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Doubly periodic waves of a discrete nonlinear Schrödinger system with saturable nonlinearity

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Abstract

A system of two discrete nonlinear Schrödinger equations of the Ablowitz-Ladik type with a saturable nonlinearity is shown to admit a doubly periodic wave, whose long wave limit is also derived. As a by-product, several new solutions of the elliptic type are provided for NLS-type discrete and continuous systems.

1 Introduction

The discrete nonlinear Schrödinger equation (d-NLS)

$$iA_t + p\frac{A(x+h,t)+A(x-h,t)-2A(x,t)}{h^2} + q|A|^2A = 0, pq \neq 0,\ p \text{ and } q \text{ real}, \tag{1.1}$$

occurs in many physical disciplines, but unfortunately is not integrable. Hence most investigations must be conducted via numerical computations or other means of approximation. In contrast, the Ablowitz–Ladik (AL) model [1],

$$iA_t + p\frac{A(x+h,t)+A(x-h,t)-2A(x,t)}{h^2} + q\frac{A(x+h,t)+A(x-h,t)}{2}|A(x,t)|^2 = 0, \tag{1.2}$$

possesses N-soliton solutions, periodic waves, an infinite number of conservation laws and a host of other properties associated with integrable systems [27, 28]. Although d-NLS occurs much more frequently than AL in physical situations, there are still some situations where AL is relevant, e.g., higher order evolution equations [17], optics [23], and protein dynamics [21]. Indeed generalizations of AL systems have received quite intensive attention in the literature [24].

The goal here is to investigate a class of coupled (or vector) Ablowitz–Ladik systems with saturable nonlinearities. By "saturable" NLS-type equation, we mean that, in the nonlinear term, the coefficient of A, typically $|A|^2$, is replaced by some expression whose limit is finite when $|A| \to +\infty$, such as $|A|^2/(1+|A|^2)$ or, in the case of two-component NLS, by $|A|^2/(|A|^2+|B|^2)$. A brief and certainly incomplete review of some existing works in the literature will provide the motivation of the present work.

We first look at recent advances on the AL system. Straightforward AL extensions of vector nonlinear Schrödinger equations exhibit solitons with elastic collision properties [2]. Indeed collision properties of chains of near identital solitons have been studied too [8, 9]. Other variants, e.g. coupled discrete evolution systems with nonlinearities resembling the AL model [4, 22], have been considered. However, they are usually not integrable, and propagation of solitons may be unstable.

A remark on saturable nonlinearity is also in order. Saturable nonlinearities probably first arose in the setting of nonlinear optics. In an optical fiber, the Kerr nonlinearity typically generates the conventional nonlinear Schrödinger equation (NLS). However, for short pulses and a high input peak pulse power, these Kerr nonlinearities cannot adequately describe the field-induced change in the refractive index. In such a case, it is sufficient to introduce a saturation term in the nonlinearity, leading in the simplest case to the one-component saturable (continuous) nonlinear Schrödinger equation [15]

$$iA_t + pA_{xx} + q\frac{|A|^2 A}{1+\mu|A|^2} = 0, \tag{1.3}$$

in which p, q, μ are real constants. Other types of nonlinearities have also been considered [20].

Its d-NLS-type discretization

$$iA_t + p\frac{A(x+h,t)+A(x-h,t)-2A(x,t)}{h^2} + q\frac{|A|^2 A}{1+\nu(qh^2/p)|A|^2} = 0, pq\nu \neq 0, \tag{1.4}$$

in which the saturation coefficient ν is dimensionless, has a considerable importance in nonlinear optics [11], but very few analytic results are known. Apart from plane waves and some perturbative solution [26], the main particular solution so far is an elliptic stationary wave solution [19], recalled as Eq. (3.5) in section 3. In particular the problem remains open to find an exact travelling wave of (1.4) with an arbitrary velocity.

When the saturation term $1/(1+\nu(qh^2/p)|A|^2)$ is replaced by an arbitrary analytic function of $|A|^2$ and, at the same time, the nonlinearity is partly discretized in the manner of Ablowitz and Ladik,

$$\begin{aligned} iA_t &+ p\frac{A(x+h,t)+A(x-h,t)-2A(x,t)}{h^2} \\ &+ q\left(F(|A|)\frac{A(x+h,t)+A(x-h,t)}{2} + G(|A|)A(x,t)\right) = 0, \end{aligned} \tag{1.5}$$

one can require the existence of a special class of moving kinks and pulses and thus determine F and G [13], with the main result that Eq. (1.4) admits no stationary pulse in the considered class. A similar inverse approach has been followed in Ref. [18].

