic procedure for solving nonlinear first-order ordinary differential equations of the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{1.1}$$

when $P(x,y)$ and $Q(x,y)$ are coprime polynomials. The Prelle-Singer (PS) method provides the form of the integrating factor when the solution of the associated system of differential equations is expressible in terms of elementary functions. Their work has been influential in providing some of the fundamental algebraic results required for the automated solution of ODEs using computer algebra. An extension of this method provides the form of an integrating factor when the solution is expressible in terms of Liouvillian functions. Recently Duarte *et al* [10, 11] have extended the technique to second-order ODEs. Essentially, their objective was to look for a wider class of possible integrating factors. To this end, they succeeded in adding the derivatives of some rational functions, to the previously known linear combinations of logarithmic derivatives.

Most recently Lakshmanan and his coworkers have generalized and used the extended Prelle-Singer method to obtain the first integrals and general solutions for a class of nonlinear equations [6, 7, 9]. They have also devised a procedure to construct a transformation which removes the time-dependent part from the first integral and provides the general solution by quadrature [8]. This procedure is shown to have a wider applicability through several examples.

In this paper we use the extended Prelle-Singer method to derive the first integrals of the Painlevé-Gambier class of ODEs. We derive a new first integral for the Painlevé-Gambier XXII equation. Using this method we also show how the known first integrals of a large class of equations, of a specific form, in the Painlevé-Gambier classification may be deduced. In addition we analyze a Liénard type equation, (second order Riccati equation) and a generalized two-dimensional Kepler system.

The organization of the paper is as follows. In Section 2 we briefly recollect the basic results involved in Darboux integrability and give a brief introduction to the Prelle-Singer method, including the relevant definitions and results related to PS method. In Section 3, we review the extended PS method, as developed by Chandrasekar *et al.* In Section 4 we discuss applications of the extended PS method to ODEs of the Painlevé-Gambier classification. Section 5 contains a discussion on second-order Liénard type equations. Finally, in Section 6 we briefly consider applications to systems of second-order ODEs and illustrate it with an example of a generalized two-dimensional Kepler system. We finish our paper with a modest outlook.

## 2 Preliminaries

Let us consider planar polynomial differential systems

$$\dot{x} = Q(x,y) \quad \text{and} \quad \dot{y} = P(x,y), \tag{2.1}$$

where $P(x,y) = \sum_{i=0}^{m} P_i(x,y), Q(x,y) = \sum_{i=0}^{m} Q_i(x,y)$ are coprime polynomials in $\mathbb{C}$ such that max $\{deg\, P, deg\, Q\} = m$ and $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous components of degree $i$. This differential system (2.1) may be described either by the vector field

$$D = Q(x,y)\frac{\partial}{\partial x} + P(x,y)\frac{\partial}{\partial y}, \tag{2.2}$$

or the differential form

$$\omega = Pdx - Qdy.$$

The corresponding phase flow is given by the solution of the first-order ordinary differential equation

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.3}$$

**Definition 2.1.** Let $U$ be an open subset of $\mathbb{K}^2$. We say that a nonconstant function $I : U \to \mathbb{K}$ is a first integral of a vector field $D$ on $U$, if and only if, $D|_U(I) = 0$.

The tangents to the trajectories of a planar polynomial differential system are defined everywhere [12]. If $f(x,y) = 0$ is the equation of an *invariant curve*, its tangent must coincide with the tangents of the trajectories. In other words, the gradient to $f$, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and $(Q,P)$ must be orthogonal over the curve $f(x,y) = 0$:

$$\dot{f} = (Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y})_{f=0} = 0.$$

**Definition 2.2.** An invariant curve $f(x,y) = 0$ is called an *algebraic curve* or Darboux polynomial of degree $m$ when $f(x,y)$ is a polynomial of degree $m$.

**Definition 2.3.** Let $D$ be the vector field associated with a differential equation. A curve $f(x,y) = 0$ is an *invariant algebraic curve* if $D[f]/f$ is a polynomial. The latter polynomial $\lambda_f = D[f]/f$ is usually called the *cofactor* of the invariant algebraic curve or *Darboux polynomial.*

## 2.1 Darboux method

The Darboux method of constructing integrating factors and first integrals of planar ODEs, relies essentially on the existence of invariant algebraic curves (or Darboux polynomials) [3]. Furthermore, the maximum degree of the invariant algebraic curves is bounded [4]. Suppose the vector field $D$ admits $s$ distinct invariant algebraic curves $f_i \quad i = 1, ..., s$.
(a) If there are $n_i \in \mathbb{C}$ not all zero, such that

$$\sum_{i=1}^{s} n_i \lambda_{f_i} = 0 \text{ then the function } \prod_{i=1}^{s} f_i^{n_i} \text{ is a first integral of the vector field } D. \tag{2.4}$$

(b) If there exists $n_i \in \mathbb{C}$ not all zero, such that

$$\sum_{i=1}^{s} n_i \lambda_{f_i} = -\mathrm{div} D \text{ then the function } \prod_{i=1}^{s} f_i^{n_i} \text{ is an integrating factor of } D. \tag{2.5}$$

These results form the essential content of Darboux integrability.

