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A new extended q-deformed KP hierarchy

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Abstract

A method is proposed in this paper to construct a new extended q-deformed KP (q-KP) hieararchy and its Lax representation. This new extended q-KP hierarchy contains two types of q-deformed KP equation with self-consistent sources, and its two kinds of reductions give the q-deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q-deformed KP hierarchy, which include two types of q-deformed KdV equation with sources and two types of q-deformed Boussinesq equation with sources. All of these results reduce to the classical ones when q goes to 1. This provides a general way to construct (2+1)- and (1+1)-dimensional q-deformed soliton equations with sources and their Lax representations.

1 Introduction

In recent years, the q-deformed integrable systems attracted many interests both in mathematics and in physics [1,8,9,11,12,15–18,20,23,24,29,30,35–37,39,40,42]. The deformation is performed by using the q-derivative ∂_q to take the place of ordinary derivative ∂_x in the classical systems, where q is a parameter, and the q-deformed integrable systems recover the classical ones as $q \to 1$. The q-deformed N-th KdV (q-NKdV or q-Gelfand-Dickey) hierarchy, the q-deformed KP (q-KP) hierarchy, and the q-AKNS-D hierarchy were constructed, and some of their integrable structures were also studied, such as the infinite conservation laws, bi-Hamiltonian structure, tau function, symmetries, Bäcklund transformation (see [12,23,35,37,39,42] and the references therein).

Multi-component generalization of an integrable model is a very important subject [3, 6, 7, 13, 19, 21, 22, 34, 38]. For example, the multi-component KP (mcKP) hierarchy given in [6] contains many physically relevant nonlinear integrable systems, such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones. Another type of coupled integrable systems is the soliton equation with self-consistent sources, which has many physical applications and can be obtained by coupling some suitable differential equations to the original soliton equation [14,26,27,31–33,41,43].

Very recently, we proposed a systematical procedure to construct a new extended KP hierarchy and its Lax representation [28]. This new extended KP hierarchy contains two types of KP equation with self-consistent sources (KPSCS-I and KPSCS-II), and its two kinds of reductions give the Gelfand-Dickey hierarchy with self-consistent sources [2] and the k-constrained KP hierarchy [5, 25]. In fact, the approach which we proposed in [28] in the framework of Sato theory can be applied to construct many other extended (2+1)-dimensional soliton hierarchies, such as BKP hierarchy, CKP hierarchy and DKP hierarchy, and provides a general way to obtain (2+1)-dimensional and (1+1)-dimensional integrable soliton hierarchies with self-consistent sources.

The KdV equation with self-consistent sources and the KP equation with self-consistent sources can describe the interaction of long and short waves (see [14, 26, 27, 31–33, 41, 43] and the references therein). In contrast with the well-studied KdV and KP equation with self-consistent sources, the q-Gelfand-Dickey hierarchy with self-consistent sources and the q-KP hierarchy with self-consistent sources have not been investigated yet. It is ineresting to consider the case of the algebra of q-pseudo-differential operator, and to see if our approach could be generalized to construct new extended q-deformed integrable systems, which would enable us to find two types of new q-deformed soliton equation with sources in a systematic way.

In this paper, we will give a systematical procedure to construct a new extended q-deformed KP (q-KP) hierarchy and its Lax representation. First, we define a new vector filed ∂_{τ_k} by a linear combination of all vector fields ∂_{t_n} in ordinary q-deformed KP hierarchy, then we introduce a new Lax type equation which consists of the τ_k-flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of ∂_{τ_k}-flow and ∂_{t_n}-flows gives rise to a new extended q-KP hierarchy. This new extended q-KP hierarchy contains two types of q-deformed KP equation with self-consistent sources (q-KPSCS-I and q-KPSCS-II), and its two kinds of reductions give the q-deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q-deformed KP hierarchy, which are some $(1 + 1)$-dimensional q-deformed soliton equation with self-consistent sources, e.g., two types of q-deformed KdV equation with self-consistent sources (q-KdVSCS-I and q-KdVSCS-II) and two types of q-deformed Boussinesq equation with self-consistent sources (q-BESCS-I and q-BESCS-II). The q-KdVSCS-II is just the q-deformed Yajima-Oikawa equation. All of these results reduce to the classical ones when $q \to 1$. Thus, the method proposed in this paper is a general way to find the $(1 + 1)$- and $(2 + 1)$-dimensional q-deformed soliton equation with self-consistent sources and their Lax representations. It should be noticed that a general setting of "pseudo-differential" operators on regular time scales has been proposed to construct some integrable systems [4, 10], where the q-differential operator is just a particular case. Our paper will be organized as follows. In section 2, we will recall some notations in the q-calculus and construct the new extended q-KP hierarchy, and then two types of q-deformed KP equation with sources will be presented. In section 3, the two kinds of reductions for the new extended q-KP hierarchy will be considered, and some $(1 + 1)$-dimensional q-deformed soliton equation with self-consistent sources will be deduced. In section 4, some conclusions will be given.

