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Changing Solitons in Classical & Quantum Integrable Defect and Variable Mass Sine-Gordon Model

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Abstract

Sine-Gordon (SG) models with position dependent mass or with isolated defects appear in many physical situations, ranging from fluxon or semi-fluxon in nonuniform Josephson junction to spin-waves in quantum spin chain with variable coupling or DNA solitons in the active promoter region. However such phenomena usually break the integrability of the model, allowing only numerical or perturbative result. We investigate two types of inhomogeneous sine-Gordon (SG) models: one with a variable mass and the other with a defect at the center and show integrability of both these models, in classical as well as in exact quantum level. The variable mass SG exhibits accelerating and shape changing exact solitons and can describe realistic problems at certain limits, while the defect SG possesses a rich class of exact solutions with creation or annihilation of solitons by the defect point. Based on our result theories for exact semi-fluxion solution in $0-\pi$-Josephson junction is proposed.

1 Introduction

Sine-Gordon (SG) model is an important nonlinear integrable field model, which along with its theoretical richness has a wide range of applications in different fields [1, 2, 3, 4, 5, 6, 7, 8, 9]. Apart from possessing all fascinating properties of classical integrable systems the SG model exhibits special properties, like relativistic invariance, integer-valued topological charge represented by solutions like kink, antikink, breather etc. [10] and most importanly the ultralocality leading to classical r-matrix formulation and the quantum integrability. Quantum integrability is guaranteed by the quantum Yang-Baxter equation (QYBE), which for the SG model yields the well known quantum $su_q(2)$ algebra [11, 12].

In realistic sytems however the SG model usually appears not in its *pure* form, but with inhomogeneities or defect, which spoil the most cherishable property of the model, e.g. the integrability. For example SG models with variable mass (VMSG) $m = m(x)$ appears in fluxon dynamics in Josephson junction with impurity [6], in DNA-soliton dynamics due to nonuniformity induced by specific base sequences in the promoter region [4], in spin wave propagation with variable interaction strength [5] etc. Semifluxons seem to arise in long JJ governed by the SG with defect, where the -ve half of the solution u is described by the standard SG, while in the +ve half the solution suffers a π-jump: $u+\pi$ [2]. However in all such inhomogeneous SG models with variable mass or defects, due to loss of integrability the solutions can be extracted only numerically or at best perturbatively [4, 6, 7, 5].

Therefore it is a challenge to build SG models with variable mass or with defects, preserving their classical and quantum integrability, and at the same time describing closely the realistic situations using their exact analytic result. We focus in the subsequent sections on two such inhomogeneous SG models: i) with space-time dependent variable mass $m(x,t)$ (VMSG) [13] and ii) with a defect at the center (DSG) [14], which have the above desirable properties.

2 Variable mass Sine-Gordon model

Solitons in the constant mass SG model, as in all integrable systems, move with constant velocity and shape. In real systems however due to nonuniformity of the media, soliton velocity, shape and amplitude might change with space-time, and a static soliton may start moving and even turn back [6, 7, 4]. This can be used also as desirable effects for fast transport, fast communication, or for a possible soliton gun [8]. However such results due to nonintegrable nature of realistic models could be achieved mostly numerically or perturbatively. We however construct an integrable VMSG model with exact solutions, which nevertheless exhibit shape changing and accelerating solitons.

2.1 VMSG model through Lax pair

Since our strategy is to respect integrability, we start from the linear spectral problem $\Phi_x(x,\lambda) = U(\lambda,x)\Phi(x,\lambda),\ \Phi_t(x,\lambda) = V(\lambda,x)\Phi(x,\lambda),$ for the SG model with its Lax pair [15]

$$U = \frac{i}{4}\left(-u_t\sigma^3 + mk_1\cos\frac{u}{2}\sigma^2 - mk_0\sin\frac{u}{2}\sigma^1\right), \tag{2.1}$$

$$V = \frac{i}{4}\left(-u_x\sigma^3 - mk_0\cos\frac{u}{2}\sigma^2 + mk_1\sin\frac{u}{2}\sigma^1\right), \tag{2.2}$$

where $k_0 = 2\lambda + \frac{1}{2\lambda},\ k_1 = 2\lambda - \frac{1}{2\lambda}$.

Note that the Lax pair contain two parameters: *mass m* and *spectral parameter* λ, which are linked to soliton width (shape) and its velocity, respectively. When they are constant, the compatibility $\Phi_{xt} = \Phi_{tx}$, or equivalently the flatness condition $U_t - V_x + [U,V] = 0$, yields the integrable SG equation. However making m or λ variable, breaks in general the integrability of the system. Interestingly we observe that, the integrability can be restored, when both these parameters vary simultaneously following the constraint: $(mk_0)_t + (mk_1)_x = 0,\ (mk_1)_t + (mk_0)_x = 0$, which yields the VMSG equation

$$u_{tt} - u_{xx} + m^2(x,t)\sin u = 0. \tag{2.3}$$

Note that the contraint can be simplified to $\kappa_t + \rho_x = 0,\ \kappa_x + \rho_t = 0$ where $\kappa = \ln m,\ \rho = \ln\lambda$ and reduced to two free field equations

$$\kappa_{tt} - \kappa_{xx} = 0,\ \ \rho_{tt} - \rho_{xx} = 0. \tag{2.4}$$

Remarkably, the set of equations (2.3-2.4) represents a new integrable relativistic system generalizing the SG model and is a reduction of the conformal affine Toda model (CATM), at the free field limit of the spectral dilatation field ρ [16].