Over the years, dynamics of pulses in saturable media has continued to attract attention. An example of a coupled waveguide, which we plan to expand on here, focuses on a saturable, nonlinear medium with cross phase modulation. The propagation of two-component spatial optical solitons carrying angular momentum has been investigated [25]. The reduction of this system to one transverse dimension defines the two-component saturable NLS

$$\begin{cases} iA_t + \lambda_1 p A_{xx} + q\dfrac{|A|^2 + \sigma_1 |B|^2}{1 + s_1\left(|A|^2 + |B|^2\right)} A = 0, \\ iB_t + \lambda_2 p B_{xx} + q\dfrac{|B|^2 + \sigma_2 |A|^2}{1 + s_2\left(|A|^2 + |B|^2\right)} B = 0, \end{cases} \tag{1.6}$$

in which $p, q, \lambda_1, \lambda_2, s_1, s_2, \sigma_1, \sigma_2$ are real constants, and this system has been shown [6] to admit at least one stationary solution (of elliptic type) with a nonconstant total intensity $|A|^2 + |B|^2$, so that the system is truly saturable. In section 5, we extrapolate this solution to a travelling wave having an arbitrary velocity. Another two-component saturable NLS,

$$\begin{cases} iA_t + \lambda_1 p A_{xx} + q\dfrac{A}{1 + s_1\left(|A|^2 + |B|^2\right)} = 0, \\ iB_t + \lambda_2 p B_{xx} + q\dfrac{B}{1 + s_2\left(|A|^2 + |B|^2\right)} = 0, \end{cases} \tag{1.7}$$

describes optically induced nonlinear photonic lattices in which discrete solitons have been observed [11, 14].

From the perspective of nonlinear science, as well as from the viewpoint of applications, it is therefore instructive to consider NLS-type equations which are discrete in space and possess a saturable nonlinearity. Given the elegant nature of the AL system, it seems plausible to study such a discrete saturable NLS with the nonlinear term discretized in the manner of AL rather than of the d-NLS. In fact preliminary studies of such systems have already begun [19, 18].

The structure of this paper is the following. In section 2 we explain how to perform computations which minimize the algebraic complexity. In section 3, we first describe a new family of stationary solutions for a single component AL with saturable nonlinearity. These new solutions are different from those given in the literature [19, 18]. They are delineated by a cyclic combination of elliptic functions in the compact and efficient notation due to Halphen.

Then in section 4, we consider a system of two coupled saturable continuous NLS and generalize a structure previously obtained [6] to an arbitrary velocity.

Finally in section 5 we introduce an AL-type discretization of system (1.6) and find exact solutions with a nonconstant total intensity $|A|^2 + |B|^2$, so that the saturability feature is effective. These solutions are stationary and of the elliptic type. To help the reader, we have gathered in Appendix all the mentioned solutions, whether existing or new.

2 Practical computation

Since the twelve Jacobi elliptic functions [3, Chap. 16] are equivalent in the complex plane, any elliptic solution can be presented as twelve equivalent complex expressions. However, in physics we are only interested in those expressions which are bounded on the real axis, i.e. which only involve the six functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ and $\mathrm{sd}, \mathrm{cd}, \mathrm{nd}$ of the canonical argument (Qx, k), in which the Jacobi modulus k lies between 0 and 1 and the Jacobi nome Q is positive [3, §18.9.9, 18.9.12],

$$m = k^2 = \frac{e_2 - e_3}{e_1 - e_3},\ Q^2 = e_1 - e_3,\ 0 < m < 1,\ 0 < Q^2,\ e_3 < e_2 < e_1. \tag{2.1}$$

In order to minimize time and effort, we will follow a two-step procedure.

1. Firstly, to find a complex (mathematical) solution $A = f(\mathrm{h}_\alpha(x), \mathrm{h}_\beta(x), \mathrm{h}_\gamma(x))$, in which (α, β, γ) is an arbitrary permutation of $(1, 2, 3)$ and $(\mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3)$ are the elliptic functions introduced by Halphen [16], defined by the differential system

$$\frac{\mathrm{d}\,\mathrm{h}_\alpha}{\mathrm{d}x} = -\,\mathrm{h}_\beta\,\mathrm{h}_\gamma,\ \mathrm{h}_\beta^2 - \mathrm{h}_\gamma^2 = -e_\beta + e_\gamma,\ e_1 + e_2 + e_3 = 0, \tag{2.2}$$

with the choice of sign $\lim_{x\to 0} x\,\mathrm{h}_\alpha(x) = +1$. The invariance under any permutation of (α, β, γ) is an important practical advantage of the Halphen notation over the Jacobi notation.