### 2.1.1 Extension of the Darboux method

If $f, g \in \mathbb{C}[x, y]$, then $e = \exp(\frac{g}{f})$ is an exponential factor of the vector field $D$ of degree $d$ if $D(e)/e$ is a polynomial of degree at most $d - 1$. Thus there are two major kinds of first integrals (1) Rational and (2) Darbouxian $\Rightarrow f^{\nu}(\exp(\frac{h}{g}))^{\mu} \quad \nu, \mu \in \mathbb{C}$.

## 2.2 The Prelle-Singer method

In 1983 Prelle and Singer [1, 2] devised a procedure which could not only determine polynomial first integrals but more importantly could be applied to systems admitting rational first integrals.

Suppose

$$\dot{x} = Q(x, y) \text{ and } \dot{y} = P(x, y) \tag{2.6}$$

is a system of first order ODEs. The vector field associated with this system is

$$D = Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}. \tag{2.7}$$

Since this is a first-order ODE, it is integrable on an open subset $U$ of $\mathbb{K}^2$, if there exists a first integral of the system on $U$.

**Definition 2.4.** A non zero function, $R : U \to \mathbb{K}$, is an integrating factor of a vector field $D$ on $U$ if and only if $D(R) = -div\,(D) \cdot R$ on $U$.

When an integrating factor is known, we can compute by quadrature, a first integral of the system up to a constant. Let us therefore assume that, we have identified a sufficient number of Darboux polynomials $f_i$ satisfying

$$D[f_i] = \lambda_i f_i, \tag{2.8}$$

where the $\lambda_i$ are cofactors. From (2.5), we have

$$\frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = -(Q_x + P_y). \tag{2.9}$$

Clearly $Q_x$ and $P_y$ are polynomials since $Q$ and $P$ are themselves polynomials; and therefore it is necessary that $f_i$ divides $D[f_i]$. If we manage to find such Darboux polynomials, then all that remains is to determine the numbers $n_i$ such that (2.9) is satisfied. This can be achieved by equating terms of various orders $x^\alpha y^\beta$ on either side and finding a consistent set of values for the $n_i$. The problem lies in determining the $f_i$. The Prelle-Singer method provides a semi algorithm for determining these whenever there exists a first integral which is an elementary function. This involves establishing bounds of different orders on the $f_i$. For example, we start with $N = 1$ and assume $f = \alpha x + \beta y + \gamma$; next we check for what values of $\alpha, \beta$ and $\gamma$, $f$ divides $D[f]$. If we fail to find such an $f$, we go to the next level and set $N = 2$ try $f = \alpha x^2 + 2\beta xy + \gamma y^2 + \delta x + \epsilon y + \mu$ and find a particular combination which divides $D[f]$ and so on. It is clear that the process is semi algorithmic by its very nature.

While the Prelle-Singer method allows for the determination of first integrals for many planar systems, it is not applicable to linear first-order ODEs having exponential integrating factors. In [10, 11] Duarte *et al* have extended the PS method to include such situations. Subsequently in a series of papers [6, 8, 9] Chandrasekar *et al* have uncovered rational and even non rational first integrals for a large class of oscillator type equations, by appropriately modifying and also extending the basic idea behind the PS procedure.

# 3 The Extended Prelle-Singer method

Consider a second-order ODE of the generic form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3.1}$$

In the existing literature such equations are called Lienard type ODEs and include a number of important physical systems:

1. $f(x) = k, \quad g(x) = w_0^2 x, \;\Rightarrow\; \ddot{x} + k\dot{x} + w_0^2 x = 0$ *damped harmonic oscillator.*
2. $f(x) = \alpha x, \quad g(x) = \beta x^3, \;\Rightarrow\; \ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$ *Modified Emden equation.*
3. $f(x) = \alpha + \beta x^2, \quad g(x) = -\gamma x + x^3, \;\Rightarrow\; \ddot{x} + (\alpha + \beta x^2)\dot{x} - (\gamma x + x^3) = 0$ *Duffing Van der Pol oscillator.*
4. $f(x) = (k_1 x^q + k_2),\; g(x) = k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x$, where $q \in \mathbb{R}$

The last case includes many systems like the anharmonic oscillator force free Helmholtz and Duffing oscillator as special cases. In [9], the authors have studied this system for $q =$ arbitrary and deduced a number of new completely integrable cases.