However, since our aim here is to apply VMSG to realistic models with given inhomogeneity, instead of dealing with the integrable system (2.3-2.4) in its general form, we restrict to particular solutions for variable mass and spectral parameter:

$$m(x,t) = m_0 f_+ f_-, \lambda = \lambda_0 \frac{f_-}{f_+} \tag{2.5}$$

compatible with the integrable VMSG (2.3). Here $f_\pm$ are arbitrary smooth functions of $x_\pm = x \pm t$, respectively. Due to explicit space-time dependent coefficient (2.5), the VMSG equation (2.3) is no longer relativistic or translational invariant. Demand of such invariance simply gives back the SG model with mass $m = const$. [17]. There also exists a nonlinear coordinate transformation: $(x,t) \to (X,T)$, which can map VMSG with variable mass to SG model with a constant mass and takes particularly simple form in the light-cone coordinates [17, 16]: $X_\pm = \int dx_\pm f_\pm^2$. Physically this means is to go to a noninertial frame of reference, which however may change the domain or make it unphysical, singularities may also arise or the boundary conditions might change. For investigating real systems with inhomogeneity inducing accelerated and shape changing solitons, which is our main focus here, one should however analyze the VMSG model in its original form. Similar situation arises in the study of accelerated solitons in plasma governed by the integrable inhomogeneous NLS equation and also in inhomogeneous Toda chain, Ablowitz -Ladik model etc. with nonisospectral flow [18, 19].

2.2 Soliton solutions and classical integrability

For extracting the exact solution for our VMSG model, we can apply Hirota's bilinearization as well as the inverse scattering (IS) method, the former being a direct method for soliton solution, while the later is an indirect method capable of giving more general solution. For soliton solution of the SG equation Hirota's solution may be expressed as $u = -2i \ln \frac{g^+}{g^-}$, where $g^\pm$ are conjugate functions with expansion in plane-wave type solutions. For the VMSG model (2.3) the same ansatz seems to work, only the plane waves should be replaced by their generalized form: $g^{(n)} = \frac{c_n}{\lambda_n} e^{\frac{i}{2}(X(\lambda_n,x,t)-T(\lambda_n,x,t))}$, where $X(\lambda_n,x,t) = \int^x dx' m(x',t) k_{1n}(x',t)$, $T(\lambda_n,x,t) = \int^t dt' m(x,t') k_{0n}(x,t')$. This gives the soliton solutions through the expansion:

$$g^\pm = 1 \pm g^{(1)}, \quad \text{for 1- kink} \; . \;\; g^\pm = 1 \pm (g^{(1)} + g^{(2)}) + s(\frac{(\theta_1 - \theta_2)}{2}) g^{(1)} g^{(2)}, \;\; \text{for 2-kink}$$

etc. with the scattering matrix $s(\theta) = tanh^2 \theta$. $\lambda_2 = -\lambda_1^* = \eta e^{i\theta}$, gives the kink-antikink bound state or the breather solution.

Similarly we can apply the IS formalism to the inhomogeneous SG model, for which the crucial step is to analyse and use the analytic properties of the Jost function Φ. Here again the asymptotic plane waves should be replaced by their generalized form. Therefore, going parallel to the standard SG one can get for the VMSG model the exact N-soliton solution (for $r(\lambda) = 0$) with discrete spectrum λ_n $n = 1, 2, \ldots N$ (zeros of $a(\lambda)$). $N = 1$-soliton (kink) solution with $\lambda_1 = i\eta$, takes the explicit form

$$u = 4 \tan^{-1}(e^\zeta), \; \zeta = \frac{i}{2}(X(i\eta,x,t) - T(i\eta,x,t)), \tag{2.6}$$

with the corresponding localized soliton $\sin \frac{u}{2} = \frac{1}{\cosh(\zeta)}$, , which we draw in Fig. 1. The variable soliton velocity is given by $v_s(x,t) = -\frac{dx}{dt} = \frac{k_1(\eta,x.t)}{k_0(\eta,x.t)}$.

For this integrable VMSG apart from the Lax pair and the exact N-soliton solusions we can find also the set of all higher conserved quantities and prove the classical integrability more explicitly by showing that it satisfies classical YBE with the same r-matrix of the SG model [20].

2.3 Quantum integrability

We explore the quantum integrability of our VMSG model following the algebraic Bethe ansatz (ABA) method for the constant mass SG case [11], since at the quantum level direct mapping from VMSG to SG becomes difficult. Quantum SG lattice Lax matrix-operator $U_j(\lambda,\mathbf{S}_j,m),\ j=1,2,\ldots,L$ involves quantum spin operators $S^3_j(u_j),S^{\pm}_j(u_j,p_j,m)$ expressed in canonical operators $u_j,\ p_j=\dot{u}_j$ and mass parameter m, which should be considered now as site dependent: m_j. We find fortunately, that the quantum $R(\frac{\lambda}{\mu})$-matrix associated with the QYBE

$$R(\frac{\lambda}{\mu})U_j(\lambda)\otimes U_j(\mu)=U_j(\mu)\otimes U_j(\lambda)R(\frac{\lambda}{\mu}),\quad j=1,2,...,n \tag{2.7}$$

for the SG model remains unchanged for its inhomogeneous extension, since this R-matrix depends only on the ratio of two spectral parameters, $\frac{\lambda(x,t)}{\mu(x,t)}$, in which x,t-dependence (2.5) enters only multiplicatively and hence cancels out. Moreover, QYBE being a local algebra (at each lattice site j) is not affected by inhomogeneity and yields the same quantum algebra $su_q(2)$, replacing only m by a site-dependent m_j in its structure constant: $[S^+_j,S^-_k]=\delta_{jk}m_j\frac{\sin\alpha 2S^3_j}{\sin\alpha}$.