2. Secondly, for each such complex solution, to convert the Halphen trio $(\mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3)$ to both bounded Jacobi trios $(\mathrm{sn}, \mathrm{cn}, \mathrm{dn})$ and $(\mathrm{sd}, \mathrm{cd}, \mathrm{nd})$ by formulae such as

$$\left\{\begin{array}{l} \mathrm{h}_1(x - x_0) = a_1\,\mathrm{dn}(Qx, k),\ \mathrm{h}_2(x - x_0) = a_2\,\mathrm{cn}(Qx, k), \\ \mathrm{h}_3(x - x_0) = a_3\,\mathrm{sn}(Qx, k), \\ a_1^2 = e_3 - e_1,\ a_2^2 = e_3 - e_2,\ a_3^2 = e_2 - e_3,\ \dfrac{a_2 a_3}{a_1} = Qk^2, \end{array}\right. \tag{2.3}$$

in which x_0 is an immaterial complex shift. Full details can be found in Ref. [5, App. B].

3 One-component saturable discrete NLS

For convenience, let us introduce the constant a_0, a kind of unit of amplitude,

$$a_0^2 = -2\frac{p}{q}. \tag{3.1}$$

Let us first consider the equation

$$iA_t + p\frac{A(x+h,t) + A(x-h,t) - 2A(x,t)}{h^2} + q\frac{|A|^2 A}{1 + \nu(qh^2/p)|A|^2} = 0, pq\nu \ne 0. \tag{1.4}$$

If one assumes for a solution of (1.4) the expression (first step as defined in section 2)

$$A = a_1\left[\mathrm{h}_\alpha(x - ct) + ib_1\frac{\mathrm{h}_\beta(x - ct)}{\mathrm{h}_\gamma(x - ct)}\right] e^{i(K(x-ct)-\omega t)}, \tag{3.2}$$

one finds six constraints among the seven introduced constants $(a_1, b_1, c, \omega, K, e_\alpha, e_\beta)$,

$$\begin{cases} c = 0,\ \omega = \dfrac{p}{h^2}\left(2 - \dfrac{1}{\nu}\right),\ \tan Kh = -b_1 \dfrac{\mathrm{h}_\beta(h)}{\mathrm{h}_\alpha(h)\,\mathrm{h}_\gamma(h)}, \\ b_1(e_\gamma - e_\beta)(e_\gamma - e_\alpha - b_1^2) = 0, \\ a_1^2 = a_0^2 \dfrac{\mathrm{h}_\gamma^3(h)}{4\nu^2 h^2\, \mathrm{h}_\beta(h) \cos Kh (b_1^2(e_\beta - e_\gamma) + \mathrm{h}_\gamma^4(h))}, \\ 2\nu = \dfrac{\mathrm{h}_\gamma(h)\left[b_1^2(\mathrm{h}_\alpha^2(h)\,\mathrm{h}_\gamma^4(h) - (e_\gamma - e_\alpha)\,\mathrm{h}_\beta^4(h)) + \mathrm{h}_\alpha^2(h)\,\mathrm{h}_\gamma^6(h)\right]}{\mathrm{h}_\beta(h)\cos Kh\left(b_1^2(e_\beta - e_\gamma) + \mathrm{h}_\gamma^4(h)\right)^2}. \end{cases} \tag{3.3}$$

The factorized form of the fourth constraint (quite similar to that obtained in [7]) defines three solutions, each depending on one arbitrary constant and requiring that ν be nonzero. In order to write them explicitly, let us introduce the nonzero constant

$$N_\alpha = \mathrm{h}_\alpha^4(h) - (e_\alpha - e_\beta)(e_\alpha - e_\gamma) = 2\,\mathrm{h}_\alpha(h)\,\mathrm{h}_\beta(h)\,\mathrm{h}_\gamma(h)\,\mathrm{h}_\alpha(2h). \tag{3.4}$$

These three solutions are

- $(b_1 = 0)$: the elliptic solution already obtained [19],

$$A = a_1\,\mathrm{h}_\alpha(x) e^{-i\omega t},\ a_1^2 = a_0^2 \frac{\mathrm{h}_\beta(h)\,\mathrm{h}_\gamma(h)}{h^2\,\mathrm{h}_\alpha^4(h)},\ 2\nu = \frac{\mathrm{h}_\alpha^2(h)}{\mathrm{h}_\beta(h)\,\mathrm{h}_\gamma(h)}, \tag{3.5}$$

