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Peristaltic MHD Flow of Third Grade Fluid with an Endoscope and Variable Viscosity

T HAYAT [a], Ebrahim MOMONIAT [b] and Fazal M MAHOMED [b]

[a] *Department of Mathematics, Quaid-i-Azam University 45320, Islamabad, Pakistan*

E-mail: t_pensy@hotmail.com

[b] *Centre for Differential Equations, Continuum Mechanics and Applications,*
School of Computational and Applied Mathematics, University of the Witwatersrand,
Johannesburg, Private Bag 3, Wits 2050, South Africa

E-mail: Ebrahim.Momoniat@wits.ac.za, Fazal.Mahomed@wits.ac.za

Abstract

This investigation deals with the mechanism of peristaltic transport of a non-Newtonian, incompressible and electrically conducting fluid with variable viscosity and an endoscope effects. The magnetic Reynolds number is taken to be small. The mechanical properties of the material are represented by the constitutive equation for a third grade fluid. The fluid fills the gap between coaxial uniform tubes, such that the inner tube is rigid and outer tube with sinusoidal wave travelling down its wall. Numerical solutions are given for large wavelength at low Reynolds number.

1 Introduction

Peristaltic is defined as a wave of relaxation contraction imparted to the walls of a flexible conduit, thereby pumping the enclosed material. The need for peristaltic pumping may arise in circumstances where it is desirable to avoid using any internal moving parts such as pistons in a pumping process. Moreover, the peristalsis is also a well known mechanism of fluid transport in biological system. Specifically it has been found to be involved in swallowing food through the esophagus, urine transport from the kidney to the bladder through the urethra, movement of chyme, transport of lymph in the lymphatic vessels and in the vasomotion of small blood vessels such as arteries, venules and capillaries. Roller and finger pumps also operate on this principle. Moreover, peristalsis has been proposed as a mechanism for the transport of spermatozoa in vasdeferens (a duct which connects the ductus epididymidis to an ampulla).

Even though peristalsis existed very well in physiology, its relevance came about mainly through the works of Kill [1]. Later several mathematical and experimental models have been developed to understand the fluid mechanical aspects of peristaltic motion. A large body of work already exists on mathematical and experimental models containing a Newtonian or non-Newtonian fluid in a channel or axisymmetric tube [2-22].

Despite the above studies, less attention has been given to the peristaltic flow of fluid with variable viscosity and an endoscope effects. Recently, Misery et al. [23] analyzed the hydrodynamic peristaltic flow of a Newtonian fluid with variable viscosity and an endoscope. The endoscope effect on peristaltic motion occurs in many clinical applications. In fact there is no difference between an endoscope and catheter from the fluid dynamics point of view, but from the physiological point of view we cannot use a catheter for small intestine.

The paper is organized in the following way. We give the basic equations in Section 2. The problem description is given in Section 3. In Section 4, we present the expressions for flow rate, pressure rise and frictional forces on the tubes. A numerical solution is presented in Section 5. Finally, the conclusions are made in Section 6.

2 Basic equations

The equations governing the conservations of mass and linear momentum for an incompressible fluid are expresed as follows:

$$\operatorname{div} \boldsymbol{V} = 0, \tag{2.1}$$

$$\rho \frac{d\boldsymbol{V}}{d\bar{t}} = -\boldsymbol{\nabla}\bar{p} + \rho \boldsymbol{f} + \operatorname{div} \boldsymbol{S}. \tag{2.2}$$

In the above equations $\boldsymbol{V}$ is the velocity vector, $\bar{p}$ is the pressure, $\bar{t}$ is the time, ρ is the density, $d/d\bar{t}$ is the material time derivative, ∇ is the spatial gradient operator, $\boldsymbol{f}$ is the body force vector per unit mass and $\boldsymbol{S}$ is the extra stress tensor. For third grade fluid with variable viscosity the constitutive equation of $\boldsymbol{S}$ is

$$\boldsymbol{S} = \sum_{i=1}^{3} \boldsymbol{S}_i, \tag{2.3}$$

where

$$\boldsymbol{S}_1 = \bar{\mu}(\overline{R})\boldsymbol{A}_1, \tag{2.4}$$

$$\boldsymbol{S}_2 = \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \tag{2.5}$$

$$\boldsymbol{S}_3 = \beta_1 \boldsymbol{A}_3 + \beta_2 (\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \beta_3 \left(tr \boldsymbol{A}_1^2\right) \boldsymbol{A}_1, \tag{2.6}$$

in which $\bar{\mu}$ is the viscosity function and $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ and β_3 are the material constants. The Rivlin-Ericksen tensors, $\boldsymbol{A}_n$, are defined through the recursion formula

$$\boldsymbol{A}_1 = \nabla \boldsymbol{V} + \nabla \boldsymbol{V}^*, \tag{2.7}$$

$$\boldsymbol{A}_n = \frac{d\boldsymbol{A}_{n-1}}{d\bar{t}} + \boldsymbol{A}_{n-1}\nabla \boldsymbol{V} + \nabla \boldsymbol{V}^* \boldsymbol{A}_{n-1}, n = 2, 3, \ldots, \tag{2.8}$$

where $(*)$ is matrix transpose. A detailed thermodynamic analysis of the stress, represented by Eq. (2.3), is given in [2,3], where it is shown that if all the motions of the fluid are to be compatible with thermodynamics in the sense that these motions satisfy the

