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Nonclassical Potential System Approach for a Nonlinear Diffusion Equation

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Abstract

In this paper we consider a class of equations that model diffusion. For some of these equations nonclassical potential symmetries are derived by using a modified system approach. These symmetries allow us to increase the number of exact known solutions. These solutions are unobtainable from classical potential symmetries derived by using the so called natural potential system nor from nonclassical symmetries of the given partial differential equation.

1 Introduction

There is no existing general theory for solving nonlinear partial differential equations and the methods of point transformations are a powerful tool. Some of the most useful point transformations are those which form a continuous group. Lie classical symmetries admitted by nonlinear partial differential equations (PDE's) are useful for finding invariant solutions. There have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole [4] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method is to require that the N order PDE

$$\Delta = \Delta\left(x,t,u,u^{(1)}(x,t),...,u^{(N)}(x,t)\right) = 0, \tag{1.1}$$

where $(x,t) \in \mathbb{R}^2$ are the independent variables, $u \in \mathbb{R}$ is the dependent variable and $u^{(l)}(x,t)$ denote the set of all partial derivatives of l order of u and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0, \tag{1.2}$$

which is associated with the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{1.3}$$

are both invariant under the transformation with infinitesimal generator (1.3). We remark that nonclassical symmetries are not symmetries of a given PDE system, they are only symmetries for a subset of all solutions, namely those which satisfy invariant surface conditions. Since the authors published [4], a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDE's in both one and several dimensions. The relationship of the reduction methods for evolution equations based on invariant surface conditions related to functional separation of variables with nonclassical and weak point symmetries is stressed in [26].

In [6, 7] Bluman introduced a method to find a new class of symmetries for a PDE. Suppose a given PDE of second order

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})=0, \tag{1.4}$$

where the subscripts denote the partial derivatives of u, can be written as a conservation law

$$\frac{D}{Dt}f(x,t,u,u_x,u_t)-\frac{D}{Dx}g(x,t,u,u_x,u_t)=0, \tag{1.5}$$

for some functions f and g of the indicated arguments. Here $\frac{D}{Dx}$ and $\frac{D}{Dt}$ are total derivative operators defined by

$$\frac{D}{Dx} = \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\ldots, \tag{1.6}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\ldots. \tag{1.7}$$

Through the conservation law (1.5) one can introduce an auxiliary potential variable v and form an auxiliary potential system

$$\begin{aligned} v_x &= f(x,t,u,u_x,u_t), \\ v_t &= g(x,t,u,u_x,u_t). \end{aligned} \tag{1.8}$$

We remark that for a given system, one can finds sets of nonlocally related systems, which include potential systems obtained from various conservation laws and subsystems obtained by excluding dependent variables [5]. For example, for many physical equations one can eliminate u from the potential system (1.8) and form an auxiliary integrated or potential equation

$$G(x,t,v,v_x,v_t,v_{xx},v_{xt},v_{tt})=0, \tag{1.9}$$

for some function G. Any Lie group of point transformations

$$\mathbf{v}=\xi(x,t,u,v)\frac{\partial}{\partial x}+\tau(x,t,u,v)\frac{\partial}{\partial t}+\phi(x,t,u,v)\frac{\partial}{\partial u}+\psi(x,t,u,v)\frac{\partial}{\partial v}, \tag{1.10}$$

admitted by (1.8) yields a nonlocal symmetry *potential symmetry* of the given PDE (1.5) if the following condition is satisfied

$$\xi_v^2+\tau_v^2+\phi_v^2\neq 0. \tag{1.11}$$

We point out that if we consider a Lie group of point transformations

$$\mathbf{w}=\xi(x,t,v)\frac{\partial}{\partial x}+\tau(x,t,v)\frac{\partial}{\partial t}+\psi(x,t,v)\frac{\partial}{\partial v}, \tag{1.12}$$

admitted by (1.9) the condition

$$\xi_v^2+\tau_v^2\neq 0 \tag{1.13}$$

is a *sufficient* but not *necessary* condition in order to yield nonlocal symmetries of (1.5).