### 3.1 Formulation

Let us briefly review the method used by the authors to deduce first integrals of oscillator type systems under very general conditions [7, 8, 9]. According to these calculations the equation of motion for the second-order ODE is written in the form:

$$\ddot{x} = \phi(x, \dot{x}). \tag{3.2}$$

This may be recast as a system of first order ODEs

$$\dot{x} = y, \quad \dot{y} = \phi(x, y) \tag{3.3}$$

or as a pair of differential one forms:

$$Sdx = Sydt \tag{3.4}$$

$$dy = \phi dt. \tag{3.5}$$

Here $S$ is an unknown function of $x, y$ which must be determined. Addition of (3.4), (3.5) leads to

$$(Sy + \phi)dt = Sdx + dy.$$

Assuming $R$ to be an integrating factor of this equation we have upon multiplication

$$R(Sy + \phi)dt - RSdx - Rdy = 0, \tag{3.6}$$

which implies that if $I(t, x, y)$ be the corresponding first integral such that

$$I_t dt + I_x dx + I_y dy = 0$$

we must have

$$I_t = R(Sy + \phi), \quad I_x = -RS, \quad I_y = -R. \tag{3.7}$$

The compatibility of these equations requires

$$I_{xy} = I_{yx}, I_{tx} = I_{xt} \ \text{ and } \ I_{ty} = I_{yt}. \tag{3.8}$$

From these conditions it is straightforward to derive the following equations

$$D[R] = -((RS) + \phi_y R), \tag{3.9}$$

$$D[RS] = -\phi_x R. \tag{3.10}$$

Two subcases may be distinguished,
A: when $I_t = 0$, that is when the system is conservative and
B: when $I_t \neq 0$ for a non conservative system.
In case of the former, it is easy to see that $S = -\frac{\phi}{y}$. Therefore one needs to determine only the unknown function $R$, which is the required integrating factor. We shall analyze case A first, since it is somewhat simpler, and postpone a discussion of the latter.

For case A, (3.9) simplifies to

$$D[R] = (\frac{\phi}{y} - \phi_y)R, \tag{3.11}$$

with

$$D = y\partial_x + \phi\partial_y.$$

Substituting the ansatz

$$R = \frac{y}{T(x,y)} \tag{3.12}$$

causes (3.11) to simplify further and it reduces to

$$D[T] = yT_x + \phi T_y = \phi_y T. \tag{3.13}$$

Let us consider an example to illustrate the method developed thus far.
**Example 1:** Consider the equation

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x) = 0.$$

This is equivalent to the system of equations

$$\dot{x} = y$$
$$\dot{y} = \phi(t,x,y) = -\left[\frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x)\right]$$

so that

$$\phi_y = -(\psi_x y + \psi_t) = -D[\psi].$$

Thus (3.13) becomes $D[\log T + \psi] = 0$ which implies $T = K\exp(-\psi)$. Hence $R = \frac{y}{K}\exp(\psi) = -I_y$ implies $I = -\frac{e^\psi}{K}\frac{y^2}{2} + \frac{J(x)}{K}$, where $K$ is a numerical constant. On the other hand $I_x = -RS$ implies $J'(x) = e^\psi(-\psi_t y - B(t,x))$. Clearly one must have $\psi_t = 0$ and $B(t,x) = B(x)$ for a time independent first integral. In that case we obtain

$$I(x,y) = -\frac{1}{K}\left[e^\psi \frac{y^2}{2} + \int^x e^\psi B dx\right]. \tag{3.14}$$

Such a first integral occurs, therefore for all equations having the generic form

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + B(x) = 0, \tag{3.15}$$

and may be a treated as a formula for deriving an time independent first integral.

## 4 First integrals of Painlevé-Gambier equations

It will be evident that the above method may be applied, in principle to a number of equations of the Painlevé-Gambier classification. We introduce a slight change of notation, for easy reference and illustrate this below.

### 4.1 Painlevé-Gambier XII equation

Let us consider the Painlevé-Gambier XII equation

$$y'' = \frac{1}{y}y'^2 + \alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}$$

Comparison with (3.15) above indicates that $\frac{1}{2}\psi_y = -\frac{1}{y}$ and hence $e^{\psi} = y^{-2}$, while $B(x,y) = -\left[\alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}\right]$. Then (3.14) yields the following first integral

$$y'^2 = \alpha y^4 + 2\beta y^3 - 2\gamma y - \delta + K_1 y^2$$

where we have set $K_1 = -2KI(y, y')$. We have checked that *all* the known $x$- independent first integrals of the Painlevé-Gambier classification [13], can be obtained from (3.14).

It is of interest to know, whether there exists other first integrals, depending perhaps on the independent variable $x$, for equations having a first integral given by the above formula. This brings us actually to a discussion of case B, i.e. ($I_t \neq 0$) of the previous section.

### 4.2 Painlevé-Gambier XXII equation

We illustrate next, the existence of an $x$ dependent first integral for equation XXII of the Painlevé-Gambier classification:

$$\frac{d^2y}{dx^2} = \frac{3y'^2}{4y} - 1. \tag{4.1}$$

A known first integral of this equation is

$$K = \left(\frac{y'^2 - 4y}{4y^{3/2}}\right), \tag{4.2}$$

which may be obtained from (3.14).
From (3.9) and (3.10), we have

$$D[R] = -(S + \phi_{y'})R \tag{4.3}$$

$$D[RS] = -\phi_y R, \tag{4.4}$$

as a result of our change in notation. Here $D = \partial_x + y'\partial_y + \phi\partial_{y'}$ with

$$\phi(x, y, y') = \frac{3y'^2}{4y} - 1 = \phi_0(y)y'^2 - 1, \quad \phi_0(y) = \frac{3}{4y}. \tag{4.5}$$