The aim of the ABA is to solve exactly the eigenvalue problem of $\mathrm{tr}T(\lambda)=\sum_n\hat{C}_n\lambda^n$, with $T=\prod^L_j U_j$, generating all higher conserved operators $\hat{C}_n$ including the Hamiltonian, with the eigenstates given as $|N>=|\lambda_1,\ldots,\lambda_N>=\prod^N_a B(\lambda_a)|0>$. $T_{12}=B(\lambda)$ acts as *creation* operator, while $T_{21}=C(\lambda)$ as *destruction* operator annihilating the pseudovaccum: $C(\lambda)|0>=0$. A crucial step in the formalism is to construct this pseudovaccum state $|0>$, which we achieve by combining the actions of the consequitive pair of Lax operators: $U_jU_{j+1}|0>$, as proposed in [11], but generalizing the procedure for site-dependent mass m_j. Thus we solve for $|0>=\prod^L_j|\Omega^{(2)}_j>$ through local pseudovaccum as $\Omega^{(2)}_j=(1+\delta^2 g_{m_1m_2}(q_1,q_2))f_{m_1m_2}(q_1,q_2)$, where g and f are generalizations: $f_{m_1m_2}=(\frac{m_2^{\frac{q_2}{\alpha}}}{m_1^{\frac{q_1}{\alpha}}})f_m,\ g_{m_1m_2}=\frac{m_2^2}{m}g_m$, over their known solution f_m,g_m for constant m [11]. Consequently the vacuum eigenvalues for the VMSG model are generalized as $A(\lambda)|0>=\alpha_{(m)}|0>,\ \ D(\lambda)|0>=\beta_{(m)}|0>$, where $\alpha_{(m)}=\prod_j a(\theta,\frac{m_j}{m_{j+1}}),\ \beta_{(m)}=\prod_j a^*(\theta,\frac{m_{j+1}}{m_j})$ with $a(\theta,\frac{m_j}{m_{j+1}})=(\frac{m_j}{m_{j+1}}+\delta^2 m_jm_{j+1}(\cosh(2\theta+i\alpha)))$. We get finally the exact eigenvalue for the conserved quantities: $trT(\lambda)$ as $\Lambda(\lambda;\lambda_1,\ldots,\lambda_N)=\alpha_{(m)}\prod^N_a f(\frac{\lambda_a}{\lambda})+\beta_{(m)}\prod^N_a f(\frac{\lambda}{\lambda_a})$, where $f(\frac{\lambda}{\mu})$ is expressed through the elements of the quantum $R(\frac{\lambda}{\mu})$-matrix for the SG model, which remains unchanged. The Bethe equations for determining the parameters $\{\lambda_a\}$ are generalized similarly.

2.4 Application to physical problems

Since our main focus is to make explicit contact with physical models, we concentrate on the shape and velocity changing soliton solution of VMSG for concrete integrable cases as shown in Fig. 1 a-d).

i) Notice that variable mass $m=m_0(x^2-t^2)^n$ remains invariant under relativistic motion and for $n=1$, yields from (2.6) the exact soliton solution $u=4\tan^{-1}(e^{\zeta}),\zeta=\frac{m}{3}(2\eta(x-t)^3+\frac{1}{2\eta}(x+t)^3)$.

The corresponding localized soliton drawn in Fig. 1a, clearly shows the intriguing change in soliton shape and width. Position-dependent mass in this case is achieved at $t \to 0$ and therefore for a short time evolution limit the above analytic solution can describe the fluxon propagation through Josephson junction with local impurity like $m_0 x^2$, as shown in Fig. 1b.

ii) Another case of physical significance $m = \sqrt{2} m_0 \cos^\alpha q(x \pm t)$, with α as arbitrary parameter gives (with $\alpha = 1$ and +ve sign) the exact kink solution $u = 4\tan^{-1}(e^\zeta),\ \zeta = m_0(k_0 x - k_1 t + \frac{1}{4q\eta}\sin 2q(x+t)),\ k_{0,1} = 2\eta \pm 1/(2\eta)$ with soliton variable velocity $v_s = m_0 d(k_1 - \frac{1}{2\eta}\cos 2q(x+t))$ and width $d = (m_0(k_0 + \frac{1}{2\eta}\cos 2q(x+t)))^{-1}$ oscillating periodically in space-time, as shown in Fig. 1c. Note that this is a particularly interesting case describing SG equation parametrically driven by a plane wave [21].

iii) Another similar integrable case is with mass $m = m_0(2\cos q(x+t)\cos q(x-t))^{\frac{\alpha}{2}}$, which at short time interval limit ($t \to 0$) gives $\approx m(x) = \tilde{m}_0(\cos qx)^\alpha$, while for evolution limited to a small space interval ($x \to 0$): $\approx m(t) = \tilde{m}_0(\cos qt)^\alpha$. Both these limits are of significant physical relevance, since a spin chain with variable coupling constant may be described by a VMSG with mass $m(x) = m_0(\cos qx)^\alpha$, where $\alpha = \frac{1}{2-K}$, with $K \geq \frac{1}{2}$ being an important parameter of the system [5]. Similarly a real oscillator chain model pumped by an alternating current [22] can be linked to a VMSG with mass $m(t) = (\cos qt)^{\frac{1}{2}}$. Therefore we may conclude that the exact soliton solutions of our VMSG model can describe analytically important physical models at certain limits, and alternatively realistic spin (or oscillator) models can be tuned to the integrable VMSG model by making its coupling strength oscillate periodically also in time (or space).

Since in physical situations the inhomogeneity of the media induces mostly position-dependent mass $m(x)$, we explore this case and conclude from our result (2.5) that, the VMSG with space-dependent variable mass can be integrable only for $m(x) = me^{\rho\ (x-x_0)}$ with $\rho = const$. This also explains why most of the realistic VMSG equations with a different position-dependent mass (see e.g. [5]) turn out to be nonintegrable. Exact kink solution for this integrable case is obtained from (2.6) as

$$u = 4\tan^{-1}(e^{\pm\zeta}),\ \zeta = \frac{1}{\rho}k_0(t)m(x),\ m(x) = \exp(\rho(x-x_0)),\ k_0(t) = \cosh(\theta - \rho(t-t_0)). \quad (2.8)$$

The corresponding variable soliton velocity and width are $v_s = tanh(\theta - \rho(t-t_0))$, and $d = \frac{1}{m(x)k_0(t)}$, showing how the shape of the soliton changes and how it accelerates, decelerates or exhibits boomeron [23] like property. This scenario is close to the predicted behavior of solitons in the dynamically active promoter zone of the T7A$_1$ DNA [4].

Another remarkable fact to notice is that, the position-dependent mass can change the boundary condition of the SG field and therefore can control the crucial topological charge Q of the kink solution. Recall that unlike fluxons in JJ, which can be described by the analytic kink solution of SG with $Q = 1$, the semifluxons with $Q = \frac{1}{2}$, has no known analytic description [2]. Using our result (2.8) we may conclude that, for $\rho > 0$, since $m(\infty) = \infty,\ m(-\infty) = 0$, the kink solution would yield $u(\infty) = 2\pi,\ u(-\infty) = \pi$, corresponding to the topological charge $Q = \frac{1}{2\pi}(u(\infty) - u(-\infty)) = \frac{1}{2}$. This fact might serve as an analytic theory based on VMSG for the semi-fluxon, observed in experiments [24]. We shall propose another possible formulation of exact semi-fluxon solution based on integrable DSG model studied in the next section.