2 New extended q-deformed KP hierarchy

In this section, we will give a procedure to construct a new extended q-KP hierarchy and its Lax representation. Then, as the examples, two types of q-deformed KP equation with self-consistent sources (q-KPSCS-I and q-KPSCS-II) will be presented explicitly.

The q-deformed differential operator ∂_q is defined as

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)},$$

which recovers the ordinary differentiation $\partial_x(f(x))$ as $q \to 1$. Let us define the q-shift operator θ as

$$\theta(f(x)) = f(qx).$$

Then we have the q-deformed Leibnitz rule

$$\partial_q^n f = \sum_{k\geq 0} \left(\begin{array}{c} n \\ k \end{array}\right)_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n \in \mathbb{Z},$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q-1}, \qquad \left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \qquad \left(\begin{array}{c} n \\ 0 \end{array}\right)_q = 1.$$

For a q-pseudo-differential operator (q-PDO) of the form

$$P = \sum_{i=-\infty}^{n} p_i \partial_q^i,$$

we decompose P into the differential part and the integral part as follows

$$P_+ = \sum_{i\geq 0} p_i \partial_q^i, \qquad P_- = \sum_{i\leq -1} p_i \partial_q^i.$$

The conjugate operation "$*$" for P is defined by

$$P^* = \sum_i (\partial_q^*)^i p_i, \qquad \partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}.$$

The q-KP hierarchy is defined by the Lax equation (see, e.g., [16])

$$\partial_{t_n} L = [B_n, L], \qquad B_n = L_+^n, \tag{2.1}$$

with Lax operator of the form

$$L = \partial_q + \sum_{i=0}^{\infty} u_i \partial_q^{-i}. \tag{2.2}$$

According to the Sato theory, we can express the Lax operator as a dressed operator

$$L = S\partial_q S^{-1}, \tag{2.3}$$

where $S = 1 + \sum_{i=1}^{\infty} S_i \partial_q^{-i}$ is called the Sato operator and S^{-1} is its formal inverse. The Lax equation (2.1) is equivalent to the Sato equation

$$S_{t_n} = -(L^n)_- S. \tag{2.4}$$

The q-wave function $w_q(x, \bar{t}; z)$ and q-adjoint wave function $w^*(x, \bar{t}; z)$ (here $\bar{t} = (t_1, t_2, t_3, \ldots)$) are defined as follows

$$w_q = S e_q(xz) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right), \tag{2.5a}$$

$$w^* = (S^*)^{-1}|_{x/q} e_{1/q}(-xz) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right), \tag{2.5b}$$

where the notation $P|_{x/t} = \sum_i p_i(x/t) t^i \partial_q^i$ (for $P = \sum_i p_i(x)\partial_q^i$) is used, and

$$e_q(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k\right).$$

It is easy to show that w_q and w_q^* satisfy the following linear systems

$$Lw_q = zw_q, \qquad \frac{\partial w_q}{\partial t_n} = B_n w_q,$$

$$L^*|_{x/q} w_q^* = z w_q^*, \qquad \frac{\partial w_q^*}{\partial t_n} = -(B_n|_{x/q})^* w_q^*.$$

It can be proved that [35]

$$T(z)_- \equiv \sum_{i \in \mathbb{Z}} L_-^i z^{-i-1} = w_q \partial_q^{-1} \theta(w_q^*). \tag{2.6}$$

For any fixed $k \in \mathbb{N}$, we define a new variable τ_k whose vector field is

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^{N} \sum_{s \geq 0} \zeta_i^{-s-1} \partial_{t_s},$$

where ζ_i's are arbitrary distinct non-zero parameters. The τ_k-flow is given by

$$\begin{aligned} L_{\tau_k} &= \partial_{t_k} L - \sum_{i=1}^{N} \sum_{s \geq 0} \zeta_i^{-s-1} \partial_{t_s} L = [B_k, L] - \sum_{i=1}^{N} \sum_{s \geq 0} \zeta_i^{-s-1} [B_s, L] \\ &= [B_k, L] + \sum_{i=1}^{N} \sum_{s \in \mathbb{N}} \zeta_i^{-s-1} [L_-^s, L] = [B_k, L] + \sum_{i=1}^{N} \sum_{s \in \mathbb{Z}} \zeta_i^{-s-1} [L_-^s, L]. \end{aligned}$$

