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Existence of Periodic Solutions of a Type of Nonlinear Impulsive Delay Differential Equations with a Small Parameter[1]

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Abstract

The Banach fixed point theorem is used to prove the existence of a unique ω periodic solution of a new type of nonlinear impulsive delay differential equation with a small parameter.

1 Introduction

It is now recognized that real world phenomena which are subject to short-time perturbations whose duration is negligible in comparison with the duration of the process are more accurately described using impulsive differential equations; see for instance [1, 3, 4, 8, 16]. Also in certain circumstances, the future state of a physical system might depend not only on the present state but also on its past history. Thus incorporating delay in the considered equations ensures that the model provides a better description for the process involved. We refer the reader to the papers [2, 5, 6, 7, 18] and the references cited therein.

In the last two decades there has been much research activity concerning the qualitative behavior of impulsive delay differential equations, see for example the papers [9, 10, 11, 13, 15, 20, 21] where stability, oscillation, controllability and periodicity of solutions of these equations have been investigated. Due to its vast importance in applications, the existence of periodic solutions, however, has attracted the interest of many authors who studied this issue by using various methods and by applying different techniques [12, 14, 20].

The aim of this paper is to investigate the existence of periodic solutions of a new type of nonlinear impulsive delay differential equation with a small parameter, of the form

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t) + \varepsilon g(t, x(t), x(t-\tau), \varepsilon), \quad t \neq \theta_i, \\ &\Delta x(\theta_i) = C_i x(\theta_i) + D_i x(\theta_{i-j}) + f_i + \varepsilon g_i(x(\theta_i), x(\theta_{i-j}), \varepsilon), \quad i \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

By employing the Banach fixed point theorem, we shall prove that equation (1.1) has a unique ω periodic solution. Our approach is based on the technique used in [16, p.37] where the existence

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of ω periodic solutions for impulsive differential equations without delay has been investigated. The equation under consideration in this paper allow delay terms not only at the continuous state but also at the fixed jumps. The main feature which distinguishes our equation from the ones in [21, 22] is the fact that the solution of equation (1.1) at the jump points will also depend on t previous data.

2 Preliminaries

Here we introduce some notations and provide some auxiliary results that will be needed in next section. Let $m,n \in \mathbb{N}$, $E \subseteq \mathbb{R}$ be an interval and $\{\theta_i\}_{i\in\mathbb{Z}}$ be a fixed sequence in E such that $\theta_{i+1} > \theta_i$ with $\lim_{i\to\infty} \theta_i = \infty$. Denote by $PLC(E,\mathbb{R}^{n\times m})$ the set of all functions $\varphi : E \to \mathbb{R}^{n\times m}$ which are piecewise left continuous for $t \in E$ having discontinuous of the first kind at $\theta_i \in E$. Let $h > 0$ and define the set $\Omega_h = \{x \in \mathbb{R}^n : \|x\| < h\}$ where $\|\cdot\|$ is any norm in $\mathbb{R}^n$.

We consider equation (1.1) with the following conditions:

(i) τ is a positive real number, j is a fixed positive integer number and $\varepsilon \in J = [-\varepsilon^*, +\varepsilon^*]$ is a small parameter;

(ii) $A,B \in PLC(\mathbb{R},\mathbb{R}^{n\times n})$, $f \in PLC(\mathbb{R},\mathbb{R}^n)$ and $g \in PLC(\mathbb{R}\times\Omega_h\times\Omega_h\times J,\mathbb{R}^n)$ are ω periodic functions in t, $\omega > \tau$;

(iii) $C_i, D_i \in \mathbb{R}^{n\times n}$, $f_i \in \mathbb{R}^n$ and $g_i \in C(\Omega_h\times\Omega_h\times J,\mathbb{R}^n)$ are p periodic sequences in i and $\{\theta_i\}_{i\in\mathbb{Z}}$ satisfies $\theta_{i+p} = \theta_i + \omega$, $p \in \mathbb{N}$;

(iv) There exist $k_1 > 0$ and $k_2 > 0$ such that

$$\begin{aligned} \|g(t,x,y,\varepsilon) - g(t,\bar{x},\bar{y},\varepsilon)\| &\le k_1(\|x-\bar{x}\| + \|y-\bar{y}\|), \\ \|g_i(x,y,\varepsilon) - g_i(\bar{x},\bar{y},\varepsilon)\| &\le k_2(\|x-\bar{x}\| + \|y-\bar{y}\|), \end{aligned}$$

for $t \in \mathbb{R}$, $i \in \mathbb{Z}$ and $x,\bar{x},y,\bar{y} \in \Omega_h$.