At $\rho \to 0$: $\zeta = \frac{1}{\rho}k_0(t)m(x) \to \zeta_0 = m(k_0(x-x_0) - k_1(t-t_0))$ and the standard SG soliton with $m = const, v_s = const.$ is recovered. Therefore we can acess the solitonic behavior in realistic models for any mass deviation from its constant value, by approximating through expansion in ρ with high orders of accuracy. Fig. 1d shows that a static soliton in a region with constant mass remains static, while an initially static soliton placed in a zone with variable mass can move with accelerated (or decelerated) motion, simulating the solitons in the DNA chain, where static DNA solitons in the inactive regions (with constant mass due to almost uniform background of two types of base pairs) remain static, while similar initially static solitons in active promoter region with variable mass (due to significant difference in the number of lighter (A-T) and heavier (G-C) base pairs) can acquire rich accelerated motion [4].

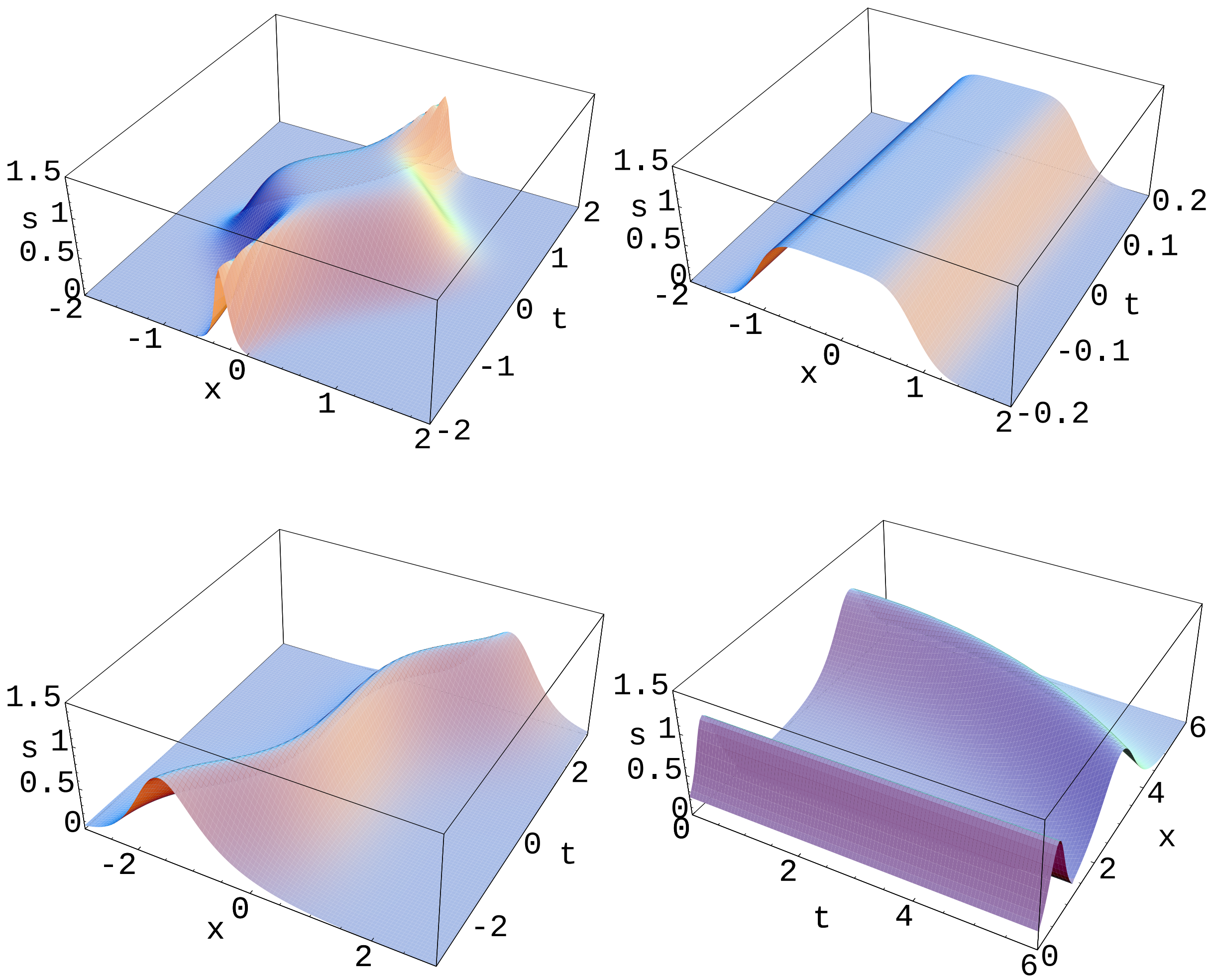

Figure 1: Exact soliton solutions: $s = \sin\frac{u}{2}(x,t)$ for the integrable VMSG equation with variable mass m. a) $m = m_0(x^2 - t^2)$ having an intriguing flattening of the soliton at the center. b) Short time interval limit of the above soliton, showing the flattening prominently. c) $m = 2m_0 \cos q(x+t)$ with oscillating behavior of the soliton. d) Static soliton in the zone $(x \leq 1.2)$ with $m = const$ and initially static soliton in the zone (at $x = 4.8$) with variable mass: $m = m_0 \exp(\rho x) \approx m(1+\rho x)$, with $\rho = 0.1$ moves backward with acceleration, resembling soliton propagation in inactive/active promoter region in a DNA chain.