Knowing that an associated system to the Boussinesq equation has the same classical symmetries as the Boussinesq equation, Clarkson [11] proposed as an open problem if an auxiliary system of the Boussinesq equation does posses more or less nonclassical symmetries than the equation itself. Bluman claims [3] that the ansatz to generate nonclassical solutions of the associated system could yield solutions of the original equation which are neither nonclassical solutions nor solutions arising from potential symmetries. In [10] two algorithms were proposed which extend the nonclassical method to a potential system (1.8) or a potential equation (1.9):

- Algorithm I Nonclassical potential system approach: The nonclassical method is applied to the associated potential system (1.8). Any Lie group of point transformations

$$\mathbf{v} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \psi(x,t,u,v)\frac{\partial}{\partial v}, \tag{1.14}$$

  admitted by (1.8) yields a nonlocal symmetry *potential symmetry* of the given PDE (1.5) if the following condition is satisfied

$$\xi_v^2 + \tau_v^2 + \phi_v^2 \neq 0. \tag{1.15}$$

- Algorithm II Nonclassical potential equation approach: The nonclassical method is applied to the associated potential equation (1.9). Any Lie group of point transformations

$$X = \xi(x,t,v)\frac{\partial}{\partial x} + \tau(x,t,v)\frac{\partial}{\partial t} + \psi(x,t,v)\frac{\partial}{\partial v} \tag{1.16}$$

  admitted by (1.9) yields a nonlocal symmetry *potential symmetry* of the given PDE (1.5) if the following condition is satisfied

$$\xi_v^2 + \tau_v^2 \neq 0. \tag{1.17}$$

Algorithm I has been considered in Bluman and Shtelen [9] and Saccomandi [28], Algorithm II has been considered in [16] for a dissipative KdV equation, but neither of these papers exhibited nonclassical potential solutions.

The nonclassical symmetries for the Burgers have been considered in [1], [24]. The *nonclassical potential* symmetries for the Burgers equation have been derived in [14] as nonclassical symmetries of the integrated equation (Algorithm II).

In [23] Priestly and Clarkson found that the solutions arising from the nonclassical symmetries of the associated potential system of the shallow water equation were obtainable by the nonclassical symmetries of the shallow water equation. Consequently, it remained as an open problem the existence of nonclassical potential symmetries, in the sense that they lead to new solutions.

The existence of nonclassical potential symmetries, in the sense that they lead to new solutions was proved in [18] and [19] for the Fokker-Planck equation

$$u_t = u_{xx} + [f(x)u]_x. \tag{1.18}$$

The classical symmetries for (1.18) were derived in [7]. The classical potential symmetries were derived by Pucci and Saccomandi in [25]. We have studied in [18], [19] the nonclassical symmetries of the Fokker-Planck equation, as well as the *nonclassical potential symmetries*. We found a class of functions $f(x)$ for which equation (1.18) does not admit, classical Lie symmetries, nonclassical symmetries nor classical potential symmetries but it admits nonclassical potential symmetries.

The diffusion processes appear in many physics processes such as plasma physics, kinetic theory of gases, solid state, metallurgy and transport in porous medium [2, 22, 27]. One of the mathematical models for diffusion processes is the nonlinear diffusion equation

$$u_t = [f(u)u_x]_x. \tag{1.19}$$

In (1.19) $u(x,t)$ is a function of position x and time t and may represent the temperature. Rosenau [27] presented a number of remarkable features of the fast diffusion processes; for $f(u) = u^n$, and $-2 \leq n \leq -1$, the family of fast diffusion (1.19) coexists with a subclass of superfast diffusions where the whole process terminates within a finite time. The special case with $n = -1$ emerges in plasma physics and reveals a surprising richness of a new physical-mathematical phenomenon. Equation (1.19) is already in a conserved form. Correspondingly, we have the so called *natural* potential system

$$\begin{aligned} v_x &= u, \\ v_t &= f(u)u_x \end{aligned} \tag{1.20}$$

and the potential equation

$$v_t = f(v_x)v_{xx}. \tag{1.21}$$

In [17], we have derived *nonclassical* and *nonclassical potential* symmetries for (1.19) with $f(u) = u^{-1}$ by considering the nonclassical symmetries of system (1.20). In [10] the nonclassical method has been applied to equation (1.21).