Closer inspection of equations (4.3) and (4.4) reveals that they are a pair of coupled first order equations in the variables $R$ and $RS$ respectively. Assuming them to admit rational solutions of the form

$$R = \frac{f}{g} \text{ and } RS = \frac{h}{g} \Rightarrow S = \frac{h}{f}, \tag{4.6}$$

these equations become

$$gD[f] - fD[g] = -(h + \phi_{y'} f) \cdot g \tag{4.7}$$

$$gD[h] - hD[g] = -\phi_y f \cdot g. \tag{4.8}$$

From a leading order analysis of the above equations, assuming $f \sim y'^{\alpha}$, $h \sim y'^{\gamma}$ and $g \sim y'^{\beta}$ and with $\phi$ as in (4.5), it follows that $\gamma = \alpha + 1$ with $\beta$ being arbitrary. This suggests the following ansatz for the $y'$ dependence of the functions $f, g$ and $h$ namely:

$$f(y, y') = f_0 + f_1 y' + f_2 y'^2,$$

$$h(y, y') = h_0 + h_1 y' + h_2 y'^2 + h_3 y'^3,$$

$$g(y, y') = g_0 + g_1 y' + g_2 y'^2 + g_3 y'^3 + g_4 y'^4. \tag{4.9}$$

Substituting them into (4.7) and equating different powers of $y'$ leads to the set of equations:

$$-g_0 f_1 + f_0 g_1 = -h_0 g_0, \tag{4.10}$$

$$g_0 F_1 - f_0 G_1 = -\{(h_1 + 2\phi_0 f_0) g_0 + h_0 g_1\}, \tag{4.11}$$

$$-g_2 f_1 + g_1 F_1 + g_0 F_2 + f_2 g_1 - f_1 G_1 - f_0 G_2 = -\{(h_2 + 2\phi_0 f_1) g_0 + (h_1 + 2\phi_0 f_0) g_1 + h_0 g_2\}, \tag{4.12}$$

$$-g_3 f_1 + g_2 F_1 + g_1 F_2 + g_0 F_3 - f_2 G_1 - f_1 G_2 - f_0 G_3$$
$$= -\{(h_3 + 2\phi_0 f_2) g_0 + (h_2 + 2\phi_0 f_1) g_1 + (h_1 + 2\phi_0 f_0) g_2 + h_0 g_3\}, \tag{4.13}$$

$$-g_4 f_1 + g_3 F_1 + g_2 F_2 + g_1 F_3 - f_2 G_2 - f_1 G_3 - f_0 G_4 = -\{(h_3 + 2\phi_0 f_2) g_1 +$$
$$(h_2 + 2\phi_0 f_1) g_2 + (h_1 + 2\phi_0 f_0) g_3 + h_0 g_4\}, \tag{4.14}$$

$$g_4 F_1 + g_3 F_2 + g_2 F_3 - f_2 G_3 - f_1 G_4 - f_0 G_5$$
$$= -\{(h_3 + 2\phi_0 f_2) g_2 + (h_2 + 2\phi_0 f_1) g_3 + (h_1 + 2\phi_0 f_0) g_4\}, \tag{4.15}$$

$$g_4 F_2 + g_3 F_3 - f_2 G_4 - f_1 G_5 = -\{(h_3 + 2\phi_0 f_2) g_3 + (h_2 + 2\phi_0 f_1) g_4\}, \tag{4.16}$$

$$g_4 f_{2y} - f_2 g_{4y} = -h_3 g_4. \tag{4.17}$$

where

$$F_1 = f_{0y} - 2f_2$$

$$F_2 = f_{1y} + \phi_0 f_1$$

$$F_3 = f_{2y} + \phi_0 f_2 \tag{4.18}$$

and

$$G_1 = g_{0y} - 2g_2,$$

$$G_2 = g_{1y} + \phi_0 g_1 - 3g_3,$$

$$G_3 = g_{2y} + 2\phi_0 g_2 - 4g_4$$

$$G_4 = g_{3y} + 3\phi_0 g_3$$

$$G_5 = g_{4y} + 4\phi_0 g_4. \tag{4.19}$$

On the other hand from (4.8) we obtain the following equations:

$$-h_0 g_1 + g_0 h_1 = 0, \tag{4.20}$$

$$h_0 G_1 - g_0 H_1 = 0, \tag{4.21}$$

$$-h_2 g_1 + h_1 G_1 + h_0 G_2 + g_2 h_1 - g_1 H_1 - g_0 H_2 = \phi_{0y} f_0 g_0 \tag{4.22}$$

$$-h_3 g_1 + h_2 G_1 + h_1 G_2 + h_0 G_3 + g_3 h_1 - g_2 H_1 - g_1 H_2 - g_0 H_3 = \phi_{0y}(f_1 g_0 + f_0 g_1), \tag{4.23}$$