3 SG model with defect

We focus now on our second project [14] of integrable sine-Gordon model with a defect (DSG) at a single point $x = 0$, described by

$$u_{tt}^{\pm} - u_{xx}^{\pm} + m_0^2 \sin u^{\pm} = 0, \quad \text{for } u^+ = u(x \geq 0,t), \; u^- = u(x \leq 0,t)). \tag{3.1}$$

This model at classical and semi-classical level was investigated from the abstract theoretical point of view [25] and the conserved quantities were found in a general framework [26]. On the other hand some forms of DSG are in use for describing an important physical situation, e.g. semi-fluxon in a long Josephson junction [24], which appears due to certain discontinuity point in such superconducting systems. However DSG models considered in this context for describing semi-fluxons are all nonintegrable systems allowing only approximate numerical solutions [2].

We construct here DSG model (3.1), show its integrability both in the classical and in exact quantum level, find systematically all higher conserved quantities including explicitl expressions for the defect Hamiltonian and momentum. As an important application we propose an analytic descriptrion of semi-fluxons within the framework of our integrable DSG.

3.1 Bridging relation and conserved quantities

Based on the result of [27] on semi-line SG our crucial observation [14] is that the functions $u^-(x,0)$ (and $u_t^-(x,0)$) can be smoothly extended from $x \leq 0$ onto the whole line by means of *bridging relation* (similarly $u^+(x,0)$ from $x \geq 0$ onto the whole line) given by the Bäcklund transformation (BT):

$$u_x^+ = u_t^- + p + q, \qquad u_t^+ = u_x^- + p - q, \;\; p = a \sin \frac{u^+ + u^-}{2}, \;\; q = a^{-1} \sin \frac{u^+ - u^-}{2} \tag{3.2}$$

with parameter a regulating the defect intensity. Therefore our concept differs conceptually from that of [25], where BT (3.2) for the DSG model was considered to be frozen only at the defect point. In our model the physical observable fields are nevertheless only u^+ along the +ve and u^- along the -ve semiaxis.

Interestingly, BT (3.2) can be represented as a *gauge transformation* relating two Lax pairs of the DSG:

$$U^{SG}(u^+) = F^0 U^{SG}(u^-)(F^0)^{-1} + F_x^0 (F^0)^{-1}, \; V^{SG}(u^+) = F^0 V^{SG}(u^-)(F^0)^{-1} + F_t^0 (F^0)^{-1} \tag{3.3}$$

where $F^0(\xi, u^+, u^-)$ is the Bäcklund matrix

$$F^0(\xi, u^+, u^-) = e^{-i\frac{\alpha}{4}\sigma^3 u^-} M(\lambda, a) e^{i\frac{\alpha}{4}\sigma^3 u^+}, \;\; M(\lambda, a) = \begin{pmatrix} \lambda & a \\ -a & \lambda \end{pmatrix}, \tag{3.4}$$

involving both fields $u^{\pm}$ and bridging between them (including the defect point $x = 0$).

3.2 Conserved quantities of the DSG

For Standard SG the infinite set of conserved quantities can be obtained from the generating function $\log a(\lambda) = \sum C_n \lambda^n, \; C_n = \int_{-\infty}^{+\infty} \rho_n(u) dx,$ where $a(\lambda) + a^*(\lambda) = Tr(T(\lambda)).$ For integrable DSG, the commuting set of conserved quantities is similarly given by

$$C_n^d = \int_{-\infty}^{0} \rho_n(u^-) dx + D_n + \int_{0}^{+\infty} \rho_n(u^+) dx \tag{3.5}$$

with a crucial additional contribution $D_n,\ n=1,2,...$ due to the defect point $x=0$ in each of the conserved quantities C_n. We intend to find these additional contributions systematically, following the faddeev-Takhtajan construction [20] coupled with the BT (3.3).

The infinite set of integrals of motion for the SG model as detailed in [20] may be given by (for $n \geq 2$)

$$C_n = i\int_{-\infty}^{+\infty}(w_{n+1}(x) - e^{-iu(x)}w_{n-1}(x))dx, \tag{3.6}$$

$$C_1 = -\frac{1}{4}\int_{-\infty}^{+\infty}(\frac{1}{2}(u_t(x)+u_x(x))^2 + 1 - \cos u(x))dx \tag{3.7}$$

where w_n are determined from the Riccati type equation obtained from the linear system of SG (2.2) as

$$\begin{aligned} w_{n+1}(x) = {} & -2iw_n^{'}(x) - \theta(x)w_n(x) + \frac{i}{2}\Sigma_{k=1}^{n}w_k(x)w_{n+1-k}(x) \\ & -\frac{i}{2}\Sigma_{k=0}^{n-1}w_k(x)e^{iu(x)}w_{n-1-k}(x) - \frac{i}{2}e^{iu}\delta_{n,1}, \end{aligned} \tag{3.8}$$

with $w_0 = i$ and $\theta(x) = u_t + u_x$.

To derive the asymptotic expansion for $\lambda \to 0$ it suffices to use the involution $(\lambda,\pi,u) \to (-\lambda^{-1},\pi,-u)$, with $\pi = u_t$, which leaves the Lax pair invariant. As a result one gets $C_{-n}(\pi,u) = (-1)^n C_n(\pi,-u),\ n=1,2,....$ In particular for the momentum P and the Hamiltonian H we have $P = -\frac{1}{2}(C_{-1}+C_1)$ and $H = \frac{1}{2}(C_{-1}-C_1)$. For extending this procedure to the DSG model, we use (3.3) to obtain a crucial bridging relation between monodromy matrices across the defect point $x=0$ as $T(0+,y,\lambda) = \frac{1}{\lambda - ia}F_0(\lambda)T(0-,y,\lambda),\quad y \neq 0$. Incorporating this relation in the above procedure of [20], we can derive systematically all D_n [14] as

$$D_1 = 2a\cos\frac{(u^+(0)+u^-(0))}{2},\ \ D_2 = w_1(0-)D_1 + iaD_1 - \frac{1}{2}D_1^2, \tag{3.9}$$

etc., where $w_1(0-)$ is solution of Riccati equation (3.8): $w_1(x) = -i\alpha(u_t^-(x)+u_x^-(x))$. For deriving D_{-n} as defect contribution in conserved quantities C_{-n} we use again the above mentioned symmetry to get $D_{-n}(\pi^-,u^-,\pi^+,u^+,a) = D_n(\pi^-,-u^-,-\pi^+,u^+,-\frac{1}{a})$ and hence obatin $D_{-1} = -\frac{2}{a}\cos\frac{(u^+(0)-u^-(0))}{2}$, and similarly all $D_{-n}, n \geq 2$ etc. using the result for higher D_n, already found, the details of which will be given separately [14].