In [10] it was pointed out that often the nonclassical method when it is applied to the potential system (Algoritm I) yields a set of undetermined determining equations while the nonclassical method it is much easier to apply to the potential equation (Algorithm II).

However we point out that a great disadvantage of Algoritm II is that condition (1.17) is a *sufficient* but *not necessary* condition in order to see if a generator is a nonclassical potential generator or not. In fact we exhibit several nonclassical potential generators which do not satisfy (1.17).

The aim of this paper is to propose a modification to the nonclassical potential system approach, in a way in which is easy to apply and we can give a *sufficient and necessary* condition in order to see if a generator is a nonclassical potential generator. Note that if the generator considered is not a nonclassical potential generator then no new solutions are found, i.e. all such solutions can be obtained from the nonclassical method applied to the given PDE (1.5).

- Modified Algorithm I Modified nonclassical potential system approach: The nonclassical method is applied to the associated potential system (1.8). Any Lie group of point transformations

$$\mathbf{v} = \xi(x,t,v)\frac{\partial}{\partial x} + \tau(x,t,v)\frac{\partial}{\partial t} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \psi(x,t,v)\frac{\partial}{\partial v}, \tag{1.22}$$

admitted by (1.8) yields a nonlocal symmetry *potential symmetry* of the given PDE (1.5) if condition (1.15) is satisfied. In [15] the nonclassical potential symmetries for the Burgers equation were derived as nonclassical symmetries, with $\xi_u = \psi_u = 0$, of the potential associated system (Modified Algorithm I).

In this paper we apply the modified potential system approach to (1.19) and, for $f(u) = 1/(au^2+bu)$, we obtain new nonclassical potential generators as well as new solutions.

2 Nonclassical symmetries for system (1.20)

We now consider the associated auxiliary system (1.20) augmented with the invariance surface condition

$$\xi v_x + \tau v_t - \psi = 0, \tag{2.1}$$

which is associated with the vector field

$$\mathbf{w} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v. \tag{2.2}$$

The nonclassical method, with $\tau \neq 0$, applied to (1.20), gives rise to nonlinear determining equations for the infinitesimals. These determining equations first appeared in Saccomandi [28] and are undetermined since they involve two equations in three unknowns $\xi(x,t,u,v)$, $\phi(x,t,u,v)$ and $\psi(x,t,u,v)$. Consequently, any $f(u)$ yields in principle an infinite number of nonclassical symmetries. In [10] the authors say that they have been unsuccessful in finding a specific solution yielding a nonclassical symmetry that is not derivable from a point symmetry admitted by the potential system (1.20). The point symmetries admitted by the potential system (1.20) are given in Bluman and Kumei [7] and Bluman *et al.* [8]. If we require that $\xi_u = \psi_u = 0$, (Modified Algorithm I) we obtain that

$$\phi = -\xi_v u^2 + (\psi_v - \xi_x)u + \psi_x \tag{2.3}$$

where $f(u)$, $\xi(x,t,v)$ and $\psi(x,t,v)$, must satisfy the following equation:

$$\begin{aligned} &[-\xi\xi_v u^3 + (\xi_v\psi - \xi\xi_x + \xi\psi_v)u^2 + (\xi_x\psi - \psi\psi_v + \xi\psi_x)u - \psi\psi_x] f'(u) \\ &\quad + [-2\xi\xi_v u^2 + (2\xi_v\psi - 2\xi\xi_x - \xi_t)u + 2\xi_x\psi + \psi_t] f(u) \\ &\quad + [\xi_{vv}u^3 + (2\xi_{xv} - \psi_{vv})u^2 + (\xi_{xx} - 2\psi_{xv})u - \psi_{xx}] f^2(u) = 0. \end{aligned} \tag{2.4}$$