- $(e_\gamma - e_\beta = 0)$: a dark one-soliton solution,

$$\begin{cases} A = a_1\left[\dfrac{\kappa}{2}\tanh\dfrac{\kappa}{2}x + ib_1\right] e^{i(Kx - \omega t)},\ \tan Kh = -b_1 \dfrac{2}{\kappa}\tanh\dfrac{\kappa h}{2} \\ a_1^2 = a_0^2\left(\dfrac{2}{\kappa h}\right)^2 \sinh^2\dfrac{\kappa h}{2}\cosh^{-4}\dfrac{\kappa h}{2}\cos^5 Kh,\ 2\nu = \cosh^2\dfrac{\kappa h}{2}\cos^{-3} Kh, \end{cases} \tag{3.6}$$

- $(e_\gamma - e_\alpha - b_1^2 = 0)$: a new elliptic solution, different from the previous one,

$$\begin{cases} A = a_1\left[\mathrm{h}_\alpha(x) + ib_1\dfrac{\mathrm{h}_\beta(x)}{\mathrm{h}_\gamma(x)}\right] e^{i(Kx - \omega t)},\ b_1^2 = e_\gamma - e_\alpha, \\ a_1^2 = a_0^2 \dfrac{\mathrm{h}_\beta(h)\,\mathrm{h}_\gamma(h) N_\gamma}{h^2 N_\alpha^2 \cos Kh},\ 2\nu = \dfrac{\mathrm{h}_\gamma(h) N_\alpha}{\mathrm{h}_\beta(h) N_\gamma \cos Kh}. \end{cases} \tag{3.7}$$

In the continuum limit, the coefficient ν goes to $1/2$ for all three solutions, so the limit for ω requires expanding ν to second order in h. Each solution goes to some solution of the continuous NLS,

- $(b_1 = 0)$:

$$e_\beta, e_\gamma = \text{arbitrary},\ A = a_0\,\mathrm{h}_\alpha(x) e^{-i\omega t},\ 2\nu = 1,\ \omega = -3pe_\alpha, \tag{3.8}$$

∘ ($e_\gamma - e_\beta = 0$):

$$\begin{cases} b_1, \kappa = \text{arbitrary},\ A = a_0 \left[\dfrac{\kappa}{2} \tanh \dfrac{\kappa}{2} x + i b_1\right] e^{i(-b_1 x - \omega t)}, \\ 2\nu = 1,\ \omega = \left(3 b_1^2 + \dfrac{\kappa^2}{2}\right) p, \end{cases} \tag{3.9}$$

∘ ($e_\gamma - e_\alpha - b_1^2 = 0$):

$$\begin{cases} e_\beta, e_\gamma = \text{arbitrary},\ A = a_0 \left[\mathrm{h}_\alpha(x) + i b_1 \dfrac{\mathrm{h}_\beta(x)}{\mathrm{h}_\gamma(x)}\right] e^{i(-b_1 x - \omega t)},\ b_1^2 = e_\gamma - e_\alpha, \\ 2\nu = 1,\ \omega = (4 e_\beta + 5 e_\gamma) p. \end{cases} \tag{3.10}$$

Following the second step of section 2, various bounded solutions represented by the complex expression (3.7) can be found in terms of bounded Jacobi functions, such as

$$A = (\mathrm{sn} + i\,\mathrm{cd}) e^{i(Kx - \omega t)},\ (\mathrm{cn} + i\,\mathrm{sd}) e^{i(Kx - \omega t)},\ \dots \tag{3.11}$$

we leave this exercise to the interested reader.

On this NLS limit, it is easier to give the precise difference between the two elliptic solutions (3.8) and (3.10), i.e. between (3.5) and (3.7). Indeed, the travelling wave reduction of NLS

$$A(x,t) = \sqrt{M(\xi)} e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct, \tag{3.12}$$

admits as general solution the elliptic functions

$$\begin{cases} M = |A|^2 = a_0^2 \left(\wp(\xi - \xi_0, g_2, g_3) - e_0\right),\ \varphi' = \dfrac{c}{2p} + \dfrac{K_1}{|A|^2},\ a_0^2 = -2\dfrac{p}{q}, \\ e_0 = \dfrac{4\omega p - c^2}{12 p^2},\ K_1^2 = -\dfrac{a_0^4}{4}(4 e_0^3 - g_2 e_0 - g_3), \end{cases} \tag{3.13}$$

therefore the above two elliptic solutions can only be particular cases of it. The first solution (3.8) is the well known Lamé function corresponding to $e_0 = e_\alpha, K_1 = 0$. As to the second solution (3.10), by establishing the ODEs for $|A|^2$ and $(\arg A)'$ one finds the values

$$e_0 = 3 E_\alpha + E_\beta,\ K_1 = -2 a_0^2 \sqrt{E_\gamma - E_\alpha}(E_\alpha - E_\beta), \tag{3.14}$$

in which the elliptic invariants E_j of (3.10) are related to those e_j of (3.8) by the Landen transformation which divides by two only one of the two periods [12, §13.23],

$$\wp(z|\omega, \omega') \to \wp\Big(z\Big|\omega, \frac{\omega'}{2}\Big). \tag{3.15}$$

This is the nonzero value of K_1 which proves that both solutions are indeed different.