Define $\tilde{B}_k$ by

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} \sum_{s \in \mathbb{Z}} \zeta_i^{-s-1} L_-^s, \tag{2.7}$$

which, according to (2.6), can be written as

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} w_q(x, \bar{t}; \zeta_i) \partial_q^{-1} \theta(w_q^*(x, \bar{t}; \zeta_i)). \tag{2.8}$$

By setting $\phi_i = w_q(x, \bar{t}; \zeta_i)$, $\psi_i = \theta(w_q^*(x, \bar{t}; \zeta_i))$, we have

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i, \tag{2.9a}$$

where ϕ_i and ψ_i satisfy the following equations

$$\phi_{i,t_n} = B_n(\phi_i), \qquad \psi_{i,t_n} = -B_n^*(\psi_i), \qquad i = 1, \cdots, N. \tag{2.9b}$$

Now we introduce a new Lax type equation given by

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i, L]. \tag{2.10a}$$

with

$$\phi_{i,t_n} = B_n(\phi_i), \qquad \psi_{i,t_n} = -B_n^*(\psi_i), \qquad i = 1, \cdots, N. \tag{2.10b}$$

We have the following lemma

Lemma 1. $[B_n, \phi\partial_q^{-1}\psi]_- = B_n(\phi)\partial_q^{-1}\psi - \phi\partial_q^{-1}B_n^*(\psi)$.

Proof. Without loss of generality, we consider a monomial: $P = a\partial_q^n$ $(n \geq 1)$. Then

$$[P, \phi\partial_q^{-1}\psi]_- = a(\partial_q^n(\phi))\partial_q^{-1}\psi - (\phi\partial_q^{-1}\psi a \partial_q^n)_-. \tag{2.11}$$

Notice that the second term can be rewritten in the following way

$$\begin{aligned}(\phi\partial_q^{-1}\psi a\partial_q^n)_- &= \phi(\theta^{-1}(\psi a))\partial_q^{n-1} - \phi\partial_q^{-1}(\partial_q\theta^{-1}(a\psi))\partial_q^{n-1})_- \\ &= (\phi\partial_q^{-1}(-\partial_q\theta^{-1}(a\psi))\partial_q^{n-1})_- = \cdots = \phi\partial_q^{-1}\left((-\partial_q\theta^{-1})^n(a\psi)\right) = \phi\partial_q^{-1}P^*(\psi),\end{aligned}$$

then the lemma is proved. ■

Proposition 1. *(2.1) and (2.10) give rise to the following new extended q-deformed KP hierarchy*

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i] = 0 \tag{2.12a}$$

$$\phi_{i,t_n} = B_n(\phi_i), \tag{2.12b}$$

$$\psi_{i,t_n} = -B_n^*(\psi_i), \quad i = 1, \cdots, N. \tag{2.12c}$$

Proof. We will show that under (2.10b), (2.1) and (2.10a) give rise to (2.12a). For convenience, we assume $N = 1$, and denote ϕ_1 and ψ_1 by ϕ and ψ, respectively. By (2.1), (2.10) and Lemma 1, we have

$$\begin{aligned}
&B_{n,\tau_k} = (L^n_{\tau_k})_+ = [B_k + \phi\partial_q^{-1}\psi, L^n]_+ \\
&= [B_k + \phi\partial_q^{-1}\psi, L^n_+]_+ + [B_k + \phi\partial_q^{-1}\psi, L^n_-]_+ \\
&= [B_k + \phi\partial_q^{-1}\psi, L^n_+] - [B_k + \phi\partial_q^{-1}\psi, L^n_+]_- + [B_k, L^n_-]_+ \\
&= [B_k + \phi\partial_q^{-1}\psi, B_n] - [\phi\partial_q^{-1}\psi, B_n]_- + [B_n, L^k]_+ \\
&= [B_k + \phi\partial_q^{-1}\psi, B_n] + B_n(\phi)\partial_q^{-1}\psi - \phi\partial_q^{-1}B_n^*(\psi) + B_{k,t_n} \\
&= [B_k + \phi\partial_q^{-1}\psi, B_n] + (B_k + \phi\partial_q^{-1}\psi)_{t_n}.
\end{aligned}$$