By a solution of (1.1) on an interval E, we mean a function $x \in PLC(E,\mathbb{R}^n)$ that satisfies (1.1). One can easily show that for any given $\sigma \ge 0$ and any given function $\phi(t) \in PLC([-\tau,0],\mathbb{R}^n)$, there is a unique solution $x(t)$ of (1.1) which satisfies

$$x(t) = \phi(t), \quad t \in [\sigma-\tau,\sigma]. \tag{2.1}$$

Equation (1.1) has been first considered in [17, 18] and in more general form in [19]. In these papers, the uniform asymptotic stability of the trivial solution and the existence of periodic solutions have been studied.

Consider the inhomogenous equation

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \ne \theta_i, \\ &\Delta x(\theta_i) = C_i x(\theta_i) + D_i x(\theta_{i-j}) + f_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.2}$$

and the corresponding homogenous equation

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau), \quad t \ne \theta_i, \\ &\Delta x(\theta_i) = C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

Definition 1. A matrix solution $X(t,\alpha)$ of (2.3) satisfying $X(\alpha,\alpha) = I$ and $X(t,\alpha) = 0$ for $t < \alpha$ is called a fundamental matrix of (2.3).

It was shown in [17, Lemma 2.2] that for $\sigma \geq 0$ the solutions of (2.2) has the form

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)-j\leq i<n(\sigma)} X(t,\theta_{i+j}^{+})D_{i+j}x(\theta_i) \\ &+ \int_{\sigma}^{t} X(t,\alpha)f(\alpha)\mathrm{d}\alpha + \sum_{n(\sigma)\leq i<n(t)} X(t,\theta_i^{+})f_i, \end{aligned} \tag{2.4}$$

where

$$n(t) = \min\{i \in \mathbb{Z} : \; \theta_i \geq t\}.$$

Define an operator $U : PLC([-\tau,0],\mathbb{R}^n) \to PLC([-\tau,0],\mathbb{R}^n) = \mathscr{B}$ through the relation $U\phi(t) = x(t+\omega;\phi)$ where $x(t;\phi)$ is a solution of (2.3) defined for $t \geq 0$ by the function ϕ given in $[-\tau,0]$. In view of (2.4), we have

$$\begin{aligned} U\phi(t) &= X(t+\omega,0)\phi(0) + \int_{-\tau}^{0} X(t+\omega,\alpha+\tau)B(\alpha+\tau)\phi(\alpha)d\alpha \\ &+ \sum_{-j\leq i<0} X(t+\omega,\theta_{i+j}^{+})D_{i+j}\phi(\theta_i). \end{aligned}$$

The operator U is compact: it maps every bounded set into a relative compact set. Indeed, from $\|\phi\| \leq M$ we obtain $\|U\phi\| \leq M_1$ and from $\omega > \tau$ we obtain $t+\omega > 0$ for $t \in [-\tau,0]$, therefore

$$\frac{d}{dt}\, x(t+\omega;\phi) = A(t)x(t+\omega) + B(t)x(t+\omega-\tau), \quad t \in (\theta_k, \theta_{k+1})$$

hence $\|\frac{d}{dt}\, x(t+\omega;\phi)\| \leq M_2$. Thus, the set of functions $\{U\phi\}$ forms a set of uniformly bounded and quasi-equicontinuous functions, consequently, on the basis of Arzela-Ascoli Lemma, it is a relative compact set.

The following lemma generalizes a fundamental result to equations of form (2.2).