Equation (2.4) coincides with the equation derived by applying the nonclassical method to the potential equation [10] and since equation (2.4) must hold for all values x, t, v and u it follows that $f(u)$ must satisfy a first order Bernouilli equation with variable coefficients. Moreover, from (2.3) we get that (1.15) is satisfied if and only if

$$\xi_v^2 + \psi_{vv}^2 + \psi_{xv}^2 \neq 0. \tag{2.5}$$

When $f(u) = 1/(au^2+bu)$ we get that ξ and ψ are related by the following conditions

$$-2a\xi\psi_v - a\xi_t + \xi_{vv} - b\xi\xi_v = 0, \tag{2.6}$$

$$-2a\xi\psi_x - \psi_{vv} + 2a\psi\psi_v - b\xi\psi_v + a\psi_t + b\xi_v\psi - b\xi_t - b\xi\xi_x + 2\xi_{vx} = 0, \tag{2.7}$$

$$2a\psi\psi_x - b\xi\psi_x - 2\psi_{vx} + b\psi\psi_v + b\psi_t + \xi_x\psi + \xi_{xx} = 0 \tag{2.8}$$

$$-\psi_{xx} + b\psi\psi_x = 0. \tag{2.9}$$

Despite the fact that the former equations are too complicated to be solved in general, special solutions can be obtained.

2.1 If $b=1$ and $a=1$

2.1.1 For

$$\xi=-2\alpha(c-2d)\tanh\left[\alpha(cv+dx)+\beta\right],\quad \tau=1\quad \psi=-2\alpha d\tanh[\alpha(cv+dx)+\beta].$$

Setting $d=1$ the similarity variable $z=x-(c-2)v$ and the family of invariant solutions is defined implicitly by

$$\log\sinh(\alpha(z+2cv-2v)+\beta)+(t+h(z))4\alpha^2(c-1)=0,$$

where h satisfies the ODE

$$4(c-1)h''-16\alpha^2(c-1)^2(h')^2+1=0, \tag{2.10}$$

whose solution is

$$h(z)=\frac{1}{4\alpha^2(c-1)}\log(\mathrm{sech}(\alpha(z+k_1)))+k_2. \tag{2.11}$$

Therefore, the family of invariant solutions is defined, implicitly, by

$$\log\sinh(\alpha(x+cv)+\beta))+\log\mathrm{sech}(\alpha(x+(2-c)v+k_1)))+(t+k_2)4\alpha^2(c-1)=0.$$

For $c\neq 0$ after setting $\alpha=b\gamma$, $c=2-\frac{1}{2\gamma}$, the family of invariant solutions can also be written as

$$e^{b^2(4\gamma^2-2\gamma)(t+k_2)}\sinh(b(2\gamma-\frac{1}{2})v+b\gamma x+\beta)\mathrm{sech}(\frac{1}{2}bv+b\gamma x+bk_1\gamma)-1=0.$$

For $c=2$ we get the explicit solution

$$v=\frac{1}{2\alpha}[\mathrm{asinh}(e^{-4\alpha^2(t+k_2)}\cosh(\alpha(x+k_1))-\alpha x-\beta].$$

2.1.2 For

$$\xi=-2\alpha(c-2d)\mathrm{cotanh}\left[\alpha(cv+dx)+\beta\right],\quad \tau=1\quad \psi=-2\alpha d\,\mathrm{cotanh}[\alpha(cv+dx)+\beta].$$

Setting $d=1$ the similarity variable $z=x-(c-2)v$ and the family of invariant solutions is defined implicitly by

$$-\log\cosh(\alpha(z+2cv-2v)+\beta)-(t+h(z))4\alpha^2(c-1)=0,$$

where h satisfies the ODE (2.10) whose solution is (2.11). Therefore, the family of invariant solutions is defined, implicitly, by

$$-\log\cosh(\alpha(x+cv)+\beta))-\log\mathrm{sech}(\alpha(x+(2-c)v+k_1)))-(t+k_2)4\alpha^2(c-1)=0.$$