■

Under (2.12b) and (2.12c), the Lax representation for (2.12a) is given by

$$\Psi_{\tau_k} = (B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i)(\Psi), \tag{2.13a}$$

$$\Psi_{t_n} = B_n(\Psi). \tag{2.13b}$$

Remark 1. The main step in our approach is to define a new Lax equation (2.10). For the extended KP hierarchy in [28], a similar formula like (2.10) can be motivated by the well-known k-constraint of KP hierarchy, which is obtained by imposing $L^k = B_k + \sum_{i=1}^{N} \phi_i\partial^{-1}\psi_i$. Here, the formula (2.10) can also be motivated by the k-constraint of q-KP hierarchy as given in [35]. This enables us to obtain the k-constrained q-KP hierarchy and the q-Gelfand-Dickey hierarchy with sources by dropping the τ_k-dependence and t_n-dependence in the new extended q-KP hierarchy (2.12) respectively (see Section 3).

Remark 2. When taking $\phi_i = \psi_i = 0$, $i = 1, \ldots, N$, then the extended q-KP hierarchy (2.12) reduces to the q-KP hierarchy.

Remark 3. Integrable systems can be constructed from the algebra of "pseudo-differential" operators on regular time scales in [4, 10], where the algebra of q-"pseudo-differential" operator is a particular case. In fact, our approach for constructing new extended integrable systems can also be generalized to the general setting as in [4, 10].

For convenience, we write out some operators here

$$B_1 = \partial_q + u_0, \qquad B_2 = \partial_q^2 + v_1\partial_q + v_0, \qquad B_3 = \partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0,$$

$$\phi_i\partial_q^{-1}\psi_i = r_{i1}\partial_q^{-1} + r_{i2}\partial_q^{-2} + r_{i3}\partial_q^{-3} + \ldots, \qquad i = 1, \ldots, N,$$

where

$$\begin{aligned}
&v_1 = \theta(u_0) + u_0, \qquad v_0 = (\partial_q u_0) + \theta(u_1) + u_0^2 + u_1, \\
&v_{-1} = (\partial_q u_1) + \theta(u_2) + u_0 u_1 + u_1\theta^{-1}(u_0) + u_2,
\end{aligned}$$

$$s_2 = \theta(v_1) + u_0, \qquad s_1 = (\partial_q v_1) + \theta(v_0) + u_0 v_1 + u_1,$$
$$s_0 = (\partial_q v_0) + \theta(v_{-1}) + u_0 v_0 + u_1 \theta^{-1}(v_1) + u_2.$$

$$r_{i1} = \phi_i \theta^{-1}(\psi_i), \qquad r_{i2} = -\frac{1}{q}\phi_i \theta^{-2}(\partial_q \psi_i), \qquad r_{i3} = \frac{1}{q^3}\phi_i \theta^{-3}(\partial_q^2 \psi_i).$$

and v_{-1} comes from $L^2 = B_2 + v_{-1}\partial_q^{-1} + v_{-2}\partial_q^{-2} + \cdots$.

Then, one can compute the following commutators

$$[B_2, B_3] = f_2 \partial_q^2 + f_1 \partial_q + f_0, \qquad [B_2, \phi_i \partial_q^{-1} \psi_i] = g_{i1}\partial_q + g_{i0} + \ldots,$$
$$[B_3, \phi_i \partial_q^{-1} \psi_i] = h_{i2}\partial_q^2 + h_{i1}\partial_q + h_{i0} + \ldots, \qquad i = 1, \ldots, N,$$

where

$$f_2 = \partial_q^2 s_2 + (q+1)\theta(\partial_q s_1) + \theta^2(s_0) + v_1 \partial_q s_2 + v_1 \theta(s_1) + v_0 s_2 - (q^2+q+1)\theta(\partial_q^2 v_1)$$
$$-(q^2+q+1)\theta^2(\partial_q v_0) - (q+1)s_2 \theta(\partial_q v_1) - s_2 \theta^2(v_0) - s_1 \theta(v_1) - s_0,$$

$$f_1 = \partial_q^2 s_1 + (q+1)\theta(\partial_q s_0) + v_1 \partial_q s_1 + v_1 \theta(s_0) + v_0 s_1 - \partial_q^3 v_1 - (q^2+q+1)\theta(\partial_q^2 v_0)$$
$$-s_2 \partial_q^2 v_1 - (q+1)s_2 \theta(\partial_q v_0) - s_1 \partial_q v_1 - s_1 \theta(v_0) - s_0 v_1,$$

$$f_0 = \partial_q^2 s_0 + v_1 \partial_q s_0 - \partial_q^3 v_0 - s_2 \partial_q^2 v_0 - s_1 \partial_q v_0,$$