Clausius-Duhem inequality and if it is assumed that the specific Helmholtz free energy is a minimum when the fluid is locally at rest, then

$$\overline{\mu} \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0. \tag{2.9}$$

A complete detail on third grade fluid is given in Dunn and Rajagopal [25]. For thermodynamically compatible fluid, Eq. (2.3) reduces to

$$\boldsymbol{S} = \overline{\mu}\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(tr\boldsymbol{A}_1^2)\boldsymbol{A}_1 \tag{2.10}$$

and for zero normal stress parameters we have

$$\boldsymbol{S} = \left[\overline{\mu} + \beta_3(tr\boldsymbol{A}_1^2)\right]\boldsymbol{A}_1. \tag{2.11}$$

In our analysis, however, we consider the model (2.3) for generality.

It is worth emphasizing that the governing equations for third grade fluids in particular and viscoelastic fluids in general are of higher order than the Navier-Stokes equations. For non-Newtonian fluids, the no-slip boundary condition is insufficient and thus, one needs additional conditions at the boundary. A detailed critical review on the boundary conditions and of the existence and uniqueness of the solution has been given in the references [24-29].

3 Problem description

We consider the flow of an incompressible, electrically conducting third grade fluid with variable viscosity through the gap between inner and outer tubes. The inner tube is rigid while the outer tube has a sinusoidal wave travelling down its wall. We choose the cylindrical coordinate system $(\overline{R}, \overline{Z})$, where the $\overline{Z}$-axis lies along the centerline of the inner and outer tubes, and $\overline{R}$ is the distance measured radially. A uniform magnetic field $\overline{B}_0$ is applied in the transverse direction to the flow. The magnetic Reynolds number is considered small and so induced magnetic field is neglected. The geometries of the wall surfaces are defined by

$$\overline{r}_1 = a_1, \tag{3.1}$$

$$\overline{r}_2 = a_2 + b\sin\frac{2\pi}{\lambda}(\overline{Z} - c\overline{t}). \tag{3.2}$$

In the above equations, a_1 is the radius of the inner tube, a_2 is the radius of the outer tube at the inlet, b is the wave amplitude, λ is the wavelength and c is the wave speed. In Figure 1 the endoscope could be represented by a line that is free to move between the boundaries $\overline{r}_1$ and $\overline{r}_2$ with the fluid.

Let $\overline{U}$ and $\overline{W}$ be the velocity components in the $\overline{R}$ and $\overline{Z}$ directions, respectively, in the Laboratory frame. Then Eqs. (2.1) and (2.2) give

$$\frac{1}{\overline{R}}\frac{\partial}{\partial\overline{R}}(\overline{R}\,\overline{U}) + \frac{\partial\overline{W}}{\partial\overline{Z}} = 0, \tag{3.3}$$

$$\rho\left[\frac{\partial}{\partial\overline{t}} + \overline{U}\frac{\partial}{\partial\overline{R}} + \overline{W}\frac{\partial}{\partial\overline{Z}}\right]\overline{U} = -\frac{\partial\overline{p}}{\partial\overline{R}} + \frac{1}{\overline{R}}\frac{\partial}{\partial\overline{R}}(\overline{R}\,\overline{S}_{\overline{R}\,\overline{R}}) + \frac{\partial}{\partial\overline{Z}}(\overline{S}_{\overline{R}\,\overline{Z}}) - \frac{\overline{S}_{\overline{\theta}\,\overline{\theta}}}{\overline{R}}, \tag{3.4}$$

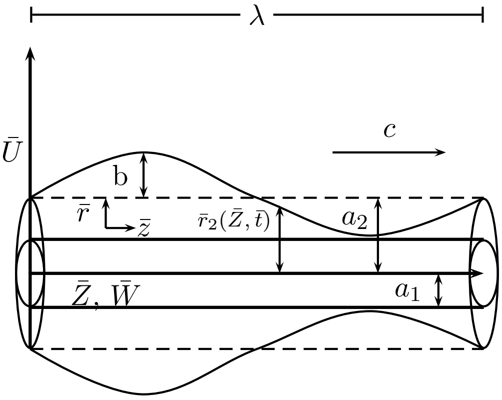

Figure 1: *Schematic of problem.*

$$\rho\left[\frac{\partial}{\partial\overline{t}}+\overline{U}\frac{\partial}{\partial\overline{R}}+\overline{W}\frac{\partial}{\partial\overline{Z}}\right]\overline{U}=-\frac{\partial\overline{p}}{\partial\overline{Z}}+\frac{1}{\overline{R}}\frac{\partial}{\partial\overline{R}}\left(\overline{R}\,\overline{S}_{\overline{R}\,\overline{Z}}\right)+\frac{\partial}{\partial\overline{Z}}\left(\overline{S}_{\overline{Z}\,\overline{Z}}\right)-\sigma B_0^2\overline{W}, \quad (3.5)$$

where σ is the electrical conductivity of the fluid.