For $c\neq 0$ and setting $\alpha=b\gamma$, $c=\frac{1}{2\gamma}$, $k_1=\frac{k}{b\gamma}$ the family of invariant solutions can also be written as

$$e^{b^2(4\gamma^2-2\gamma)(t+k_2)}\cosh(b(2\gamma-\frac{1}{2})v+b\gamma x+k)\mathrm{sech}(\frac{1}{2}bv+b\gamma x+\beta)-1=0.$$

For $c = 2$ we get the explicit solution

$$v = \frac{1}{2\alpha}[\text{acosh}(e^{-4\alpha^2(t+k_2)}\cosh(\alpha(x+k_1)) - \alpha x - \beta].$$

2.1.3 For

$$\xi = 2\alpha(c-2d)\tan\,[\alpha(cv+dx)+\beta], \quad \tau = 1, \quad \psi = 2\alpha d\tan[\alpha(cv+dx)+\beta],$$

setting $d = 1$ the similarity variable $z = x-(c-2)v$ and the family of invariant solutions is defined implicitly by

$$\log\,\sin(\alpha(z+2cv-2v)+\beta)+(t+h(z))4\alpha^2(c-1) = 0,$$

where h satisfies the ODE

$$4(c-1)h'' + 16\alpha^2(c-1)^2(h')^2 + 1 = 0$$

whose solution is

$$h(z) = k_2 - \frac{1}{4\alpha^2(c-1)}\log(\sec(\alpha(z+k_1))).$$

Therefore, the family of invariant solutions is defined, implicitly, by

$$\log\,\sin(\alpha(x+cv)+\beta)) + \log\,\sec(\alpha(x+(2-c)v+k_1))) - (t+k_2)4\alpha^2(c-1) = 0.$$

2.1.4 For

$$\xi = -2\alpha(c-2d)\text{cotan}\,[\alpha(cv+dx)+\beta], \quad \tau = 1, \quad \psi = -2\alpha d\,\text{cotan}[\alpha(cv+dx)+\beta],$$

setting $d = 1$ the similarity variable $z = x-(c-2)v$ and the family of invariant solutions is defined implicitly by

$$\log(\cos(2\alpha(z+2cv-2v)+2\beta)+1) - (t+h(z))8\alpha^2(c-1) - \log(2) = 0.$$

where h satisfies the ODE (2.10).

2.1.5 For

$$\xi = [2\alpha\tan\,(\alpha(v+ct)+b)) - c], \quad \tau = 1, \quad \psi = c,$$

the similarity variable $z = v - ct$ and the family of invariant solutions is defined implicitly by

$$\log\,\cos^2(\alpha(z+2ct+b)+2c(x+ct-h(z)) = 0,$$

where f satisfies the ODE

$$ch'' + c^2(h')^2 + 2c^2h' + c^2 + \alpha^2 = 0.$$

Therefore, the family of invariant solutions is defined, implicitly, by

$$\log\,\cos(2\alpha(v+ct+b)+1) + 2\log\,\sec(\alpha(v-ct+k_1))) - \log(2) + c(2(x+v-k_2)) = 0.$$

2.1.6 For

$$\xi = -2\alpha\text{cotan}\,(\alpha(v+ct)+b)) - c, \quad \tau = 1, \quad \psi = c,$$

the similarity variable $z = v - ct$ and the family of invariant solutions is defined implicitly by

$$-\log\,\sin(\alpha(z+2ct+b) - c(x+ct+h(z)) = 0,$$

where f satisfies the ODE

$$ch'' - c^2(h')^2 + 2c^2h' - c^2 - \alpha^2 = 0,$$

whose solution is

$$h = \frac{1}{c}[\log\,\sec(a(z+k_1)) + cz] + k_2.$$

Therefore, the family of invariant solutions is defined, implicitly, by

$$-\log\,\sin(\alpha(v+ct+b)) - \log\,\sec(\alpha(v-ct+k_1))) - c(x+v+k_2) = 0.$$