$$g_{i1} = \theta^2(r_{i1}) - r_{i1}, \qquad g_{i0} = (q+1)\theta(\partial_q r_{i1}) + \theta^2(r_{i2}) + v_1 \theta(r_{i1}) - r_{i1}\theta^{-1}(v_1) - r_{i2},$$

$$h_{i2} = \theta^3(r_{i1}) - r_{i1}, \qquad h_{i1} = (q^2+q+1)\theta^2(\partial_q r_{i1}) + \theta^3(r_{i2}) + s_2 \theta^2(r_{i1}) - r_{i1}\theta^{-1}(s_2).$$

$$h_{i0} = (q^2+q+1)\theta(\partial_q^2 r_{i1}) + (q^2+q+1)\theta^2(\partial_q r_{i2}) + \theta^3(r_{i3}) + (q+1)s_2 \theta(\partial_q r_{i1})$$
$$+s_2 \theta^2(r_{i2}) + s_1 \theta(r_{i1}) - r_{i1}\theta^{-1}(s_1) + \frac{1}{q} r_{i1}\theta^{-2}(\partial_q s_2) - r_{i2}\theta^{-2}(s_2) - r_{i3}.$$

Now, we list some examples in the new extended q-KP hierarchy (2.12).

Example 1 (The first type of q-KPSCS (q-KPSCS-I)). For $n = 2$ and $k = 3$, (2.12) yields the first type of q-deformed KP equation with self-consistent sources (q-KPSCS-I) as follows

$$-\frac{\partial s_2}{\partial t_2} + f_2 = 0, \tag{2.14a}$$

$$\frac{\partial v_1}{\partial \tau_3} - \frac{\partial s_1}{\partial t_2} + f_1 + \sum_{i=1}^{N} g_{i1} = 0, \tag{2.14b}$$

$$\frac{\partial v_0}{\partial \tau_3} - \frac{\partial s_0}{\partial t_2} + f_0 + \sum_{i=1}^{N} g_{i0} = 0, \tag{2.14c}$$

$$\phi_{i,t_2} = B_2(\phi_i), \qquad \psi_{i,t_2} = -B_2^*(\psi_i), \qquad i = 1, \ldots, N. \tag{2.14d}$$

The Lax representation for (2.14) is

$$\Psi_{\tau_3} = (\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0 + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(\Psi), \tag{2.15a}$$

$$\Psi_{t_2} = (\partial_q^2 + v_1\partial_q + v_0)(\Psi). \tag{2.15b}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-KPSCS-I (2.14) reduces to the first type of KP equation with self-consistent sources (KPSCS-I) which reads as [31, 32]

$$u_{1,t_2} - u_{1,xx} - 2u_{2,x} = 0, \tag{2.16a}$$

$$2u_{1,\tau_3} - 3u_{2,t_2} - 3u_{1,x,t_2} + u_{1,xxx} + 3u_{2,xx} - 6u_1u_{1,x} + 2\partial_x \sum_{i=1}^{N} \phi_i\psi_i = 0, \tag{2.16b}$$

$$\phi_{i,t_2} - \phi_{i,xx} - 2u_1\phi_i = 0, \tag{2.16c}$$

$$\psi_{i,t_2} + \psi_{i,xx} + 2u_1\psi_i = 0, \qquad i = 1, \ldots, N. \tag{2.16d}$$

Example 2 (The second type of q-deformed KPSCS (q-KPSCS-II)). For $n = 3$ and $k = 2$, (2.12) yields the second type of q-deformed KP equation with self-consistent sources (q-KPSCS-II) as follows

$$\frac{\partial s_2}{\partial \tau_2} - f_2 + \sum_{i=1}^{N} h_{i2} = 0, \tag{2.17a}$$

$$\frac{\partial s_1}{\partial \tau_2} - \frac{\partial v_1}{\partial t_3} - f_1 + \sum_{i=1}^{N} h_{i1} = 0, \tag{2.17b}$$

$$\frac{\partial s_0}{\partial \tau_2} - \frac{\partial v_0}{\partial t_3} - f_0 + \sum_{i=1}^{N} h_{i0} = 0, \tag{2.17c}$$

$$\phi_{i,t_3} = B_3(\phi_i), \qquad \psi_{i,t_3} = -B_3^*(\psi_i), \qquad i = 1, \ldots, N. \tag{2.17d}$$