In the Laboratory frame $(\overline{R},\overline{Z})$, the flow in the gap between inner and outer tubes is unsteady. However, if observed in a coordinate system $(\overline{r},\overline{z})$ moving at the wave speed c (wave frame), it can be treated as steady. The coordinates and velocities in two frames are related through

$$\overline{z} = \overline{Z}-c\overline{t}, \quad \overline{r}=\overline{R}, \quad (3.6)$$

$$\overline{u} = \overline{U}, \quad \overline{w}=\overline{W}-c, \quad (3.7)$$

where $\overline{u}$ and $\overline{w}$ are the velocity components in the directions of $\overline{r}$ and $\overline{z}$, respectively. The boundary conditions in the wave frame are

$$w = -c, \quad u=0 \quad \text{at} \quad r=r_1=a_1, \quad (3.8)$$

$$w = -c, \quad u=0 \quad \text{at} \quad r=r_2=a_2+b\sin\frac{2\pi}{\lambda}z. \quad (3.9)$$

It would be expedient to simplify the governing equations by introducing non-dimensional variables. The following variables could thus be introduced:

$$r = \frac{r}{a_2}, \quad r_1=\frac{r_1}{a_2}=\frac{a_1}{a_2}=\epsilon<1,$$

$$z = \frac{z}{\lambda}, \quad \mu(r)=\frac{\mu(r)}{\mu_0}, \quad u=\frac{\lambda u}{a_2 c}, \quad w=\frac{w}{c}, \quad (3.10)$$

$$p = \frac{a_2^2 p}{c\lambda\mu_0}, \quad t=\frac{ct}{\lambda}, \quad \boldsymbol{S}=\frac{a_2}{\mu_0 c}\mathbf{S},$$

$$r_2 = \frac{r_2}{a_2}=1+\phi\sin 2\pi z, \quad (3.11)$$

where μ_0 is the viscosity on the endoscope and ϵ is the radius ratio. We note here that the viscosity is a property of the fluid and depends on the mechanical properties of the fluid. By non-dimensionalising the model equation converting the problem to a Laboratory frame we have removed the dependence of the model on the physical parameters. We make the assumption that the viscosity depends on the radial coordinate to make meaningful progress in solving the model equation.

The Reynolds number Re, the wave number δ, the amplitude ratio ϕ and the Hartmann number M are defined by

$$\begin{aligned} Re &= \frac{\rho c\, a_2}{\mu_0}, \quad \delta = \frac{a_2}{\lambda} \ll 1, \quad \phi = \frac{b}{a_2} < 1, \\ M &= \sqrt{\frac{\sigma}{\mu_0}} B_0\, a_2. \end{aligned} \tag{3.12}$$

With the help of Eqs. (3.6), (3.7) and (3.10) - (3.12), the Eqs. (3.3) - (3.5) and boundary conditions (3.8) and (3.9) become of the following form:

$$Re\delta^3 \left[u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z} \right] u = -\frac{\partial p}{\partial r} + \frac{\delta}{r}\frac{\partial}{\partial r}(rS_{rr}) + \delta^2 \frac{\partial}{\partial z}(S_{rz}) - \delta\frac{S_{\theta\theta}}{r}, \tag{3.13}$$

$$Re\delta \left[u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z} \right] w = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\, S_{rz}) + \delta\frac{\partial}{\partial z} S_{zz} - M^2(w+1), \tag{3.14}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{3.15}$$

$$w = -1, \quad u = 0 \quad \text{at} \quad r = r_1 = \epsilon, \tag{3.16}$$

$$w = -1, \qquad \text{at} \quad r = r_2, \tag{3.17}$$

where

$$\lambda_1 = \frac{\alpha_1 c}{\mu_0 a_2}, \ \lambda_2 = \frac{\alpha_2 c}{\mu_0 a_2}, \ \gamma_1 = \frac{\beta_1 c^2}{\mu_0 a_2^2}, \gamma_2 = \frac{\beta_2 c^2}{\mu_0 a_2^2}, \ \gamma_3 = \frac{\beta_3 c^2}{\mu_0 a_2^2}. \tag{3.18}$$