2.1.7 For

$$\xi = -2\alpha\tanh\,(\alpha(v+ct)+b)) - c, \quad \tau = 1, \quad \psi = c,$$

the similarity variable $z = v - ct$ and the family of invariant solutions is defined implicitly by

$$-\log\,\cosh(\alpha(z+2ct+b) - c(x+ct+h(z)) = 0,$$

where f satisfies the ODE

$$ch'' - c^2(h')^2 + 2c^2h' - c^2 + \alpha^2 = 0, \tag{2.12}$$

whose solution is

$$h = \frac{1}{c}[\log\,\text{sech}(a(z+k_1)) + cz] + k_2. \tag{2.13}$$

Therefore, the family of invariant solutions is defined, implicitly, by

$$-\log\,\cosh(\alpha(v+ct+b)) - \log\,\text{sech}(\alpha(v-ct+k_1))) - c(x+v+k_2) = 0.$$

2.1.8 For

$$\xi = -2\alpha\text{cotanh}\,(\alpha(v+ct)+b)) - c, \quad \tau = 1, \quad \psi = c,$$

the similarity variable $z = v - ct$ and the family of invariant solutions is defined implicitly by

$$-\log\,\sinh(\alpha(z+2ct+b) - c(x+ct+h(z)) = 0,$$

where f satisfies (2.12) whose solution is (2.13). Therefore, the family of invariant solutions is defined, implicitly, by

$$-\log\,\sinh(\alpha(v+ct+b)) - \log\,\text{sech}(\alpha(v-ct+k_1))) - c(x+v+k_2) = 0.$$

2.1.9 For

$$\xi = \frac{2(2-c)}{x+cv}, \quad \tau = 1 \quad \psi = \frac{-2}{x+cv},$$

the similarity variable $z = x + (2-c)v$ and the family of invariant solutions is defined implicitly by

$$v^2 + xv + 2(t + f(z)) = 0$$

where h satisfies the ODE $h'' = 0$. Therefore, the family of invariant solutions is defined, by

$$v = \pm\frac{1}{2}[(x^2 - 4ck_1x - 8(t+k_2) + 4(c-2)^2k_1^2)^{1/2} - x + (2c-4)k_1].$$

2.1.10 For

$$\xi = \frac{2(2b-1)}{bx+v}, \quad \tau = 1, \quad \psi = \frac{-2b}{bx+v},$$

the similarity variable $z = x + (2-c)v$ and the family of invariant solutions is defined implicitly by

$$v^2 + xv + 2(t + h(z)) = 0,$$

where h satisfies $h'' = 0$. Therefore, the family of invariant solutions is defined, by

$$v = \pm\frac{1}{2}[(x^2 - 4k_1x - 8(t+k_2) + 4(2b-1)^2k_1^2)^{1/2} - x + 2(1-2b)k_1].$$

We point out that the corresponding generators **2.1.1**,...,**2.1.4** for $c = 2d$ do not satisfy (1.17) however they are nonclassical potential generators due to the fact that in all of them (2.5), and consequently (1.15) are satisfied.

2.2 $b = 0$ and $a = 1$

the equation becomes $v_t = \dfrac{v_{xx}}{v_x^2}$ and may be linearized and transformed into the linear heat equation. Hence a nonclassical point symmetry of the linear diffusion equation is a nonclassical nonlocal symmetry of (1.19).

2.3 $a = 0$ and $b = 1$

Equation (1.19) has been considered in [17] and [20]. In [17], although unfortunately there appear some misprints in the generators, we have derived the following nonclassical potential reductions and solutions:

2.3.1

$$\xi = 2\tan(v + kt + k_1), \quad \tau = 1, \quad \psi = k,$$

The similarity variable $x = v - kt$ and the family of invariant solutions is defined implicitly by

$$\frac{\log\sec(z - 2v - c)}{k} - x - h(z) = 0,$$

where h satisfies the ODE

$$kh'' - k^2(h')^2 - 1 = 0.$$

Therefore, setting $\zeta = v + kt$, the family of invariant solutions is defined, implicitly, by

$$-x + \frac{\log\sec(\zeta + k_1)}{k} - \frac{\log\sec(z + k_2)}{k} - k_3 = 0.$$