The Lax representation for (2.17) is

$$\Psi_{\tau_2} = (\partial_q^2 + v_1\partial_q + v_0 + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(\Psi), \tag{2.18a}$$

$$\Psi_{t_3} = (\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)(\Psi). \tag{2.18b}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-KPSCS-II (2.17) reduces to the second type of KP equation with self-consistent sources (KPSCS-II) which reads as [31]

$$u_{1,\tau_2} - u_{1,xx} - 2u_{2,x} + \partial_x \sum_{i=1}^{N} \phi_i\psi_i = 0, \tag{2.19a}$$

$$3u_{2,\tau_2} + 3u_{1,x,\tau_2} - 2u_{1,t_3} - u_{1,xxx} + 6u_1u_{1,x} - 3u_{2,xx} + 3\partial_x \sum_{i=1}^{N} \phi_{i,x}\psi_i = 0, \tag{2.19b}$$

$$\phi_{i,t_3} - \phi_{i,xxx} - 3u_1\phi_{i,x} - 3(u_{1,x} + u_2)\phi_i = 0, \tag{2.19c}$$

$$\psi_{i,t_3} - \psi_{i,xxx} - 3u_1\psi_{i,x} + 3u_2\psi_i = 0, \qquad i = 1, \ldots, N. \tag{2.19d}$$

3 Reductions

The new extended q-deformed KP hierarchy (2.12) admits reductions to several well-known q-deformed $(1+1)$-dimensional systems.

3.1 The n-reduction of (2.12)

The n-reduction is given by

$$L^n = B_n \qquad \text{or} \qquad L^n_- = 0, \tag{3.1}$$

then (2.5) implies that

$$B_n(\phi_i) = L^n\phi_i = \zeta_i^n\phi_i, \tag{3.2a}$$
$$-B_n^*(\psi_i) = -L^{n*}\psi_i = -\zeta_i^n\psi_i. \tag{3.2b}$$

By using Lemma 1 and (3.2), we can see that the constraint (3.1) is invariant under the τ_k flow

$$\begin{aligned}(L^n_-)_{\tau_k} &= [B_k, L^n]_- + \sum_{i=1}^N [\phi_i\partial_q^{-1}\psi_i, L^n]_- \\ &= [B_k, L^n_-]_- + \sum_{i=1}^N [\phi_i\partial_q^{-1}\psi_i, L^n_+]_- + \sum_{i=1}^N [\phi_i\partial_q^{-1}\psi_i, L^n_-]_- \\ &= \sum_{i=1}^N [\phi_i\partial_q^{-1}\psi_i, B_n]_- = -\sum_{i=1}^N (\phi_{i,t_n}\partial_q^{-1}\psi_i + \phi_i\partial_q^{-1}\psi_{i,t_n}) \\ &= -\sum_{i=1}^N (\zeta_i^n\phi_i\partial_q^{-1}\psi_i - \zeta_i^n\phi_i\partial_q^{-1}\psi_i) = 0.\end{aligned} \tag{3.3}$$

The equations (3.1) and (2.4) imply that $S_{t_n} = 0$, so $(L^k)_{t_n} = 0$, which together with (3.3) means that one can drop t_n dependency from (2.12) and obtain

$$B_{n,\tau_k} = [(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^N \phi_i\partial_q^{-1}\psi_i, B_n], \tag{3.4a}$$
$$B_n(\phi_i) = \zeta_i^n\phi_i, \tag{3.4b}$$
$$B_n^*(\psi_i) = \zeta_i^n\psi_i, \quad i = 1, \cdots, N. \tag{3.4c}$$

The system (3.4) is the q-deformed Gelfand-Dickey hierarchy with self-consistent sources.

Example 3 (The firs type of q-deformed KdVSCS (q-KdVSCS-I)). For $n = 2$ and $k = 3$, (3.4) presents the first type of q-deformed KdV equation with self-consistent sources (q-KdVSCS-I)

$$v_{1,\tau_3} + f_1 + \sum_{i=1}^{N} g_{i1} = 0, \tag{3.5a}$$

$$v_{0,\tau_3} + f_0 + \sum_{i=1}^{N} g_{i0} = 0, \tag{3.5b}$$

$$u_2 + \theta(u_2) + \partial_q(u_1) + u_0 u_1 + u_1 \theta^{-1}(u_0) = 0, \tag{3.5c}$$

$$(\partial_q^2 + v_1\partial_q + v_0)(\phi_i) - \zeta^2\phi_i = 0, \tag{3.5d}$$

$$(\partial_q^2 + v_1\partial_q + v_0)^*(\psi_i) - \zeta^2\psi_i = 0, \qquad i = 1, \cdots, N, \tag{3.5e}$$

with the Lax representation

$$\Psi_{\tau_3} = (\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0 + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(\Psi),$$