$$h_3 G_1 + h_2 G_2 + h_1 G_3 + h_0 G_4 + g_4 h_1 - g_3 H_1 -$$
$$g_2 H_2 - g_1 H_3 - g_0 H_4 = \phi_{0y}(f_2 g_0 + f_1 g_1 + f_0 g_2), \tag{4.24}$$

$$h_3 G_2 + h_2 G_3 + h_1 G_4 + h_0 G_5 - g_4 H_1 - g_3 H_2 - g_2 H_3 - g_1 H_4 = \phi_{0y}(f_2 g_1 + f_1 g_2 + f_0 g_3), \tag{4.25}$$

$$h_3 G_3 + h_2 G_4 + h_1 G_5 - g_4 H_2 - g_3 H_3 - g_2 H_4 = \phi_{0y}(f_2 g_2 + f_1 g_3 + f_0 g_4), \tag{4.26}$$

$$h_3 G_4 + h_2 G_5 - g_4 H_3 - g_3 H_4 = \phi_{0y}(f_2 g_3 + f_1 g_4), \tag{4.27}$$

$$h_3 G_5 - g_4 H_4 = \phi_{0y} f_2 g_4, \tag{4.28}$$

where

$$H_1 = h_{0y} - 2h_2,$$
$$H_2 = h_{1y} + \phi_0 h_1 - 3h_3,$$
$$H_3 = h_{2y} + 2\phi_0 h_2$$
$$H_4 = h_{3y} + 3\phi_0 h_3. \tag{4.29}$$

To solve the system of first order coupled PDEs given by (4.10)-(4.17) and (4.20)-(4.28) we observe that, one can satisfy one half of each set identically, by making a second ansatz, namely

$$f_{odd} = g_{odd} = h_{even} = 0. \tag{4.30}$$

It then follows that

$$H_1 = H_3 = G_2 = G_4 = 0,$$

and from (4.20) we find

$$h_1 = 0. \tag{4.31}$$

Taking this in to account we are now left with the following equations from the set (4.10)-(4.17):

$$g_0(f_{0y} - 2f_2) - f_0(g_{0y} - 2g_2) = -2\phi_0 f_0 g_0, \tag{4.32}$$

$$g_2(f_{0y} - 2f_2) + g_0(f_{2y} + 2\phi_0 f_2) - f_2(g_{0y} - 2g_2) - f_0(g_{2y} + 2\phi_0 g_2 - 4g_4)$$
$$= \{(h_3 + 2\phi_0 f_2)g_0 + 2\phi_0 f_0 g_2\} \tag{4.33}$$

$$g_4(f_{0y} - 2f_2) + g_2(f_{2y} + 2\phi_0 f_2) - f_2(g_{2y} + 2\phi_0 g_2 - 4g_4) - f_0(g_{4y} + 4\phi_0 g_4)$$
$$= \{(h_3 + 2\phi_0 f_2)g_2 + 2\phi_0 f_0 g_4\} \tag{4.34}$$

$$g_4 f_{2y} - f_2 g_{4y} = h_3 g_4. \tag{4.35}$$

On the other hand from the set of equations (4.20)-(4.28), with $h_1 = 0$ we obtain the following four equations:

$$3h_3 = \phi_{0y} f_0, \tag{4.36}$$

$$h_3(g_{0y} + g_2 - 3\phi_0 g_0) - g_0 h_{3y} = \phi_{0y}(f_2 g_0 + f_0 g_2), \tag{4.37}$$

$$h_3(g_{2y} - g_4 - \phi_0 g_2) - g_2 h_{3y} = \phi_{0y}(f_2 g_2 + f_0 g_4), \tag{4.38}$$

$$h_3(g_{4y} + \phi_0 g_4) - g_4 h_{3y} = \phi_{0y} f_2 g_4. \tag{4.39}$$

Since $\phi_0 = \frac{3}{4y}$ it follows $\phi_{0y} = -\frac{\phi_0}{y}$ and upon rearranging (4.39), we have

$$h_3 g_{4y} - g_4 h_{3y} = -\phi_0 g_4 (h_3 + \frac{f_2}{y}).$$

Making the assumption that the coefficients of the highest powers of $y'$ in the expressions for $g, h$ are constants, say $g_4 = \mu$ and $h_3 = \nu$ so that $g_{4y} = h_{3y} = 0$, one obtains the following relation determining the coefficient of the $y'^2$ in $f$:

$$h_3 + \frac{f_2}{y} = 0 \Rightarrow f_2 = -\nu y. \tag{4.40}$$

While from (4.36) we obtain

$$f_0 = -4\nu y^2. \tag{4.41}$$

The remaining equations (4.37) and (4.38) determine the coefficients $g_0$ and $g_2$, from solutions of the following coupled linear equations:

$$g_{0y} - \frac{3}{y} g_0 = 2g_2, \tag{4.42}$$

$$g_{2y} - \frac{3}{2y} g_2 = 4\mu. \tag{4.43}$$

These conditions are consistent with the set of equations (4.32)-(4.35), as may be verified. Furthermore, the solutions of (4.42) and (4.43) are easy to construct and are given by

$$g_2 = -8\mu y \;\text{ and }\; g_0 = 16\mu y^2.$$

Hence we finally obtain

$$R = \frac{f}{g} = \frac{-\nu y(4y + y'^2)}{\mu(y'^2 - 4y)^2} \;\text{ and }\; S = \frac{h}{f} = \frac{\nu y'^3}{-\nu y(4y + y'^2)}. \tag{4.44}$$