We note that a closed form solution of the dynamical equations for arbitrary values of all parameters seems to be impossible. According, we carry out the investigation on the basis of the long wavelength at low Reynolds number assumption. The large wavelength analysis is applicable for the flow of semen in vas deferens, movement of chyme in small intestine and transport of semen in the ductus afferents of the male reproductive tract. Under long wavelength consideration, Eqs. (3.13) and (3.14) yield

$$-\frac{\partial p}{\partial r} = 0, \tag{3.19}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(rS_{rz}) - M^2(w+1), \tag{3.20}$$

where

$$S_{rz} = \mu(r)\frac{\partial w}{\partial r} + 2(\gamma_2 + \gamma_3)\left(\frac{\partial w}{\partial r}\right)^3. \tag{3.21}$$

Eq. (3.19) indicates that p is independent of r. Hence p is only a function of z.
From Eqs. (3.20) and (3.21) we can write

$$\frac{dp}{dz} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\mu(r)\frac{\partial w}{\partial r} + 2\Gamma\left(\frac{\partial w}{\partial r}\right)^3\right\}\right] - M^2(w+1), \tag{3.22}$$

in which the Deborah number Γ is

$$\Gamma = \gamma_2 + \gamma_3. \tag{3.23}$$

4 Rate of volume flow

The dimensional rate of fluid flow in the Laboratory frame is given an

$$Q = 2\pi \int_{\bar{r}_1}^{\bar{r}_2} \overline{W}\,\overline{R} d\overline{R}. \tag{4.1}$$

In above equation $\overline{r}_1$ is a constant and $\overline{r}_2$ depends upon $\overline{Z}$ and $\overline{t}$.
The rate of fluid in wave frame is

$$\overline{q} = 2\pi \int_{\overline{r}_1}^{\overline{r}_2} \overline{w}\,\overline{r}\, d\overline{r}. \tag{4.2}$$

With the help of Eqs. (3.6) and (3.7) and integration the two rates are related by the following equation

$$\overline{Q} = \overline{q} + \pi c\left(\overline{r}_2^2 - \overline{r}_1^2\right). \tag{4.3}$$

The time mean flow over a period $T_1(=\lambda/c)$ at a fixed Z-position is given by

$$\hat{Q} = \frac{1}{T_1}\int_0^{T_1} \overline{Q} d\overline{t}. \tag{4.4}$$

The above expression finally yields

$$\hat{Q} = \overline{q} + \pi c\left(a_2^2 - a_1^2 + \frac{b^2}{2}\right) \tag{4.5}$$

which may be written as,

$$\frac{\hat{Q}}{2\pi c a_2^2} = \frac{\overline{q}}{2\pi c a_2^2} + \frac{1}{2}\left(1 - \epsilon^2 + \frac{\phi^2}{2}\right). \tag{4.6}$$

The above expression in nondimensional variables is

$$\Theta = F + \frac{1}{2}\left(1 - \epsilon^2 + \frac{\phi^2}{2}\right), \tag{4.7}$$

where

$$F = \frac{\overline{q}}{2\pi a_2^2 c} = \int_{r_1}^{r_2} rwdr, \tag{4.8}$$

$$\Theta = \frac{\hat{Q}}{2\pi c a_2^2}, \tag{4.9}$$

are the non-dimensional flow rate and time mean flow, respectively.

In non-dimensional variables, the pressure rise ΔP_λ and the frictional forces on the inner $F_\lambda^{(i)}$ and outer $F_\lambda^{(0)}$ tubes are respectively given by

$$\Delta P_\lambda = \int_0^1 \left(\frac{dp}{dz}\right) dz, \tag{4.10}$$

$$F_\lambda^{(i)} = \int_0^1 r_1^2 \left(-\frac{dp}{dz}\right) dz, \tag{4.11}$$

$$F_\lambda^{(o)} = \int_0^1 r_2^2 \left(-\frac{dp}{dz}\right) dz. \tag{4.12}$$

5 Numerical Solution

We use `bvp4c` in MATLAB to solve (3.22) numerically. The dotted lines in the figures are the boundaries r_1 and r_2. In Figures 2, 3, 4 and 5 we have chosen

$$\frac{dp}{dz} = z, \qquad \mu(r) = 1. \tag{5.1}$$

For this choice of the pressure gradient we find that

$$\Delta P_\lambda = \frac{1}{2}, \tag{5.2}$$

$$F_\lambda^{(i)} = -\frac{\epsilon^2}{2}, \tag{5.3}$$

$$F_\lambda^{(o)} = -\frac{1}{2} + \frac{\phi}{\pi} - \frac{\phi^2}{4}. \tag{5.4}$$

In Figures 6, 7, 8 and 9 we have chosen

$$\frac{dp}{dz} = z, \qquad \mu(r) = \exp(-r^2). \tag{5.5}$$

We have chosen to show the results for the case when $\mu(r)$ is a Gaussian profile because these results provided us with the best comparison with the case $\mu(r) = 1$. Numerical calculations with other functional forms of the viscosity were undertaken.