We have checked that the saturable discrete NLS (1.4) does not admit a solution depending on two elliptic functions with different periods such as the one [5] admitted by the discrete integrable NLS of Ablowitz and Ladik.

4 Two-component saturable continuous NLS

We next consider the two-component saturable continuous NLS

$$\begin{cases} iA_t + \lambda_1 p A_{xx} + q\dfrac{|A|^2 + \sigma_1 |B|^2}{1 + s_1\left(|A|^2 + |B|^2\right)} A = 0, \\ iB_t + \lambda_2 p B_{xx} + q\dfrac{|B|^2 + \sigma_2 |A|^2}{1 + s_2\left(|A|^2 + |B|^2\right)} B = 0, \end{cases} \tag{1.6}$$

In the continuum case, when one requires the total intensity $|A|^2+|B|^2$ to be nonconstant, so that the system (1.6) truly display saturability, then only one elliptic solution is known [6], in which the moduli $|A|, |B|$ have a zero velocity c. However, if one assumes

$$\begin{cases} A = a_1\, \mathrm{h}_\beta(x-ct)\, \mathrm{h}_\gamma(x-ct) e^{i(K_a(x-ct) - \omega_a t)}, \\ B = a_2\, \mathrm{h}_\beta(x-ct)\, \mathrm{h}_\alpha(x-ct) e^{i(K_b(x-ct) - \omega_b t)},\ e_\gamma \ne e_\alpha, \end{cases} \tag{4.1}$$

the above mentioned solution, which contained no arbitrary parameter at all, is extended to a new solution with one arbitrary parameter (the velocity c), defined by the algebraic relations

$$\begin{cases} c = \text{ arbitrary},\ a_1^2 = a,\ a_2^2 = -a, \\ a = 9a_0^4 \dfrac{\dfrac{\lambda_1}{1-\sigma_1} + \dfrac{\lambda_2}{1-\sigma_2}}{\dfrac{1}{\lambda_1 s_1} - \dfrac{1}{\lambda_2 s_2} - \dfrac{1}{2}\left(\dfrac{1-\sigma_1}{\lambda_1 s_1} - \dfrac{1-\sigma_2}{\lambda_2 s_2}\right)}, \\ e_\gamma - e_\alpha = -\dfrac{1-\sigma_1}{3a_0^2\lambda_1 s_1} = \dfrac{1-\sigma_2}{3a_0^2\lambda_2 s_2} \ne 0, \\ e_\gamma + e_\alpha = \dfrac{1-\sigma_1^2}{9a_0^2 s_1} - 2\dfrac{a_0^2\lambda_1^2}{a} = \dfrac{1-\sigma_2^2}{9a_0^2 s_2} + 2\dfrac{a_0^2\lambda_2^2}{a}, \\ K_a = \dfrac{c}{2p\lambda_1},\ K_b = \dfrac{c}{2p\lambda_2}, \\ \omega_a = -\dfrac{c}{2}K_a + \dfrac{2p}{s_1 a_0^2} - 3p\lambda_1 e_\alpha - \dfrac{18p\lambda_1^2 a_0^2}{a(1-\sigma_1)}, \\ \omega_b = -\dfrac{c}{2}K_b + \dfrac{2p}{s_2 a_0^2} - 3p\lambda_2 e_\gamma + \dfrac{18p\lambda_2^2 a_0^2}{a(1-\sigma_2)}. \end{cases} \tag{4.2}$$

and there exists exactly one constraint among the fixed coefficients,

$$\frac{1-\sigma_1}{\lambda_1 s_1} + \frac{1-\sigma_2}{\lambda_2 s_2} = 0, \tag{4.3}$$

which implies $s_1 \ne s_2$. In particular, the dispersion relations take the form

$$\omega_j = -\frac{c}{2}K_j + \text{constant}_j,\ j = a, b. \tag{4.4}$$

Since the three roots e_α are always real, this complex solution defines three bounded solutions (in terms of $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$), identical to those mentioned in [6, Eqs. (12)–(13)], i.e. $(A, B) = (\mathrm{sn}\,\mathrm{dn}, \mathrm{cn}\,\mathrm{dn})$, $(\mathrm{cn}\,\mathrm{sn}, \mathrm{dn}\,\mathrm{sn})$, $(\mathrm{dn}\,\mathrm{cn}, \mathrm{sn}\,\mathrm{cn})$, with minor adjustments for the additional dependence on c.