It is now straightforward to obtain the corresponding first integral as

$$I(x, y, y') = \frac{\nu}{4\mu}\left(x - \frac{4yy'}{y'^2 - 4y}\right). \tag{4.45}$$

## 5 Lienard type equations

We discuss next equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{5.1}$$

Instead of writing this as a system of first-order equations of the usual form

$$\dot{x} = y$$

$$\dot{y} = \phi(x, y)$$

where $\phi = -(f(x)y + g(x))$, we re-write it as

$$\dot{x} = v - r\frac{g(x)}{f(x)} \tag{5.2}$$

$$\dot{v} = -\frac{1}{r}f(x)v, \tag{5.3}$$

subject to the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \frac{1}{r}\left(1 - \frac{1}{r}\right)f(x), \quad r \neq 0, 1. \tag{5.4}$$

Here $r$ is a parameter. In order to determine a first integral for the system (5.2)and (5.3), we follow the same formulation as outlined in section (3.1) and demand that the one form

$$R[S(v - r\frac{g}{f}) - \frac{1}{r}fv]dt - RSdx - Rdv = 0, \tag{5.5}$$

be exact. This means there exists a function $I(t, x, v)$ such that

$$I_t = R[S(v - r\frac{g}{f}) - \frac{1}{r}fv]$$

$$I_x = -RS \quad \text{and} \quad I_v = -R. \tag{5.6}$$

If we are interested in a time independent first integral, so that $I_t = 0$, we immediately obtain

$$S = \frac{fv}{r\left(v - r\frac{g}{f}\right)}. \tag{5.7}$$

From the compatibility of (5.6), using the above expression for $S$, we have the following equation for determining the integrating factor $R$, *viz*

$$R_x + \frac{fv/r}{\left(r\frac{g}{f} - v\right)}R_v = -\frac{g}{\left(r\frac{g}{f} - v\right)^2}R. \tag{5.8}$$

Next we make the ansatz

$$R = \frac{\left(r\frac{g}{f} - v\right)}{T(x, v)}, \tag{5.9}$$

and after inserting it into (5.8), obtain the following equation for determining $T(x,v)$,

$$X[T] := \left(r\frac{g}{f} - v\right)\frac{\partial T}{\partial x} + \frac{fv}{r}\frac{\partial T}{\partial v} = fT. \tag{5.10}$$

As Chandrasekar *et al* have shown, it is not necessary to obtain the general solution of (5.10). Any particular solution of it is sufficient to determine a first integral, when it exists. In principle this leads to a considerable simplification, which must not be underestimated. For the problem of determining a particular solution of $T$, we shall use the technique of Darboux polynomials. Notice that if $f(x)$ be a polynomial, then in view of (5.4), we conclude that $g/f$ must also be a polynomial. For the vector field $X$ as defined in (5.10) we find that

$$X[h_1] = X[v] = \frac{f}{r}h_1 \tag{5.11}$$

and

$$X[h_2] = X\left[\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right] = \frac{(r-1)}{r}fh_2. \tag{5.12}$$

In other words, $h_1 = v$ and $h_2 = \frac{g}{f} - \frac{(r-1)}{r(r-2)}v$ are Darboux polynomials of the vector field $X$ with cofactors $\lambda_1 = \frac{f}{r}$ and $\lambda_2 = \frac{(r-1)}{r}f$ respectively. Consequently, for $T(x,v) = h_1^{n_1}h_2^{n_2}$, we can find rational numbers such that $X[T] = fT$ namely $n_1 = n_2 = 1$. Thus we have the following particular solution of (5.10):

$$T(x,v) = v\left(\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right). \tag{5.13}$$

This completes the determination of the integrating factor $R$ as

$$I_v = -R = -\frac{rg/f - v}{v/r(rg/f - (r-1)v/(r-2))} \quad \text{and} \quad I_x = -RS = \frac{f}{rg/f - (r-1)v/(r-2)}. \tag{5.14}$$

The corresponding first integral is given by

$$I(x,v) = \log\left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right]^{\frac{r}{r-1}}, \quad r \neq 0,1,2, \tag{5.15}$$

which essentially means that

$$C(x,v) = \left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right], \quad r \neq 0,1,2 \tag{5.16}$$

is a constant of motion.

Of course one could have obtained this first integral in a much more simpler way, by observing that (5.10) admits a solution $T = v^r$. This in turn gives $R = (rg/f - v)/v^r$ and $RS = -fv^{1-r}/r$ from which one gets the first integral (5.16).