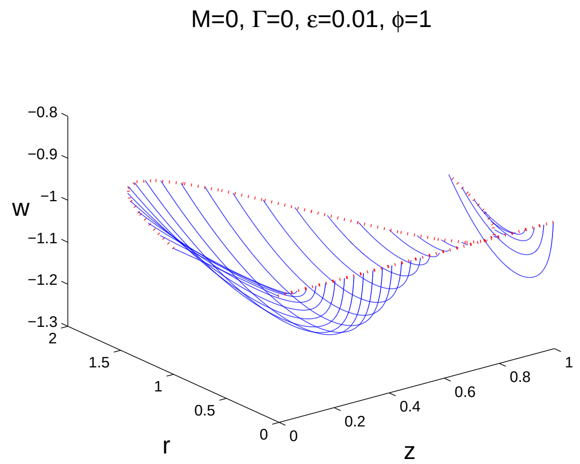

Figure 2: *Plot for $M = 0$ and $\Gamma = 0$.*

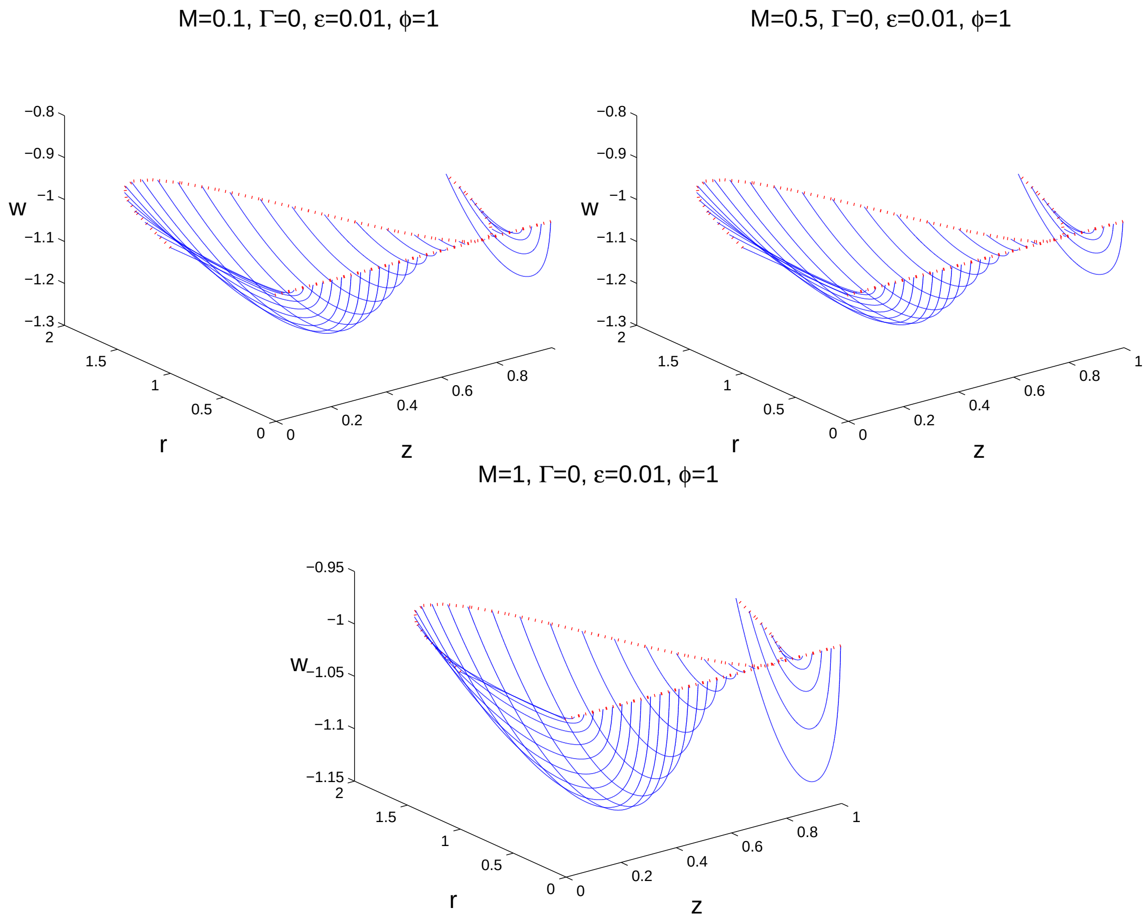

Figure 3: *Plot showing variation in M for fixed Γ.*

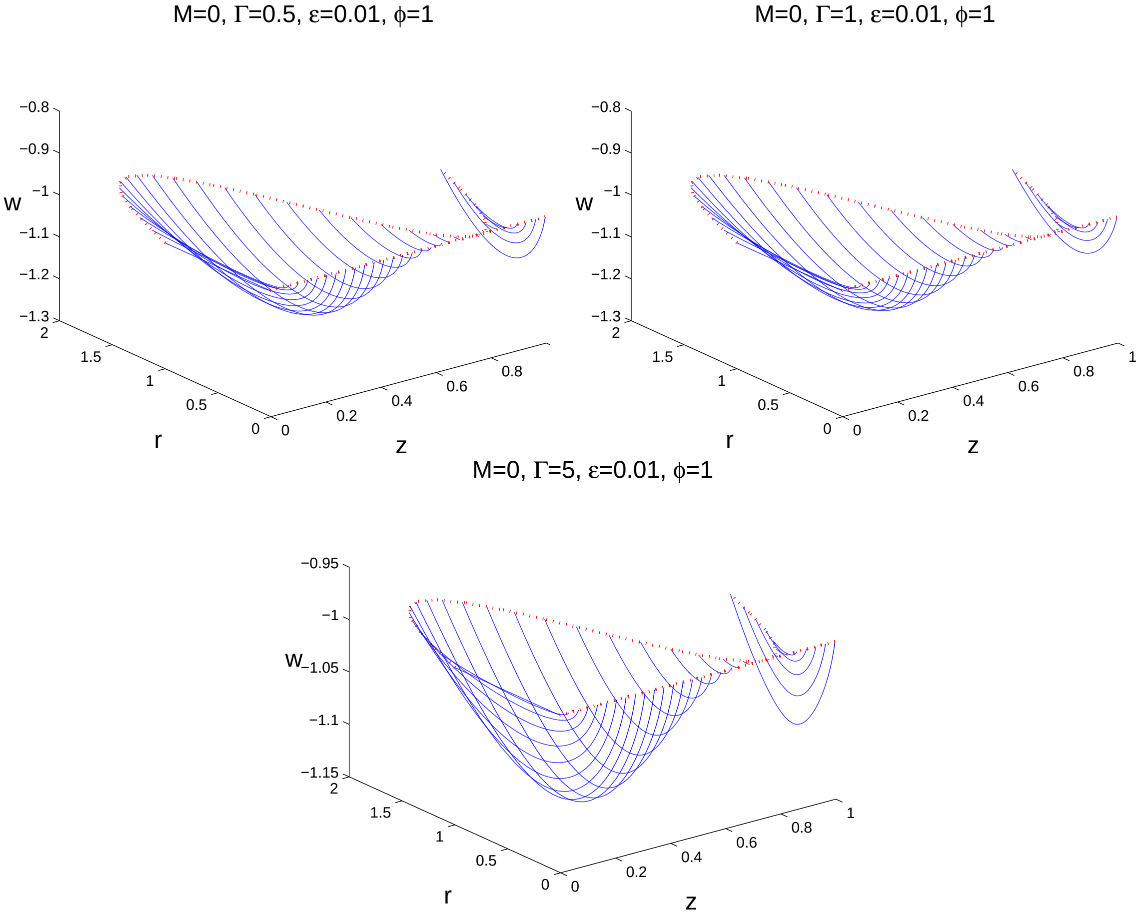