The long wave limit of this solution is $(e_\beta - e_\alpha)(e_\beta - e_\gamma) = 0$, i.e.

$$\begin{cases} A = \alpha_1 \operatorname{sech} k(x-ct) \tanh k(x-ct) e^{i(K_a(x-ct) - \omega_a t)}, \\ B = \alpha_2 \operatorname{sech}^2 k(x-ct) e^{i(K_b(x-ct) - \omega_b t)}, \end{cases} \tag{4.5}$$

with the relations

$$\begin{cases} c = \text{ arbitrary}, \ \alpha_1^2 = \alpha_2^2 = \dfrac{\lambda_1 - \lambda_2}{\lambda_2 s_2}, \ k^2 = \dfrac{1}{3a_0^2 s_2}\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right), \\ K_a = \dfrac{c}{2p\lambda_1}, \ K_b = \dfrac{c}{2p\lambda_2}, \\ \omega_a = -\dfrac{c^2}{4p\lambda_1} - \dfrac{2p}{6s_2 a_0^2}\left(1 - \dfrac{\lambda_1}{\lambda_2}\right), \ \omega_b = -\dfrac{c^2}{4p\lambda_2} + \dfrac{4p}{3s_2 a_0^2}\left(1 - \dfrac{\lambda_2}{\lambda_1}\right), \end{cases} \tag{4.6}$$

and the two fixed constraints

$$1 - \sigma_1 = \frac{s_1}{s_2}\left(1 - \frac{\lambda_1}{\lambda_2}\right), \ 1 - \sigma_2 = 1 - \frac{\lambda_2}{\lambda_1}. \tag{4.7}$$

5 Two-component saturable discrete NLS of A-L type

Let us finally consider an AL-type discretization of system (1.6), namely

$$\begin{cases} iA_t + \lambda_1 p \dfrac{A(x+h,t) + A(x-h,t) - 2A(x)}{h^2} \\ \qquad + q \dfrac{|A|^2 + \sigma_1 |B|^2}{1 + s_1\left(|A|^2 + |B|^2\right)} \dfrac{A(x+h,t) + A(x-h,t)}{2} = 0, \\ iB_t + \lambda_2 p \dfrac{B(x+h,t) + B(x-h,t) - 2B(x)}{h^2} \\ \qquad + q \dfrac{|B|^2 + \sigma_2 |A|^2}{1 + s_2\left(|A|^2 + |B|^2\right)} \dfrac{B(x+h,t) + B(x-h,t)}{2} = 0, \end{cases} \tag{5.1}$$

The expression (4.1) where x is now a discrete variable $x = nh$ is still a solution of (5.1), but with much less freedom among the parameters. It is defined by

$$\begin{cases} c = 0, \ a_1^2 = a, \ a_2^2 = -a, \ K_a = 0, \ K_b = 0, \\ s_1 = +\dfrac{1}{a(e_\alpha - e_\gamma)(\mathrm{h}_\gamma^2(h) - 3e_\alpha)}, \ \lambda_1 s_1 = \dfrac{h^2 \, \mathrm{h}_\beta^4(h)}{a_0^2 \, \mathrm{h}_\alpha^2(h)(3\,\mathrm{h}_\alpha^2(h) + e_\alpha - e_\beta)}, \\ s_2 = -\dfrac{1}{a(e_\gamma - e_\alpha)(\mathrm{h}_\alpha^2(h) - 3e_\gamma)}, \ \lambda_2 s_2 = \dfrac{h^2 \, \mathrm{h}_\beta^4(h)}{a_0^2 \, \mathrm{h}_\gamma^2(h)(3\,\mathrm{h}_\gamma^2(h) + e_\gamma - e_\beta)}, \\ \\ \omega_a = -\dfrac{2pa(e_\gamma - e_\alpha)\,\mathrm{h}_\beta(h)(\mathrm{h}_\gamma^2(h) - 3e_\alpha)(\mathrm{h}_\beta^2(h)(\mathrm{h}_\beta(h) - \mathrm{h}_\gamma(h)) + 2(e_\alpha - e_\beta)\,\mathrm{h}_\gamma(h))}{a_0^2\,\mathrm{h}_\alpha^2(h)(3\,\mathrm{h}_\alpha^2(h) + e_\alpha - e_\beta)}, \\ \omega_b = +\dfrac{2pa(e_\alpha - e_\gamma)\,\mathrm{h}_\beta(h)(\mathrm{h}_\alpha^2(h) - 3e_\gamma)(\mathrm{h}_\beta^2(h)(\mathrm{h}_\beta(h) - \mathrm{h}_\alpha(h)) + 2(e_\gamma - e_\beta)\,\mathrm{h}_\alpha(h))}{a_0^2\,\mathrm{h}_\gamma^2(h)(3\,\mathrm{h}_\gamma^2(h) + e_\gamma - e_\beta)}, \\ 1 - \sigma_1 = \dfrac{(e_\gamma - e_\alpha)(\mathrm{h}_\alpha^2(h) - e_\alpha + e_\gamma)}{\mathrm{h}_\alpha^2(h)(3\,\mathrm{h}_\alpha^2(h) + e_\alpha - e_\beta)}, \ 1 - \sigma_2 = \dfrac{(e_\alpha - e_\gamma)(\mathrm{h}_\gamma^2(h) - e_\gamma + e_\alpha)}{\mathrm{h}_\gamma^2(h)(3\,\mathrm{h}_\gamma^2(h) + e_\gamma - e_\beta)}. \end{cases} \tag{5.2}$$