Lemma 1. *Let conditions (i)–(iii) (excluding conditions on ε, g and g_i) be satisfied. Then, equation (2.2) has a unique ω periodic solution if and only if equation (2.3) does not have ω periodic solutions different from the zero solution.*

Proof. Let $x_0(t;\phi)$ be the solution of (2.2) defined for $t \geq 0$ by the function ϕ given in $[-\tau,0]$. From conditions (i)–(iii) (excluding conditions on ε, g and g_i), it follows that $x_0(t+\omega;\phi)$ is likewise a solution of (2.2) defined for $t+\omega \geq \tau$. Then $x_0(t+\omega;\phi) = x_0(t;\phi)$ for all $t \geq -\tau$ and hence the solution is periodic. Thus, the periodicity condition of the solution is written as $x_0(t+\omega;\phi) = \phi(t)$ for $t \in [-\tau,0]$. Let V be the operator defined by $V\phi = x_0(t+\omega;\phi)$; the function ϕ is an initial function for a periodic solution of the equation if and only if $V\phi = \phi$, in other words, the periodic solutions of the equation correspond to the fixed points of the operator V. Let $z(t;\phi)$ be the solution of (2.3), defined for $t \geq 0$ by the initial function ϕ given in $[-\tau,0]$. Then

$$x_0(t;\phi) = z(t;\phi) + \int_0^t X(t,\alpha)f(\alpha)d\alpha + \sum_{0\leq i<n(t)} X(t,\theta_i)f_i.$$

If U is the operator defined by the relation $U\phi = z(t+\omega;\phi)$, we have

$$V\phi = U\phi + \int_0^{t+\omega} X(t+\omega,\alpha)f(\alpha)d\alpha + \sum_{0\leq i<n(t+\omega)} X(t+\omega,\theta_i)f_i, \tag{2.5}$$

where $n(\omega) = p$. The periodicity condition implies that

$$(I-U)\phi = \int_0^{t+\omega} X(t+\omega,\alpha) f(\alpha) d\alpha + \sum_{0\le i<n(t+\omega)} X(t+\omega,\theta_i) f_i,$$

or,

$$x_0(t) = \phi(t) = (I-U)^{-1}\Big\{ \int_0^{t+\omega} X(t+\omega,\alpha) f(\alpha) d\alpha + \sum_{0\le i<n(t+\omega)} X(t+\omega,\theta_i) f_i \Big\}. \tag{2.6}$$

The operator U is compact, hence, $I-U$ has inverse if and only if the equation $(I-U)\phi = 0$ has no nontrivial periodic solutions of periodic ω. The proof is complete. ∎

The following result provides the representation of solutions of equation (1.1). It is an immediate consequence of Lemma 2.2 in [17] and hence we omit the proof.

Lemma 2. *Let $X(t,\alpha)$ be a fundamental matrix of (2.3) and $\sigma \ge 0$ be a real number. If $x(t)$ is a solution of (1.1), then*

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)-j\le i<n(\sigma)} X(t,\theta_{i+j}^{+})D_{i+j}x(\theta_i) \\ &+ \int_{\sigma}^{t} X(t,\alpha)\big[f(\alpha)+\varepsilon g(\alpha,x(\alpha),x(\alpha-\tau),\varepsilon)\big]\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)\le i<n(t)} X(t,\theta_i^{+})\big[f_i+\varepsilon g_i(x(\theta_i),x(\theta_{i-j}),\varepsilon)\big]. \end{aligned} \tag{2.7}$$

3 The Main Results

Choose $\rho > 0$ such that $\rho + m < h$. For each function $x \in PLC(\mathbb{R},\mathbb{R}^n)$, we define the norm

$$\|x\|_\omega = \sup_{t\in[0,\omega]} |x(t)|.$$

Set

$$M = \sup_{t,\alpha\in[0,\omega]} |X(t,\alpha)|,\ b_1 = \sup_{\alpha\in[0,\omega]} |B(\alpha)|,\ b_2 = \sup_{i=1,\dots,p} |B_i|$$

and

$$\begin{aligned} \mu &= \sup_{t\in[0,\omega],|\varepsilon|\le\varepsilon^*} \Big| \int_\sigma^t X(t,\alpha)g(\alpha,x_0(\alpha),x_0(\alpha-\tau),\varepsilon)\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)\le\theta_i<n(t)} X(t,\theta_i^{+})g_i(x_0(\theta_i),x_0(\theta_{i-j}),\varepsilon)\Big|. \end{aligned}$$