Figure 4: *Plot showing variation in* Γ *for fixed* M*.*

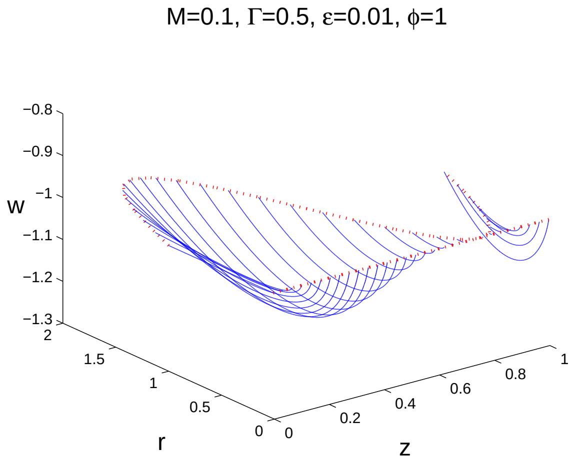

Figure 5: *Plot showing variation in* M *and* Γ*.*

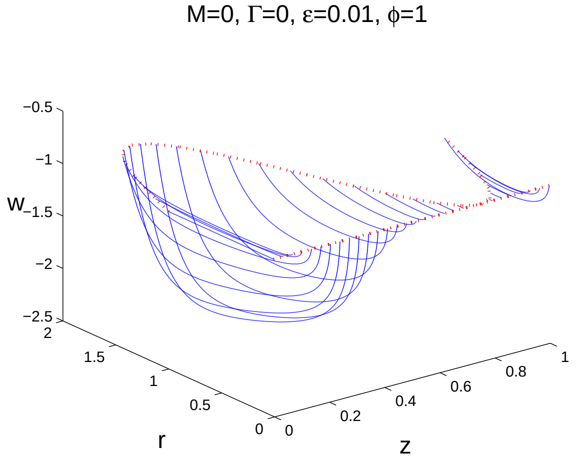

Figure 6: *Plot for* $M = 0$ *and* $\Gamma = 0$.