As a result we obtain the Hamiltonian (energy) for the DSG model as

$$H^{(def)} = \int_{-\infty}^{0}H(u^-)dx + \int_0^{\infty}H(u^+)dx + H_d(0) \tag{3.10}$$

where $H(u) = \frac{1}{2}(u_x^2+u_t^2) + m_0^2(1-\cos u)$ is the standard SG energy-density, while extra contributions from defect at $x=0$ is

$$H_d(0) = -(2a\cos\frac{u^+(0)+u^-(0)}{2} + 2a^{-1}\cos\frac{u^+(0)-u^-(0)}{2}), \tag{3.11}$$

and the momentum as

$$P^{(def)} = \int_{-\infty}^{0}P(u^-)dx + \int_0^{\infty}P(u^+)dx + P_d(0), \tag{3.12}$$

where $P(u) = u_x u_t$ is the standard SG result, while

$$P_d(0) = -2a\cos\frac{u^+(0)+u^-(0)}{2} + 2a^{-1}\cos\frac{u^+(0)-u^-(0)}{2}. \tag{3.13}$$

is the defect contribution.

Note that inspite of additional terms in the Hamiltonian at the defect point $x = 0$ the equations for $u^\pm$ at this point still give the same SG equation. This can be shown by carefully considering the extra terms, which come from the integral part and cancel the additional defect contribution.

3.3 Classical and Quantum integrability of DSG through Yang-Baxter equation

A semiclassical treatment through factorizable S-matrix has been given in [25] for the DSG model. We present here the exact quantum integrable extension of this model following closely the approach of standard quantum SG model [11], also detailed in the previous section. For this we construct first an exact lattice regularized version of our quantum DSG model with a discrete monodromy matrix

$$T^N_{-N}(\lambda) = T^{N+}(\lambda)F^d_0(\lambda, u^+_0, u^-_0)T^{N-}(\lambda) \tag{3.14}$$

where

$$T^{N+}(\lambda) = U^+_N(\lambda, u^+_N)\cdots U^+_1(\lambda, u^+_1),\ \ T^{N-}(\lambda) = U^-_{-1}(\lambda, u^-_{-1})\cdots U^-_{-N}(\lambda, u^-_{-N}) \tag{3.15}$$

with $U^\pm_j(\lambda, u^\pm_j),\ j = \pm1, \ldots \pm N$ being the discrete quantum Lax operator of the lattice SG model defined along both sides of the defect, while $F^d_0(\lambda, u^+_0, u^-_0)$ is the Lax operator at the defect point $j = 0$. Since for quantum integrability each of the Lax operators involved in (3.14) must satisfy the QYBE with trigonometric R^{trig}-matrix, each of them should be a particular realization of the ancestor model defined in [28]:

$$L^{trig}_{anc}(\lambda) = \begin{pmatrix} \lambda\hat{c}^{(+)}_1 e^{i\alpha s^3} + \lambda^{-1}\hat{c}^{(-)}_1 e^{-i\alpha s^3} & 2\sin\alpha s^{(-)}_q \\ 2\sin\alpha s^{(+)}_q & \lambda\hat{c}^{(+)}_2 e^{-i\alpha s^3} + \lambda^{-1}\hat{c}^{(-)}_2 e^{i\alpha s^3} \end{pmatrix}, \tag{3.16}$$

with the quantum spin operators generating a generalized quantum algebra

$$[s^{(+)}_q, s^{(-)}_q] = \left(\hat{M}^{(+)}\sin(2\alpha s^3) - i\hat{M}^{(-)}\cos(2\alpha s^3)\right)\frac{1}{\sin\alpha}, \tag{3.17}$$

$$[s^3, s^{(\pm)}_q] = \pm s^{(\pm)}_q, [\hat{M}^{(\pm)}, \cdot] = 0. \tag{3.18}$$

Here the deforming operators $\hat{M}^{(\pm)} = \frac{1}{2}(\hat{c}^{(+)}_1\hat{c}^{(-)}_2 \pm \hat{c}^{(-)}_1\hat{c}^{(+)}_2)$ are expressed through $\hat{c}^{(\pm)}_a,\ a = 1, 2,$ which are mutually commuting and central. The superscripts $(\pm)$ here are ovbiously different from $\pm$ lebeling the fields along $\pm$ ve semiaxis in the DSG model.

$U^\pm_j$ in (3.15)) should be the exact lattice Lax operator of the SG model, while F^d_0 a similar lattice version of the SG Bäcklund matrix (3.4) and both of them must be derivable from the ancestor model (3.16). Note that a reduction at $\hat{c}^{(\pm)}_a = \mp i\Delta, a = 1, 2,$ takes (3.18) to $su_q(2)$ and the related generators represented in canonical variables $[u^\pm_j, p^\pm_k] = i\delta_{jk}$, to

$$s^3 = \frac{u^\pm}{2},\ s^{(+)}_q = e^{-ip^\pm}g(u^\pm, \Delta),\ s^{(-)}_q = (s^{(+)}_q)^\dagger, \tag{3.19}$$

where

$$g(u,\Delta) = \left(1 + 2\Delta^2 \cos\alpha(u+\frac{1}{2})\right)^{\frac{1}{2}} \frac{1}{\sin\alpha}. \tag{3.20}$$

recover from (3.16) the known quantum Lax operator U_j^{SG} for the lattice SG model [11].