Fix $\varepsilon \in [-\varepsilon_0,+\varepsilon_0]$ where $\varepsilon_0 \in (0,\varepsilon^*)$ is such that

$$\gamma = 2\varepsilon_0 M(k_1\omega + k_2 p)[M\|(I-U)^{-1}\|_\tau(b_1\tau + b_2 j + 1) + 1] < 1, \tag{3.1}$$

and

$$\varepsilon_0\mu[M\|(I-U)^{-1}\|_\tau(b_1\omega+b_2p)+1]\le(1-\gamma)\rho, \tag{3.2}$$

where $\|x\|_\tau=\sup_{t\in[-\tau,0]}|x(t)|$. The set $\mathscr{B}_\omega=\{x\in PLC(\mathbb{R},\mathbb{R}^n):\ x(t)$ is ω periodic$\}$ with the norm $\|x\|_\omega$ is evidently a Banach space.

Theorem 1. *Assume that*

(a1) Conditions (i)–(iv) hold;

(a2) Equation (2.3) has no nontrivial solutions;

(a3) The unique ω periodic solution $x_0(t)$ of equation (2.2) is such that

$$m=\sup_{0\le t\le\omega}|x_0(t)|<h. \tag{3.3}$$

Then there exists $\varepsilon_0\in(0,\varepsilon^*)$ such that for each $\varepsilon\in[-\varepsilon_0,+\varepsilon_0]$ equation (1.1) has a unique ω periodic solution $x(t,\varepsilon)$. Moreover, $x(t,0)=x_0(t)$ and

$$\lim_{\varepsilon\to0}x(t,\varepsilon)=x_0(t)\ \ \textit{uniformly in } t\in\mathbb{R}. \tag{3.4}$$

Proof. Let $E=\{x\in\mathscr{B}_w:\|x-x_0\|\le\rho\}$. We define the operator $F:E\to\mathscr{B}_\omega$, $x\to Fx=u$ by the formula

$$\begin{aligned}
u(t) &= X(t,\sigma)u(\sigma)+\int_{\sigma-\tau}^{\sigma}X(t,\alpha+\tau)B(\alpha+\tau)u(\alpha)\mathrm{d}\alpha\\
&+\sum_{n(\sigma)-j\le i<n(\sigma)}X(t,\theta_{i+j}^+)B_{i+j}u(\theta_i)\\
&+\int_\sigma^t X(t,\alpha)\big[f(\alpha)+\varepsilon g(\alpha,x(\alpha),x(\alpha-\tau),\varepsilon))\big]\mathrm{d}\alpha\\
&+\sum_{n(\sigma)\le i<n(t)}X(t,\theta_i^+)\big[f_i+\varepsilon g_i(x(\theta_i),x(\theta_{i-j}),\varepsilon)\big].
\end{aligned} \tag{3.5}$$

We note that if $x\in E$ then $\|x\|\le\|x-x_0\|+\|x_0\|\le\rho+m<h$ and the operator F is well defined. Moreover, form (3.5) it follows that the function $Fx(t)$ is ω periodic and $Fx\in\mathscr{B}_\omega$.

Let $Fx_0=u_0$. Then from (2.4) and (3.5) we obtain

$$\begin{aligned}
u_0(t)-x_0(t) &= X(t,\sigma)\big[u_0(\sigma)-x_0(\sigma)\big]\\
&+\int_{\sigma-\tau}^{\sigma}X(t,\alpha+\tau)B(\alpha+\tau)\big[u_0(\alpha)-x_0(\alpha)\big]\mathrm{d}\alpha\\
&+\sum_{n(\sigma)-j\le i<n(\sigma)}X(t,\theta_{i+j}^+)B_{i+j}\big[u_0(\theta_i)-x_0(\theta_i)\big]\\
&+\varepsilon\Big[\int_\sigma^t X(t,\alpha)g(\alpha,x(\alpha),x(\alpha-\tau),\varepsilon)\mathrm{d}\alpha\\
&+\sum_{n(\sigma)\le i<n(t)}X(t,\theta_i^+)g_i(x(\theta_i),x(\theta_{i-j}),\varepsilon)\Big].
\end{aligned} \tag{3.6}$$