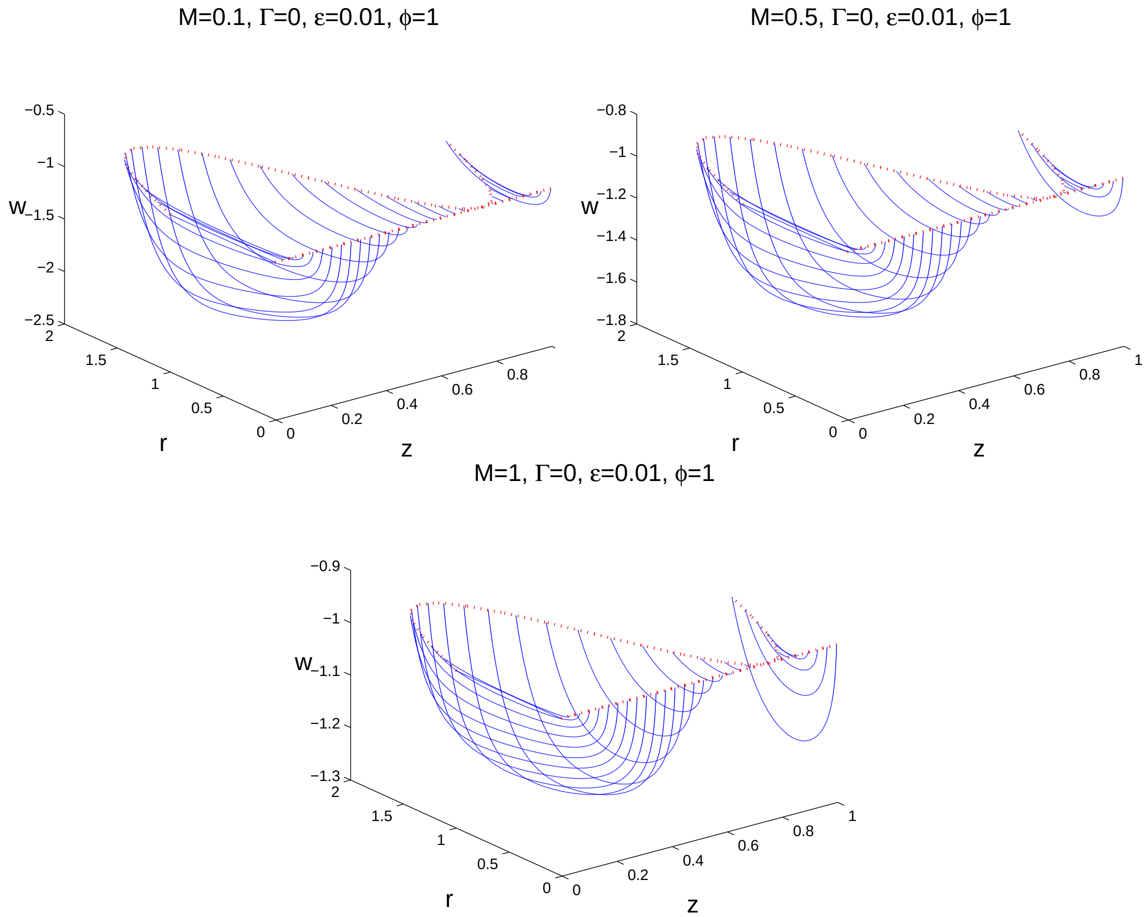

Figure 7: *Plot showing variation in* M *for fixed* Γ.