Now for constructing the dicrete BT operator F_0^d similarly from (3.16), we choose $\hat{c}_a^{(+)} = 1,\ \hat{c}_a^{(-)} = 0,\ a = 1,2$, giving $\hat{M}^{(\pm)} = 0$. This reduces (3.18) to an algebra $[s_q^{(+)}, s_q^{(-)}] = 0, \quad [s^3, s_q^{(\pm)}] = \pm s_q^{(\pm)}$, with a consistent realization in the form

$$e^{i\alpha s^3} = e_-,\ s_q^{(+)} = (s_q^{(-)})^\dagger = ae_+ P_-^{-1},\ \text{ where }\ e_\pm = e^{i\frac{\alpha}{4}(u_0^+ \pm u_0^-)}, P_- = e^{i2(p_0^+ - p_0^-)} \tag{3.21}$$

with commutation realations $[e_+, P_-] = 0,\ e_- P_- = e^{-i\alpha} P_- e_-$. (3.16) therefore reduces finally to the explicit solution

$$F_0^d(\lambda) = P_-^{\frac{1}{2}\sigma^3} F_0(\lambda) P_-^{-\frac{1}{2}\sigma^3} \tag{3.22}$$

Note that both the above discrete Lax operators obtained as realizations of the quantum integrable (3.16) by construction must satisfy QYBE exactly with R^{trig}-matrix. Consequently, (3.14) with (3.15) represent a quantum integrable discrete DSG model.

At the classical limit when $R \to r$ the QYBE reduces to classical YBE (CYBE) $\{U_j(\lambda), \otimes U_j(\mu)\}_{PB} = \delta_{jk}[r(\lambda,\mu), U_j(\lambda), \otimes U_k(\mu)]$. We can deriectly check that both the discrete Lax operators (3.22) and exact lattice U_j^{SG} satisfiy CYBE exactly, proving the classical integrability of the DSG explicitly.

Following the formulation of quantum SG model [11] we can apply the Algebraic Bethe ansatz method to the lattice regularized quantum DSG constructed above and solve in principle its eigenvalue problem exactly. For constructing the crucial pseudovaccum $|0>^\pm = \prod_{j=\pm 1}^{\pm} N |\Omega_j>$, we observe that the approach should be the same as in the standard SG model, at all sites except the defect point, yielding $C^\pm(\lambda)|0>^\pm = 0$ along the $\pm$-semiaxis. However the defect point would play a nontrivial role, since after crossing this point, say from the left the pseudovaccum property gets lost due to nontriangular matrix form of $F_0^d|\Omega_0>$. Instead of annihilating the local vaccum $|\Omega_0>$, the defect at $j = 0$ would turn it to a state $O|\Omega_0>$ at the lower left corner of the matrix $F_0^d|\Omega_0>$, where operator $O = -ae^{i(2(p_0^+ - p_0^-) + \frac{\alpha}{4}(u_0^- + u_0^+))}$, and creating at the same time its conjugate state $-O^\dagger|\Omega_0>$ at the upper right corner. This is expected to lead to the creation/annihilation of quantum states by the defect point similar to that with classical solitons as we will observe below. This tricky point however needs careful and separate analysis and should be dealt with elsewhere.

It is crucial to check that the discrete DSG we constructed and solved above should yield the same DSG field model we are investigating here, at the continuum limit with lattice const. $\Delta \to 0$. Note that at this limit the canonical variables go to canonical fields: $u_j^\pm \to u^\pm(x), p_j^\pm \to \Delta p^\pm(x)$, with $[u^\pm(x), p^\pm(y)] = i\delta(x-y)$. Therefore for extracting the limit we have to scale $p_j^\pm$ giving $e^{ip_j^\pm} \approx 1 + i\Delta p^\pm(x)$. It is easy to check that this would yield from the Lax operator of the lattice SG model: $\sigma^1 U_j^{SG\pm} \to (1 + \Delta U^{SG\pm}(x)) + O(\Delta^2)$ the corresponding field Lax operator $U^{SG\pm}(x)$ given by (2.2).

Concentrating on the defect point $j = 0$ at the same limit, we get the expansion :

$$F_0^d(\lambda, u_0^+, u_0^-) \to F^0(\lambda, u^+(0), u^-(0)) + \Delta F^1(\lambda, u^+(0), u^-(0)) \tag{3.23}$$

which clearly yields only F^0 at $\Delta \to 0$, i.e. recovers the same BT matrix (3.4) at the continuum limit, which was our essential requirement. Therefore collecting all nontrivial terms we get finally the continuum limit of (3.14)

$$T(\lambda) = \left(e^{\int_0^{+\infty} U^{SG+}(\lambda,x')dx'}\right) F^0(\lambda, u^+(0), u^-(0)) \left(e^{\int_{-\infty}^0 U^{SG-}(\lambda,x')dx'}\right) \tag{3.24}$$

yielding our original DSG field model we are investigating.

3.4 Changing solitons in DSG and possible semifluxon formation

The connection between $U^\pm$ given by BT (3.3) suggests that the corresponding Jost solutions, which in turn yields the soliton solutions through inverse scattering method, are linked by the BT matrix (3.4), i.e. by a multiplicative polynomial of first order in the spectral parameter λ. This concludes intriguingly that the role of the defect is to create or annihilate a soliton! That is $u^- = const.$ ('vacuum') solution would create due to (3.2) a 1-kink solution for u^+ or similarly u^- as 1-kink propagating through the defect point may be annihilated to a vacuum or transformed into 2-kink, given by u^+. Even richer solutions can occur in the DSG generalizing this situation , i.e. u^- propagating as N-kink after passing through the defect point at $x = 0$ may turn into a $N+1$-kink (through creation of soliton) or into a $N-1$-kink (through annihilation of soliton), described by the solution u^+. We would like to clarify that though the $u^\pm$ fields can be prolonged formally to each others domain, the physically observable fields are u^- in the domain $x < 0$ and u^+ in the semiaxis $x > 0$, while at the defect point both of them are present linked by the BT (3.2). Two such possible scenario are shown in Fig. 2 a-b). A pertinent question arises here regarding the obvious violation of topological charge in this SG model with a defect. One should note however that the topological charge arises in the SG model as a degree of mapping from $S^1 \to S^1$, while the coordinate-axis with a defect or a discontinuity point (like a puncture in the sphere) can not be mapped into a smooth sphere or S^1, violating thus the concept of the topological charge itself. Therefore in DSG the soitons seem to be no loger topological and hence their number may change. This is also the reason why we can describe in this framework semi-fluxons, as shown below, which are nontopoogical entries. The possibility of creation/annihilation of solitons in DSG has also been indicated in [25] through their analysis of soliton scattering processes. Our analysis of this problem depending on the boundary conditions of the fields $u^\pm$ at space infinities will be detailed elsewhere [14].