It follows that

$$\|u_0-x_0\|_\omega\le M\|u_0-x_0\|_\tau+Mb_1\tau\|u_0-x_0\|_\tau+Mb_2j\|u_0-x_0\|_\tau+|\varepsilon|\mu.$$

In view of (2.6) and (2.7), one can see that

$$\|u_0 - x_0\|_\tau \le |\varepsilon| S\mu,$$

where $S = \|(I-U)^{-1}\|_\tau$. This implies that

$$\|Fx_0 - x_0\|_\omega \le |\varepsilon|\mu[MS(b_1\tau + b_2 j + 1) + 1]. \tag{3.7}$$

Hence, from (3.1) and (3.2), we have

$$\|Fx_0 - x_0\|_\omega \le (1-\gamma)\rho < \rho. \tag{3.8}$$

Let $x, y \in E$ and $Fx = u,\ Fy = v$. Then from (3.5), we get

$$\begin{aligned} u(t) \ - \ v(t) &= X(t,\sigma)[u(\sigma) - v(\sigma)] \\ &+ \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)[u(\alpha)-v(\alpha)]\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)-j\le i<n(\sigma)} X(t,\theta_{i+j}^{+})B_{i+j}\big[u(\theta_i)-v(\theta_i)\big] \\ &+ \varepsilon\Big\{\int_{\sigma}^{t} X(t,\alpha)\big[g(\alpha,x(\alpha),x(\alpha-\tau,\varepsilon)-g(\alpha,y(\alpha),y(\alpha-\tau),\varepsilon)\big]\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)\le i<n(t)} X(t,\theta_i^{+})\big[g_i(x(\theta_i),x(\theta_{i-j}),\varepsilon)-g_i(y(\theta_i),y(\theta_{i-j}),\varepsilon)\big]\Big\}. \end{aligned} \tag{3.9}$$

Using (3.9) and condition (iv) we obtain the estimate

$$\|u-v\|_\omega \le M(b_1\omega + b_2 p + 1)\|u-v\|_\tau + |\varepsilon|M(2k_1\omega + 2k_2 p)\|x-y\|_\omega.$$

In view of (2.6) and (2.7), one can see that

$$\|u-v\|_\tau = |\varepsilon| SM(2k_1\omega + 2k_2 p)\|x-y\|_\omega.$$

This implies that

$$\|Fx - Fy\|_\omega \le 2|\varepsilon|M(k_1\omega + k_2 j)[MS(b_1\tau + b_2 j + 1)]\|x-y\|_\omega. \tag{3.10}$$

In virtue of (3.1), we have

$$\|Fx - Fy\|_\omega \le \gamma\|x-y\|_\omega. \tag{3.11}$$

If $x \in E$ then $Fx \in E$ since by (3.8) and (3.11) we obtain

$$\|Fx - x_0\|_\omega \le \|Fx - Fx_0\|_\omega + \|Fx_0 - x_0\|_\omega \le \gamma\|x-x_0\|_\omega + (1-\gamma)\rho \le \rho.$$

Hence $FE \subseteq E$ and F is a contraction. Then by the Banach fixed point theorem, the operator F has a unique fixed point $\widetilde{x} = x(t,\varepsilon) \in E$ such that $F\widetilde{x} = \widetilde{x}$, that is,

$$\begin{aligned} \widetilde{x}(t) &= X(t,\sigma)\widetilde{x}(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)\widetilde{x}(\alpha)\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)-j\le i<n(\sigma)} X(t,\theta_{i+j}^{+})B_{i+j}\widetilde{x}(\theta_i) \\ &+ \int_{\sigma}^{t} X(t,\alpha)\big[f(\alpha) + \varepsilon g(\alpha,\widetilde{x}(\alpha),\widetilde{x}(\alpha-\tau),\varepsilon)\big]\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)\le i<t} X(t,\theta_i^{+})\big[f_i + \varepsilon g_i(\widetilde{x}(\theta_i),\widetilde{x}(\theta_{i-j}),\varepsilon)\big]. \end{aligned} \tag{3.12}$$