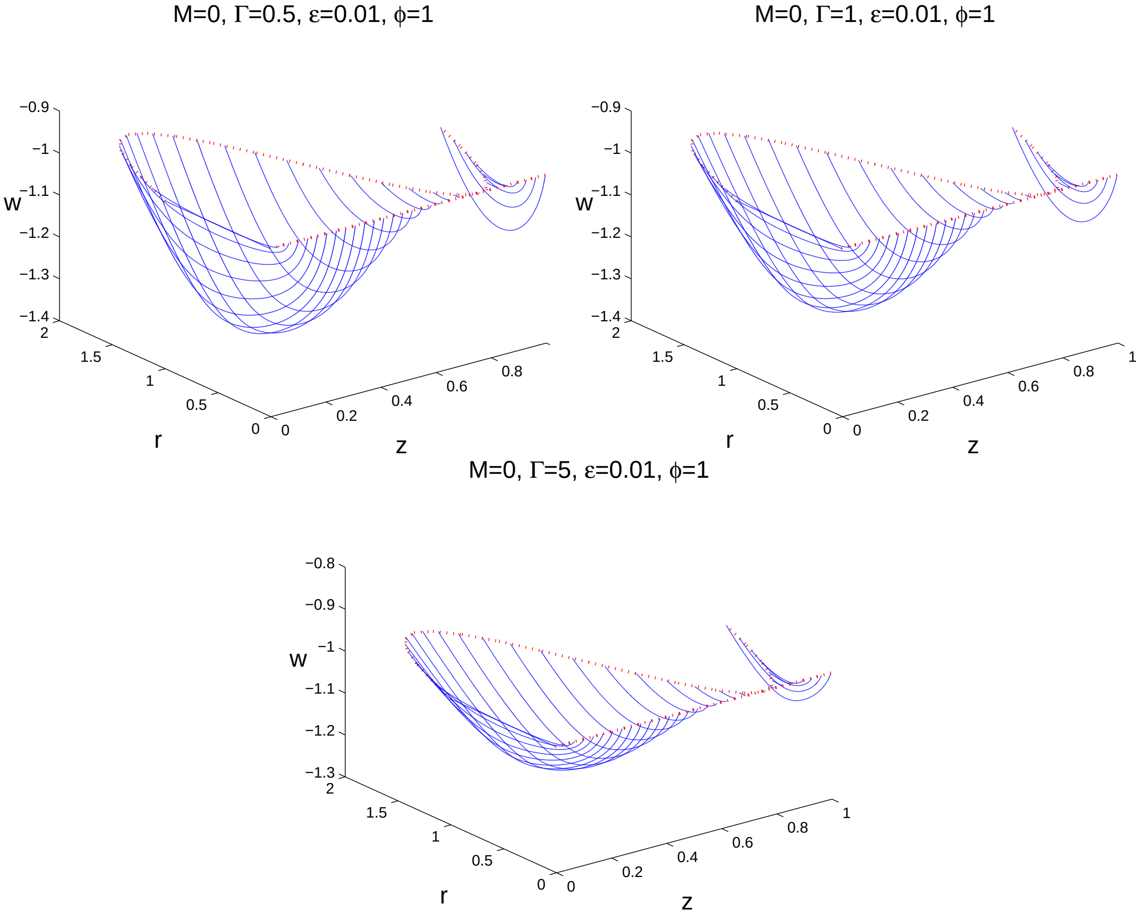

Figure 8: *Plot showing variation in* Γ *for fixed* M*.*

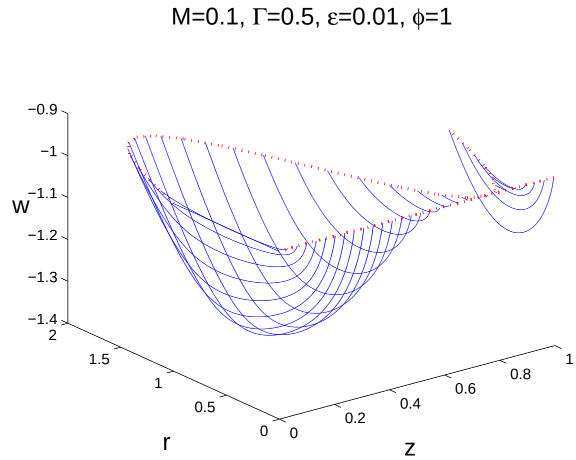

Figure 9: *Plot showing variation in* M *and* Γ.

6 Concluding remarks

In Hayat et al. [30] we have shown that the pressure rise increases while the frictional forces decrease with increasing Hartmann number. In this paper we have kept the pressure rise and frictional force along the inner wall constant. The frictional force on the outer wall varies quadratically with increasing ϕ. By fixing the angle ϕ we are fixing the frictional force on the outer wall as well. We have compared graphs for constant viscosity and variable viscosity with a Gaussian profile.

In Figures 2, 3, 4 and 5 we plot the effects of variation in Hartmann and Deborah numbers on the velocity w where the viscosity is constant. We note that increasing the Hartmann number decreases the magnitude of the velocity. Increasing the Deborah number increases the magnitude of the velocity. In Figure 5 when both the Hartmann and Deborah numbers are non-zero the magnitude of the velocity is not as affected when compared with zero Hartmann and Deborah numbers in Figure 2. The Hartmann and Deborah numbers not only affect the magnitude of the velocity but also the shape of the velocity surface.

In Figures 6, 7, 8 and 9 we consider the effects of a variable viscosity with a Gaussian distribution. The effects of varying the Hartmann and Deborah numbers are the same as indicated above. The magnitude of the velocity is significantly higher with variable viscosity. This can clearly be seen when comparing Figures 2 and 7. Also, the shape of the velocity surface is significantly altered. The velocity profile is now "fatter" and wider. In conclusion we note that the main contributions of a variable viscosity is in changing the magnitude of velocity and shape of the velocity surface.

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