2.3.2 From generator

$$\xi = 2\tan(x + v), \quad \tau = 1, \quad \psi = 2\tan(x + v),$$

we obtain the independent variable $z = v - x$ and the implicit solution

$$-t - h(z) + \frac{\log(\sin(z-2v))}{4} = 0, \tag{2.14}$$

where h satisfies

$$4h'' + 16h'^2 + 1 = 0.$$

Consequently we get the implicit solution

$$\frac{\log(-\sin(v+x))}{4} + \frac{\log(\sec(v-x))}{4} - t - k_3 = 0.$$

In [21] it also appear in explicit form

$$\begin{aligned} v &= k_4 - \operatorname{atan}\left(\frac{e^{4k_3}\tan(x) + e^{4t}}{e^{4t}\tan(x) + e^{4k_3}}\right), \\ u &= \frac{e^{8t} - e^{8k_3}}{4e^{4t+4k_3}\cosh(x)\sinh(x) + e^{8t} - e^{8k_3}}. \end{aligned} \tag{2.15}$$

3 Nonclassical symmetries for PDE (1.19)

To obtain nonclassical symmetries of (1.19), we require that the PDE (1.19) and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0, \tag{3.1}$$

which is associated with the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{3.2}$$

are both invariant under the transformation with infinitesimal generator (3.2). We can distinguish two different cases:
Case $\tau \neq 0$,without loss of generality, we may set $\tau(x,t,u) = 1$. The nonclassical method applied to (1.19) recover the Lie classical symmetries.
Case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and we get that the determining equation for the infinitesimal ϕ is

$$f''(u)\phi^3 + f'(u)(3\phi\phi_x + 2\phi^2\phi_u) + f(u)(\phi_{xx} + 2\phi\phi_{xu} + \phi^2\phi_{uu}) - \phi_t = 0 = 0. \tag{3.3}$$

This determining equation first appeared in [10]. In principle, any $f(u)$ yields solutions of (3.3) Thus we proceed, by making an ansatz on the form of $\phi(x,t,u)$, to solve (3.3).
3.1 For $f(u) = \frac{1}{u^2+u}$. Choosing $\phi = (u^2+u)(\alpha(x,t)u + \beta(x,t))$, with $\alpha = -\xi$ and $\beta = \psi$ after substituting into the determining equation and splitting with respect to u we obtain that functions ξ and ψ must satisfy the following conditions

$$\begin{aligned} \xi_t &= 0, & (3.4)\\ -2\xi\psi_x + \psi_t - \xi_t - \xi\xi_x &= 0, & (3.5)\\ 2\psi\psi_x - \xi\psi_x + \psi_t + \xi_x\psi + \xi_{xx} &= 0, & (3.6)\\ -\psi_{xx} + \psi\psi_x &= 0. & (3.7) \end{aligned}$$

These conditions are precisely conditions (2.6-2.9) if we assume that ξ and ψ do not depend on v. Consequently we can state:

$$\mathbf{w} = \xi(x,t)\partial_x + \partial_t + \psi(x,t)\partial_v + (\psi_x - \xi_x u)\partial_u$$

is a generator for system (1.20) *if and only if*

$$\mathbf{v} = \partial_x + (-\xi(x,t)u + \psi(x,t))\left(u^2 + u\right)\partial_u$$

is a generator for equation (1.19). Consequently if we set $c = 0$ in generators 2.1.1,...,2.1.4 these generators yield solutions that can be derived from nonclassical symmetries with $\tau \equiv 0$ of the original PDE (1.19) .

4 Concluding remarks

We propose a modification to the nonclassical potential system approach, which unlike the non-classical potential equation approach gives a *sufficient and necessary* condition in order to see if a generator is a nonclassical potential generator or not. We prove that the nonlinear diffusion equation (1.19) when $f(u) = 1/(au^2 + bu)$, with a and b arbitrary constants admits nonclassical potential symmetries. These symmetries can be derived from the corresponding nonclassical symmetries of the associated potential system (1.20) by requiring $\xi_u = \psi_u = 0$. We show that if we assume that $\xi_v = \psi_v = 0$ these generators yield nonclassical potential solutions of (1.19) which can be derived from the nonclassical symmetries with $\tau \equiv 0$ of the original PDE (1.19).

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