$$(\partial_q^2 + v_1\partial_q + v_0)(\Psi) = \lambda\Psi, \qquad u_2 + \theta(u_2) + \partial_q(u_1) + u_0 u_1 + u_1\theta^{-1}(u_0) = 0.$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-KdVSCS-I (3.5) reduces to the first type of KdV equation with self-consistent sources (KdVSCS-I) which reads as

$$u_2 = -\frac{1}{2} u_{1,x},$$

$$u_{1,\tau_3} - 3u_1 u_{1,x} - \frac{1}{4} u_{1,xxx} + \partial_x \sum_{i=1}^{N} \phi_i \psi_i = 0,$$

$$\phi_{i,xx} + 2u_1\phi_i - \zeta^2\phi_i = 0,$$

$$\psi_{i,xx} + 2u_1\psi_i - \zeta^2\psi_i = 0, \qquad i = 1, \cdots, N.$$

The first type of KdV equation with self-consistent sources (KdVSCS-I) can be solved by the inverse scattering method [27, 33] or by the Darboux transformation (see [26] and the references therein).

Example 4 (The first type of q-BESCS (q-BESCS-I)). For $n = 3$ and $k = 2$, (3.4) presents the first type of q-deformed Boussinesq equation with self-consistent sources (q-BESCS-I)

$$s_{2,\tau_2} - f_2 + \sum_{i=1}^{N} h_{i2} = 0, \tag{3.6a}$$

$$s_{1,\tau_2} - f_1 + \sum_{i=1}^{N} h_{i1} = 0, \tag{3.6b}$$

$$s_{0,\tau_2} - f_0 + \sum_{i=1}^{N} h_{i0} = 0, \tag{3.6c}$$

$$(\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)(\phi_i) - \zeta^3\phi_i = 0, \tag{3.6d}$$

$$(\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)^*(\psi_i) - \zeta^3\psi_i = 0, \qquad i = 1, \cdots, N, \tag{3.6e}$$

with the Lax representation

$$\Psi_{\tau_2} = (\partial_q^2 + v_1\partial_q + v_0 + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(\Psi), \qquad (\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)(\Psi) = \lambda\Psi. \tag{3.7}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-BESCS-I (3.6) reduces to the first type of Boussinesq equation with self-consistent sources (BESCS-I) which reads as

$$\begin{aligned}
&-2u_{2,x} - u_{1,xx} + u_{1,\tau_2} + \partial_x \sum_{i=1}^{N} \phi_i\psi_i = 0,\\
&3u_{2,\tau_2} - 3u_{2,xx} + 3u_{1,x,\tau_2} + 6u_1u_{1,x} - u_{1,xxx} + 3\partial_x \sum_{i=1}^{N} \phi_{i,x}\psi_i = 0,\\
&\phi_{i,xxx} + 3u_1\phi_{i,x} + 3(u_{1,x} + u_2)\phi_i - \zeta^3\phi_i = 0,\\
&\psi_{i,xxx} + 3u_1\psi_{i,x} - 3u_2\psi_i + \zeta^3\psi_i = 0, \qquad i = 1, \cdots, N.
\end{aligned}$$

3.2 The k-constrained hierarchy of (2.12)

The k-constraint is given by [5, 25]

$$L^k = B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i. \tag{3.8}$$

By using the above k-constraint, it can be proved that L and B_n are independent of τ_k. By dropping τ_k dependency from (2.12), we get

$$\left(B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i\right)_{t_n} = \left[(B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i)_+^{\frac{n}{k}}, B_k + \sum_{i=1}^{N} \phi_i\partial_q^{-1}\psi_i\right], \tag{3.9a}$$

$$\phi_{i,t_n} = (B_k + \sum_{j=1}^{N} \phi_j\partial_q^{-1}\psi_j)_+^{\frac{n}{k}}(\phi_i), \tag{3.9b}$$

$$\psi_{i,t_n} = -(B_k + \sum_{j=1}^{N} \phi_j\partial_q^{-1}\psi_j)_+^{\frac{n}{k}*}(\psi_i), \quad i = 1, \cdots, N, \tag{3.9c}$$

which is the constrained q-deformed KP hierarchy. Some solutions of the constrained q-deformed KP hierarchy can be represented by q-deformed Wronskian determinant (see [12] and the references therein).

Remark 4. In [4, 10], the k-constrained q-KP hierarchy can be constructed from the q-KP hierarchy by imposing the k-constraint. Here, the k-constrained q-KP hierarchy is obtained directly from the *extended* q-KP hierarchy (2.12) by dropping the τ_k dependence due to the k-constraint.