The velocity c now vanishes (while it is arbitrary in the continuous case) and the fixed parameters $\lambda_1, \lambda_2, \sigma_1, \sigma_2, s_1, s_2$ obey three constraints (as compared to one in the continuous case). In particular the relation

$$\frac{1}{s_1} - \frac{1}{s_2} = -2a(e_\alpha - e_\gamma)^2, \tag{5.3}$$

requires s_1 and s_2 to be different if the elliptic functions remain nondegenerate.

In order to remove the constraint $c = 0$, we have tried the assumption (inspired by the relations (2.2))

$$\begin{cases} A = -a_1 \dfrac{\mathrm{h}_\alpha(\xi + h/2) - \mathrm{h}_\alpha(\xi - h/2)}{h} e^{i(K_a \xi - \omega_a t)}, \ \xi = x - ct, \\ B = -a_2 \dfrac{\mathrm{h}_\gamma(\xi + h/2) - \mathrm{h}_\gamma(\xi - h/2)}{h} e^{i(K_b \xi - \omega_b t)}, \ e_\gamma \neq e_\alpha, \end{cases} \tag{5.4}$$

but the system admits no such solution at all.

The long wave limit of the solution (4.1), (5.2) is $(e_\beta - e_\alpha)(e_\beta - e_\gamma) = 0$, and it leads to the same expression (4.5) as in the continuous case.

6 Discussion and conclusion

A system of coupled discrete evolution equations, with cubic nonlinear terms of the Ablowitz–Ladik type and the presence of saturable nonlinearities, has been studied. For the single component case, a new type of periodic waves is formulated in terms of the elliptic functions introduced by Halphen, which maintain a high degree of symmetry. This family of periodic patterns is different from those established earlier in the literature.

For the two-component case, solutions in terms of single and products of elliptic functions are found, which both correspond roughly to sinusoidal structures. As expected in any complicated nonlinear system, significant constraints on the properties of the system, i.e. coefficients of dispersion, cross phase modulation, and saturation, must exist for analytical progress to be made.

A very important issue which has not been addressed in the present work is the stability of these wave patterns. Computational tests must be conducted which will be performed in the near future.

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References

[1] M.J. Ablowitz and J.F. Ladik, Nonlinear differential–difference equations, J. Math. Phys. **16** (1975) 598–603.

[2] M.J. Ablowitz, Y. Ohta and A.D. Trubatch, On discretizations of the vector nonlinear Schrödinger equation, Phys. Lett. A **253** (1999) 287–304.

[3] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Tenth printing (Dover, New York, 1972).

[4] A. Bülow, D. Hennig and H. Gabriel, Soliton interaction for a nonlinear discrete double chain, Physical Review E **59** (1999) 2380–2392.

[5] K.-W. Chow, R. Conte and Neil Xu, Analytic doubly periodic wave patterns for the integrable discrete nonlinear Schrödinger (Ablowitz-Ladik) model, Phys. Lett. A **346** (2006) 422–429.

[6] K.-W. Chow, B.A. Malomed and K. Nakkeeran, Exact solitary- and periodic-wave modes in coupled equations with saturable nonlinearity, Phys. Lett. A **359** (2006) 37–41.

[7] R. Conte and K.-W. Chow, Periodic waves of a discrete higher order nonlinear Schrödinger equation, Commun. Theor. Phys. **46** (2006) 961–965. http://arXiv.org/abs/nlin.PS/0604010

[8] S.V. Dmitriev, P.G. Kevrekidis, B.A. Malomed and D.J. Frantzeskakis, Two-soliton collisions in a near-integrable lattice system, Physical Review E **68** (2003) 056603–1–7.

[9] E.V. Doktorov, N.P. Matsuka and V.M. Rothos, Dynamics of the Ablowitz–Ladik soliton train, Physical Review E **69** (2004) 056607–1–7.