### 5.1 A Liénard type nonlinear oscillator – the second order Riccati equation

We illustrate the above method with a well known example:

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0. \tag{5.17}$$

Here $f(x) = \alpha x$ and $g(x) = \beta x^3$. The condition (5.4) gives a quadratic equation for the parameter $r$, with solution

$$\frac{1}{r} = \frac{1}{2}\left[1 \pm \sqrt{1 - 8\beta/\alpha^2}\right].$$

If we choose the value of $r$, then this solution determines a relation between the parameters $\alpha$ and $\beta$ of the equation; conversely, given the parameters it fixes the value of $r$ . For example, the choice $r = 3$ yields $\beta = \frac{\alpha^2}{9}$. Thus setting $\alpha = 3k$ we have $\beta = k^2$ and the equation becomes

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0. \tag{5.18}$$

This particular form is often called the second Riccati equation (and is also the Painlevé-Gambier equation VI with q(Z)=0 of [13]). Its first integral from (5.16) is

$$C_1(x, v) = \frac{kx^2 - 2v}{v^2}. \tag{5.19}$$

The phase flow for the equation, under these circumstances, as determined from (5.2) and (5.3) is

$$\frac{dv}{dx} = \frac{2kxv}{kx^2 - 2v},$$

which may be separated by using the above expression for $C_1$ *viz*

$$\frac{dv}{dx} = \frac{2kx}{C_1 v} \quad \Rightarrow \quad \frac{1}{2}C_1 v^2 - kx^2 = K_2$$

where $K_2$ is an integration constant.
If we desire to express $C_1(x, v)$ in terms of $x$ and the actual velocity $\dot{x}$, then we simply eliminate $v$ using (5.2) to get

$$C_1(x, \dot{x}) = -\frac{2\dot{x} + kx^2}{(\dot{x} + kx^2)^2},$$

which coincides with the results in [5]. In fact it has been shown by Cariñena *et al* that this first integral plays the role of the Hamiltonian for (5.18).

## 6 A generalized 2D- Kepler system

In [7], the authors considered a system of second order ODE's of the generic form

$$\ddot{x} = \frac{P_1}{Q_1} = \phi_1 \ \text{ and } \ \ddot{y} = \frac{P_2}{Q_2} = \phi_2$$

where it is assumed that $\phi_i (i = 1, 2)$ depend on $t, x, \dot{x}, y, \dot{y}$ in general. They illustrate the general procedure and finish off with the following example of the two-dimensional Kepler problem.

$$\ddot{x} = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\ddot{y} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}. \tag{6.1}$$

Their analysis yielded the following first integrals:

$$I_1 = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{\sqrt{x^2+y^2}}$$

$$I_2 = y\dot{x} - x\dot{y}$$

$$I_3 = \dot{x}(y\dot{x} - x\dot{y}) - \frac{y}{\sqrt{x^2+y^2}}. \tag{6.2}$$

corresponding to the Hamiltonian, the angular momentum and the Runge Lenz vector respectively.
We shall consider a system which is similar to this, but of the form:

$$\ddot{x} = -\frac{x(x^2+b)}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{xg_1(x,y)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\ddot{y} = -\frac{y(3x^2+2y^2+b)}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{yg_2(x,y)}{(x^2+y^2)^{\frac{3}{2}}} \tag{6.3}$$

where

$$g_1(x,y) = (x^2+b), \text{ and } g_2(x,y) = (3x^2+2y^2+b). \tag{6.4}$$

As shown in [7], if $I$ be a first integral of the coupled system such that

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0$$

and if we write the coupled system of equations as:

$$(\phi_1 + S_1\dot{x})\, dt - S_1 dx - d\dot{x} = 0 \tag{6.5}$$

$$(\phi_2 + S_2\dot{y})\, dt - S_2 dy - d\dot{y} = 0 \tag{6.6}$$

then we must have

$$I_t = R_1(\phi_1 + S_1\dot{x}) + R_2(\phi_2 + S_2\dot{y})$$

$$I_x = -R_1 S_1$$

$$I_y = -R_2 S_2$$

$$I_{\dot{x}} = -R_1$$

$$I_{\dot{y}} = -R_2. \tag{6.7}$$

Here $R_1, R_2$ represent the respective integrating factors of the system of equations (6.5, 6.6). Compatibility of the set of equations (6.7) then yields the following:

$$D[S_1] = -\phi_{1x} - \frac{R_2}{R_1}\phi_{2x} + \frac{R_2}{R_1}S_1\phi_{2\dot{x}} + S_1\phi_{1\dot{x}} + S_1^2$$

$$D[S_2] = -\phi_{2y} - \frac{R_1}{R_2}\phi_{1y} + \frac{R_1}{R_2}S_2\phi_{1\dot{y}} + S_2\phi_{2\dot{y}} + S_2^2$$

$$D[R_1] = (R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}} + R_1S_1)$$

$$D[R_2] = -(R_2\phi_{2\dot{y}} + R_1\phi_{1\dot{y}} + R_2S_2)$$

$$S_1R_{1y} = R_1S_{1y} + S_2R_{2x} + R_2S_{2x}$$

$$R_{1x} = \frac{\partial}{\partial\dot{x}}(R_1S_1),\ R_{2y} = \frac{\partial}{\partial\dot{y}}(R_2S_2)$$

$$R_{1y} = \frac{\partial}{\partial\dot{x}}(R_2S_2),\ R_{2x} = \frac{\partial}{\partial\dot{y}}(R_1S_1),\ R_{1\dot{y}} = R_{2\dot{x}}$$