Example 5 (The second type of q-KdVSCS (q-KdVSCS-II)). For $n = 3$ and $k = 2$, (3.9) gives rise to the second type of q-deformed KdV equation with self-consistent sources

(q-KdVSCS-II).

$$v_{1,t_3} + f_1 - \sum_{i=1}^{N} h_{i1} = 0, \tag{3.10a}$$

$$v_{0,t_3} + f_0 - \sum_{i=1}^{N} h_{i0} = 0, \tag{3.10b}$$

$$u_2 + \theta(u_2) + \partial_q(u_1) + u_0 u_1 + u_1\theta^{-1}(u_0) - \sum_{i=1}^{N} r_{i1} = 0, \tag{3.10c}$$

$$\phi_{i,t_3} = (\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)(\phi_i), \tag{3.10d}$$

$$\psi_{i,t_3} = -(\partial_q^3 + s_2\partial_q^2 + s_1\partial_q + s_0)^*(\psi_i), \qquad i = 1, \cdots, N. \tag{3.10e}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-KdVSCS-II (3.10) reduces to the second type of KdV equation with self-consistent sources (KdVSCS-II or Yajima-Oikawa equation) which reads as

$$u_2 = -\frac{1}{2}u_{1,x} + \frac{1}{2}\sum_{i=1}^{N}\phi_i\psi_i,$$

$$u_{1,t_3} = \frac{1}{4}u_{1,xxx} + 3u_1u_{1,x} + \frac{3}{4}\sum_{i=1}^{N}(\phi_{i,xx}\psi_i - \phi_i\psi_{i,xx}),$$

$$\phi_{i,t_3} = \phi_{i,xxx} + 3u_1\phi_{i,x} + \frac{3}{2}u_{1,x}\phi_i + \frac{3}{2}\phi_i\sum_{j=1}^{N}\phi_j\psi_j,$$

$$\psi_{i,t_3} = \psi_{i,xxx} + 3u_1\psi_{i,x} + \frac{3}{2}u_{1,x}\psi_i - \frac{3}{2}\psi_i\sum_{i=1}^{N}\phi_j\psi_j, \qquad i = 1, \cdots, N.$$

Example 6 (The second type of q-BESCS (q-BESCS-II)). For $n = 2$ and $k = 3$, (3.9) gives rise to the second type of q-deformed Boussinesq equation with self-consistent sources (q-BESCS-II))

$$s_{2,t_2} - f_2 = 0, \tag{3.11a}$$

$$s_{1,t_2} - f_1 - \sum_{i=1}^{N} g_{i1} = 0, \tag{3.11b}$$

$$s_{0,t_2} - f_0 - \sum_{i=1}^{N} g_{i0} = 0, \tag{3.11c}$$

$$\phi_{i,t_2} = (\partial_q^2 + v_1\partial_q + v_0)(\phi_i), \tag{3.11d}$$

$$\psi_{i,t_2} = -(\partial_q^2 + v_1\partial_q + v_0)^*(\psi_i), \qquad i = 1, \cdots, N. \tag{3.11e}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the q-BESCS-II (3.11) reduces to the second type of Boussinesq equation with self-consistent sources (BESCS-II) which reads as

$$
\begin{aligned}
&-2u_{2,x} - u_{1,xx} + u_{1,t_2} = 0,\\
&3u_{2,t_2} - 3u_{2,xx} + 3u_{1,x,t_2} + 6u_1u_{1,x} - u_{1,xxx} - 2\partial_x \sum_{i=1}^{N} \phi_i\psi_i = 0,\\
&\phi_{i,t_2} = \phi_{i,xx} + 2u_1\phi_i,\\
&\psi_{i,t_2} = -\psi_{i,xx} - 2u_1\psi_i, \qquad i = 1, \cdots, N.
\end{aligned}
$$

4 Conclusions

A method is proposed in this paper to construct a new extended q-deformed KP (q-KP) hiearchy and its Lax representation. This new extended q-KP hierarchy contains two types of q-deformed KP equation with self-consistent sources (q-KPSCS-I and q-KPSCS-II), and its two kinds of reductions give the q-deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q-deformed KP hierarchy. Thus, the reductions of the new extended q-KP hierarchy may give some q-deformed $(1+1)$-dimensional soliton equation with self-consistent sources, e.g., the two types of q-deformed KdV equation with self-consistent sources (including q-deformed Yajima-Oikawa equation) and two types of q-deformed Boussinesq equation with self-consistent sources. All of these results reduce to the classical ones when $q \to 1$. The method proposed in this paper is a general way to find $(1+1)$- and $(2+1)$-dimensional q-deformed soliton equation with self-consistent sources and their Lax representations.

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