[10] N.K. Efremidis, J. Hudock, D.N. Christodoulides, J.W. Fleischer, O. Cohen and M. Segev, Two-dimensional optical lattice solitons, Phys. Rev. Lett. **91** (2003) 213906-1-4.

[11] N.K. Efremidis, S. Sears, D.N. Christodoulides, J.W. Fleischer and M. Segev, Discrete solitons in photorefractive optically induced photonic lattices, Phys. Rev. E **66** (2002) 046602-1-5.

[12] E. Erdélyi (ed.), *Higher transcendental functions* (McGraw-Hill, New York, 1953).

[13] S. Flach, Y. Zolotaryuk and K. Kladko, Moving lattice kinks and pulses: An inverse method, Phys. Rev. E **59** (1999) 6105–6115.

[14] J.W. Fleischer, M. Segev, N.K. Efremidis and D.N. Christodoulides, Observation of two-dimensional discrete solitons in optically induced nonlinear photonic lattices, Nature **422** (2003) 147–150.

[15] S. Gatz and J. Herrmann, Soliton propagation in materials with saturable nonlinearity, J. Opt. Soc. Am. B **8** (1991) 2296–2302.

[16] G.-H. Halphen, *Traité des fonctions elliptiques et de leurs applications* (Gauthier-Villars, Paris, 1886, 1888, 1891). `http://gallica.bnf.fr/document?O=N007348`

[17] M. Hisakado, The Davey-Stewartson system and the Bäcklund transformations, Journal of the Physical Society of Japan **67** (1998) 3038–3043.

[18] A. Khare, K.Ø. Rasmussen, M. Salerno, M.R. Samuelsen and A. Saxena, Discrete nonlinear Schrödinger equations with arbitrarily high-order nonlinearities, Physical Review E **74** (2006) 016607.

[19] A. Khare, K.Ø. Rasmussen, M.R. Samuelsen and A. Saxena, Exact solutions of the saturable discrete nonlinear Schrödinger equation, J. Phys. A **38** (2005) 807–814.

[20] W. Królikowski, B. Luther-Davies, G. McCarthy and A. Błędowski, Analytical solution for photorefractive screening solitons, Phys. Rev. E **61** (2000) 2010–2014.

[21] K. Kundu, Perturbative study of classical Ablowitz–Ladik type soliton dynamics in relation to enery transport in α-helical proteines, Physical Review E **61** (2000) 5839–5851.

[22] B.A. Malomed and J. Yang, Solitons in coupled Ablowitz-Ladik chains, Phys. Lett. A **302** (2002) 163–170.

[23] A. Mohamadou, F. Fopa and T.C. Kofané, Modulational instability and spatial structures of the Ablowitz–Ladik equation, Optics Communications **266** (2006) 648–655.

[24] D.E. Pelinovsky, Translationally invariant nonlinear Schrödinger lattices, Nonlinearity **19** (2006) 2695–2716.

[25] J.K. Salgueiro and Y.S. Kivshar, Single- and double-vortex vector solitons in self-focusing nonlinear media, Physical Review E **70** (2004) 056613.

[26] M. Stepić, D. Kip, L. Hadžievski and A. Maluckov, One-dimensional bright discrete solitons in media with saturable nonlinearity, Phys. Rev. E **69** (2004) 066618-1-7.

[27] V.E. Vekslerchik, Finite-genus solutions for the Ablowitz-Ladik hierarchy, J. Phys. A **32** (1999) 4983–4994.

[28] J.A. Zagrodzinski, Periodic processes and dispersion relations for Ablowitz–Ladik equations, Chaos, solitons and fractals **11** (2000) 145–152.

Appendix. Summary of solutions

Table 1. Various types of NLS equations considered in the text. Types= (discrete or continuous), (one- or two-component), (saturable or not), (AL or non-AL). Y=yes, N=no, B=both, ell=elliptic, tri=trigonometric. A star (*) denotes apparently new results.

discr/cont	components	saturable	AL	equation	solution	ref
C	1	Y		(1.3)	quadrature	[15]
C	2	Y		(1.6)	ell	[6]
					ell (4.1), (4.2)	* c arb
					ell (4.5), (4.6)	* c arb
C	2	Y		(1.7)	none	
D	1	N	N	(1.1)	none	
D	1	N	Y	(1.2)	many	[1]
D	1	Y	N	(1.4)	3sol (3.2), (3.3)	
					ell (3.2), (3.5)	[19]
					tri (3.6)	
					ell (3.2), (3.7)	*
D	1	B	B	(1.5)	sech, tanh	[13]
D	2	Y	Y	(5.1)	ell (4.1), (5.2)	* $c = 0$