Here $D$ represents the vector field

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \phi_1\frac{\partial}{\partial\dot{x}} + \phi_2\frac{\partial}{\partial\dot{y}}$$

The problem is basically to find solutions (particular) satisfying these equations. The explicit details of how these may be simplified and reduced to a more manageable form are contained in [7]. For the system (6.3) one particular solution is the following:

$$R_1 = \dot{x}, \quad R_2 = \dot{y}, \quad S_1 = \frac{x(x^2+b)}{\dot{x}(x^2+y^2)^{\frac{3}{2}}}, \quad S_2 = \frac{y(3x^2+2y^2+b)}{\dot{y}(x^2+y^2)^{\frac{3}{2}}}.$$

With these values of $R_i, S_i$ $(i = 1, 2)$ we obtain the following first integral:

$$I(x, y, \dot{x}, \dot{y}) = -\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{x^2 + 2y^2 - b}{\sqrt{x^2+y^2}}\right]. \tag{6.8}$$

In fact it is easy to verify that this integral is actually the Hamiltonian. However, we have not yet been able to deduce the analogs of the angular momentum or the Lenz vector for this case.

## 7 Conclusion

In this article we have shown how the extended Prelle-Singer (PS) method, as developed by Lakshmanan and his coworkers, can be used to deduce first integrals of certain special classes of second order nonlinear ODEs. In particular, we have focused on the Painlevé-Gambier type ODEs. We have found an additional first integral for the Painlevé-Gambier XXII equation. This appears to be a new result. Using this method we have also derived a formula for the first integrals of a particular sub-class of equations of the Painlevé-Gambier classification. In addition, we have used a novel transformation to analyze the second order Riccati equation (a special case of the Painlevé-Gambier VI equation). Finally we have applied the extended PS method to derive a first integral, of a modified form of the 2D Kepler problem, which is an example of a system of second-order ODEs.

It would be interesting to extend the analysis to the third and higher order nonlinear oscillator equations. Hopefully the proposed method would work for both scalar and multicomponent equations of arbitrary order. Using the first integrals we expect to study appropriate Lagrangians and Hamiltonians. In fact a quantized description can be developed using these Hamiltonian forms which can be mapped onto known quantum mechanical toy models for the damped systems.

Finally we have observed that the method described here for determination of first integrals is closely related to finding solutions of the adjoint symmetry equation. In our forthcoming paper we hope to give a detail mapping and an algorithmic formulation for these class of systems.

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# References

[1] M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations.* Trans. Amer. Math. Soc. 279 (1983), no. 1, 215–229.

[2] M.F. Singer, *Liouvillian first integrals of differential equations.* Trans. Amer. Math. Soc. 333 (1992), no. 2, 673–688.

[3] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré.*, Bull. Sci. Math. (2) 2 (1878), 60–96, 123–144, 151–200.

[4] J.P. Juanolou, *Équations de Pfaff algébriques.* (French)[Algebraic Pfaffian equations] Lecture Notes in Mathematics, 708, Springer, Berlin, 1979. v+255 pp.

[5] J.F. Cariñena, M.F. Rañada and M. Santander, *Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability.* J. Math. Phys. 46 (2005), no. 6, 062703, 18 pp.

[6] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan,*On the general solution of the modified Emden-type equation* $\ddot{x}+\alpha x\dot{x}+\beta x^3=0$.J. Phys. A 40 (2007), 4717-4727.

[7] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *Extended Prelle-Singer method and integrability/solvability of a class of nonlinear $n$th order ordinary differential equations.* J. Nonlinear Math. Phys. 12 (2005), suppl. 1, 184–201.

[8] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *A unification in the theory of linearization of second-order nonlinear ordinary differential equations.* J. Phys. A 39 (2006), no. 3, L69–L76.

[9] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *A simple and unified approach to identify integrable nonlinear oscillators and systems.* J. Math. Phys. 47 (2006), no. 2, 023508, 37 pp.

[10] L.G.S. Duarte, S. E. S. Duarte, L. A. C. da Mota and J. E. F. Skea, *Solving second-order ordinary differential equations by extending the Prelle-Singer method.* J. Phys. A 34 (2001), no. 14, 3015–3024.

[11] L.G.S. Duarte, S. E. S. Duarte, L. A. C. da Mota, *Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution.* J. Phys. A 35 (2002), no. 4, 1001–1006.

[12] A. Goriely, *Integrability and nonintegrability of dynamical systems.* Advanced Series in Nonlinear Dynamics, 19. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+415 pp.

[13] E.L. Ince, *Ordinary differential equations*, Dover, New York. 1956.