It follows that $x(t,\varepsilon)$ is the unique ω periodic solution of equation (1.1). Obviously, $x(t,0)=x_0(t)$. Moreover, from (3.8) and (3.10) we obtain the estimate

$$\|\widetilde{x}-x_0\|_\omega \le \frac{|\varepsilon|\mu[MS(b_1\tau+b_2j+1)+1]}{1-2|\varepsilon|M(k_1\omega+k_2q)[MS(b_1\tau+b_2j+1)+1]}$$

which proves (3.4). ■

Define successively the sequence of functions $x_n(t)=x_n(t,\varepsilon)$, $n=0,1,2,\ldots$, where $x_0(t,\varepsilon)=x_0(t)$ is the unique ω periodic solution of equation (2.2) and x_{n+1} is the unique ω periodic solution of equation

$$\begin{aligned} &x'_{n+1}(t)=A(t)x_{n+1}(t)+B(t)x_{n+1}(t-\tau)+f(t)+\varepsilon g(t,x_n(t),x_n(t-\tau),\varepsilon),\ t\neq\theta_i,\\ &\Delta x_{n+1}(\theta_i)=C_ix_{n+1}(\theta_i)+D_ix_{n+1}(\theta_{i-j})+f_i+\varepsilon g_i(x_n(\theta_i),x_n(\theta_{i-j}),\varepsilon),\ i\in\mathbb{Z}. \end{aligned} \tag{3.13}$$

In view of Lemma 1 and Lemma 2, equation (3.13) has a unique ω periodic solution of the form

$$\begin{aligned} x_{n+1}(t) &= X(t,\sigma)x_{n+1}(\sigma)+\int_{\sigma-\tau}^{\sigma}X(t,\alpha+\tau)B(\alpha+\tau)x_{n+1}(\alpha)\mathrm{d}\alpha\\ &+ \sum_{n(\sigma)-j\le k<n(\sigma)}X(t,\theta_{k+j}^+)D_{k+j}x_{n+1}(\theta_k)\\ &+ \int_\sigma^t X(t,\alpha)\big[f(\alpha)+\varepsilon g(\alpha,x_n(\alpha),x_n(\alpha-\tau),\varepsilon)\big]\mathrm{d}\alpha\\ &+ \sum_{n(\sigma)\le k<n(t)}X(t,\theta_k^+)\big[g_k+\varepsilon g_k(x_n(\theta_k),x_n(\theta_{k-j}),\varepsilon)\big]. \end{aligned} \tag{3.14}$$

Corollary 1. Let the assumptions of Theorem 1 be fulfilled. Then

$$\lim_{n\to\infty}x_n(t,\varepsilon)=x(t,\varepsilon)\ \text{ uniformly in } t\in\mathbb{R},\ \varepsilon\in[-\varepsilon_0,+\varepsilon_0], \tag{3.15}$$

where $x_n(t,\varepsilon)$ are the unique ω periodic solutions of equation (3.13).

Proof. It is easy to verify that the functions $x_n=x_n(t,\varepsilon)$ defined by (3.14) are such that $x_n=Fx_{n-1}$, $n=1,2,\ldots$. Then using (3.11) we obtain

$$\|x_n-\widetilde{x}\|_\omega\le\frac{\gamma^n}{1-\gamma}\|x_1-x_0\|_\omega,\ n=1,2,\ldots.$$

The last inequality proves (3.15). ■

Remark 1. In view of (3.10), it follows that

$$\|x_n-\widetilde{x}\|_\omega\le\frac{\beta^n}{1-\beta}\|x_1-x_0\|_\omega,\ \ n=1,2,\ldots, \tag{3.16}$$

where

$$\beta=2|\varepsilon|M(k_1\tau+k_2j)[MS(b_1\tau+b_2j+1)+1].$$

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