Experimental detection of semifluxons in early days [29] has reached now high level of sophistication, where FM and AFM arrays of semifluxons are observed in recent clean experiements [24]. However unlike fluxons exact analytic theory of semi-fluxons seems to be not available yet. A standard theoretical approach is to take the critical current as $I_s = I_c \sin u$, in the -ve half axis in a SG model, while $I_s = I_c \sin(u + \pi)$ in the +ve half with a π-jump of its field, suffered after the discontinuty or defect point at $x = 0$, Another approach is to consider instead a damped driven SG equation with additional terms like : $u^\pm_{tt} - u^\pm_{xx} + m_0^2 \sin u^\pm = \gamma + \alpha\theta_{xx} + \beta\dot{u}$, which are activated at the discontinuty point $x = 0$ [2]. However these appoaches are numerical and approximate, since both the related defect SG models as described above are nonintegrable with no analytic solution.

Using the crucial BT between $u^\pm$ fields and the intriguing *creation/annihilation* of solitons by the defect point in our integrable DSG model together with a control of the soliton phase shift, we can formulate a possible exact theory of the semifluxon dynamics allowing analytic solutions. The creation of semifluxon arrays can be explained similarly by extending our construction to

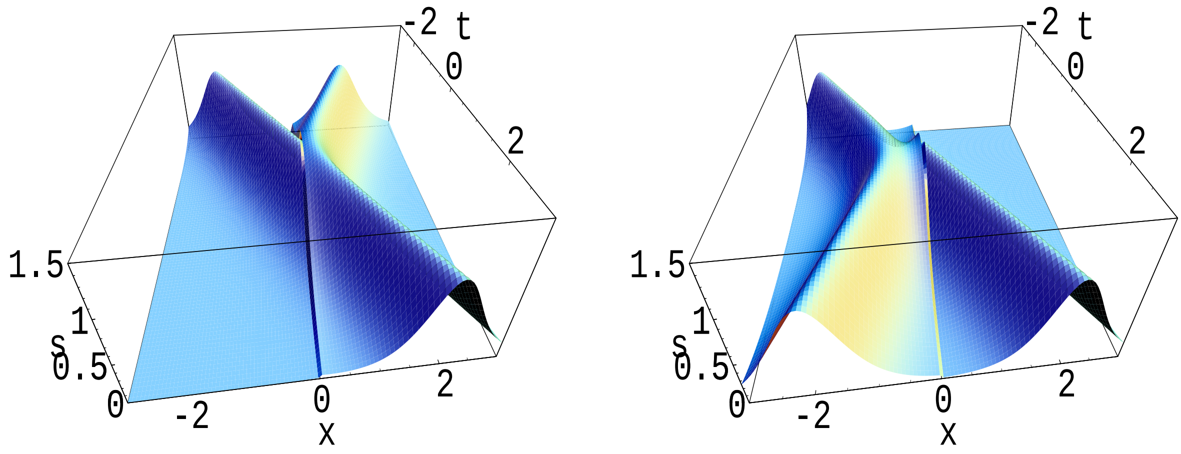

Figure 2: Soliton a) creation (from 1 to 2-kink) and b) annihilation (2 to 1-kink) at the defect point $x = 0$ in the integrable defect SG

integrable muti-defect SG. Experimental semifluxons may be described by the SG model with a defect at $x = 0$, in the interval $-L \leq x \leq +L$, where L may be considered to be large for a long Josepson junction. We take $L \rightarrow \infty$ to get our integrable DSG model, for creating an exact solution with a $0 - \pi$ jump we consider a constant soution $u^+ = \pi$, along $x \geq 0$. Note that simply $u^- = 0$ is not a consistent solution, which should be constructed now from Backlund image (3.2) yielding the equation $u_x^- = k_0(a) \cos \frac{u^-}{2}$ $, k_0(a) = (a + a^{-1})$. Integrating this equation we get the exact kink-solution $u^- = 4tan^{-1}(\frac{e^{\psi}-1}{e^{\psi}+1})$, where $\psi = \frac{k_0(a)}{2}(x + x_0)$. It is crucial to note that x_0 comes as an arbitrary integration constant, for any value of which u^- is an exact solution of above BT equation. For semi-fluxon solution we shall consider the influence of x_0 as the effective intervention through the discontinuity point and choose $x_0 = L$. This yields $\psi(x = -L) = 0$ and consequently $u^-(x = -L) = 4tan^{-1}(0) = 0$. Since for our model we should take $L \rightarrow \infty$, we get $u^-(-\infty) = 0, u^-(0) = 4tan^{-1}(1) = \pi$, and as constant solution also $u^+(+\infty) = \pi$. This gives clearly the topological charge of the solution on the whole axis as $Q = \frac{1}{2\pi}(u^+(\infty) - u^-(\infty)) = \frac{1}{2}$, describing a $0 - \pi$ transition of the field and hence a semifluxon!

Note that here $u_x^-(-L) = k_0(a) = const., \ L \rightarrow \infty$ and $\neq 0$ as in usual SG model, which however is not required here since we use only BT for generating our exact soliton solution. Hamiltonian density may be shifted by a const. to make the energy finite.

4 Concluding Remarks

We have investigated inhomogeneous SG models with variable mass and variable soliton velocity (VMSG) and with a defect at a point (DSG). We have shown that integrability can be preserved for both these models at the classical and at the exact quantum level with the construction of exact soliton solutions exhibiting unusual changing properties. In the VMSG model the exact solitons can change velocity, shape, width and amplitude, simulating under certain limiting conditions soliton dynamics in some real systems of physica importance. In the DSG model on the other hand solitons can change their number during their propagation, by creating or annihilating solitons while moving across the defect point. Physical evidence of this intriguing fact is yet to be confirmed in experiments. Based on our result on VMSG and DSG models we propose two different mechanisms for semikink formation, which may be considered as possible exact solutions of semi-fluxons, observed in recent experiments.

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