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Global Dissipative Solutions of the Generalized Camassa-Holm Equation

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Abstract

A new approach to the analysis of wave-breaking solutions to the generalized Camassa-Holm equation is presented in this paper. Introduction of a set of variables allows for solving the singularities. A continuous semigroup of dissipative solutions is also built. The solutions have non-increasing H^1 energy and thus energy loss occurs only through wave breaking.

1 Introduction

A generalization of the Camassa-Holm equation, derived in 1998 by Dai [22, 23], reads as

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \quad t > 0, \quad x \in \mathbb{R}. \tag{1.1}$$

The equation describes far-field, finite length, finite amplitude radial deformation waves in cylindrical compressible hyperelastic rods, u representing the radial stretch relative to a pre-stressed state. In the physical derivation of (1), the material parameter γ is fixed and ranges from -29.47 to 3.41 cf. [15] but throughout this paper we allow for any real value of γ. For any $\gamma \in \mathbb{R}$ the equation (1) is Hamiltonian (with operator $\frac{1}{2}\partial_x(1-\partial_x^2)^{-1}$ and functional $\int_{\mathbb{R}}(u^3 + \gamma u u_x^2)dx$ - see the discussion in [15]). Special particular cases of (1) are of great interest. For example, with $\gamma = 1$ we obtain the Camassa-Holm equation [9, 26], a model for the propagation of shallow water waves, with $u(t,x)$ standing for the water's free surface over a flat bed. The Camassa-Holm equation is a completely integrable infinite-dimensional Hamiltonian system [10, 13], and models the existence of permanent [11] as well as that of breaking waves [12]. For an investigation of the periodic setting, see [17, 18]. The solitary waves of the Camassa-Holm equation are orbitally stable peaked solitons [14, 15]. These peaked waves are analogous to the exact traveling wave solutions of the governing equations for water waves representing waves of greatest height, see the discussion in [20, 21]. For $\gamma = 0$ the equation (1) becomes the regularized long wave equation [3], another celebrated model for the propagation of shallow water waves over a flat bed. Notice that the only equation in the class (1) that is an integrable Hamiltonian system is the Camassa-Holm equation ($\gamma = 1$) cf. [25]. Also, in the periodic setting, this is the only case when the equation is a re-expression of the geodesic flow of the diffeomorphism group over the unit-circle, see [19]. The solitary wave solutions of (1) were investigated in [27, 29], while the Cauchy problem is studied in [15], [30], [32]-[34]: the equation is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, and the only way singularities can develop is if $\inf_{x\in\mathbb{R}}\{\gamma u_x(t,x)\}$ becomes $-\infty$ in finite time (see [15]), a situation known to occur under various conditions. In particular, for $\gamma = 0$ all solutions are global.

In view of the possible development of singularities in finite time, it is natural to wonder about the behavior of a solution after the occurrence of wave breaking. Recently, this issue has been discussed for the Camassa-Holm equation [5] and for the hyperelastic rod equation in [31]. In this paper, we adapt the device of Bressan and Constantin [6] to solve the singularities of solutions to (1) and we establish the existence of a semigroup of global solutions with non-increasing H^1 energy.

2 The derivation of equation

Given the equation (with $\gamma > 0$)

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \quad t > 0, \quad x \in \mathbb{R},$$

a simple manipulation leads us to

$$(1 - \partial_x^2)(u_t + \gamma uu_x) = -\partial_x \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right)$$

and

$$u_t + \gamma uu_x = -\partial_x (1 - \partial_x^2)^{-1} \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right).$$

The latter formula can be recast as

$$u_t + \left(\frac{\gamma}{2} u^2 \right)_x + P_x = 0, \tag{2.1}$$

where

$$P = \frac{1}{2} e^{-|x|} * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right).$$

This is the form of the hyperelastic rod equation that will be of use in the present investigation.

The Cauchy problem associated to (2.1) by means of the data $u(0,\cdot) = \overline{u} \in H^1(\mathbb{R})$ has a *local solution* in the following sense, see [31, 5]: the function $u = u(t,x)$, defined on $[0,T] \times \mathbb{R}$ for a small $T > 0$, is Hölder continuous, $u(t,\cdot) \in H^1(\mathbb{R})$ for all $t \in [0,T]$ and the map $t \mapsto u(t,\cdot)$ from $[0,T]$ to $L^2(\mathbb{R})$ is Lipschitzian of order 1 and verifies the L^2 – equality

$$\frac{du}{dt} = -\gamma uu_x - P_x$$

for almost every $t \in [0, T]$. It has been established in [31] that this solution can be continuated indefinitely to the right of T even if it develops a singularity in T, that is $\lim_{t \nearrow T} \{\inf_{x \in \mathbb{R}} u_x(t, x)\} = -\infty$, by requiring that the energy functional

$$E(t) = \int_{\mathbb{R}} [u^2(t,x) + u_x^2(t,x)]dx$$

is constant for almost every $t \geq 0$. In fact, to see that this is the common circumstance for smooth solutions of (1.1), assume that $u \in C_t^1 \cap C_x^2$ with $\lim_{x \to \pm\infty} u(t,x) = 0$ at all times and $u_x \in L^\infty \cap L^2$. This will allow for formal differentiation of various integrals. Starting with

$$u_t + \gamma u u_x = -P_x, \tag{2.2}$$

after a differentiation with respect to x followed by a multiplication by u_x, we get

$$u_{tx} + \gamma(u_x^2 + u u_{xx}) = -P_{xx} = \frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2 - P \tag{2.3}$$

and respectively

$$\partial_t \left(\frac{u_x^2}{2} \right) + \gamma \partial_x \left(\frac{u u_x^2}{2} \right) + \frac{\gamma - 3}{2} \partial_x \left(\frac{u^3}{3} \right) = -P u_x. \tag{2.4}$$

Again, multiplying (2.2) by u, we get

$$\begin{aligned} \partial_t \left(\frac{u^2}{2} \right) + \gamma \partial_x \left(\frac{u^3}{3} \right) &= -P_x u \\ &= -(Pu)_x + P u_x. \end{aligned}$$

By addition,

$$\frac{1}{2}\partial_t(u^2 + u_x^2) + \gamma \partial_x \left(\frac{u u_x^2}{2} \right) + \frac{3}{2}(\gamma - 1)\partial_x \left(\frac{u^3}{3} \right) + \partial_x(Pu) = 0.$$

Since $P \in L^\infty(\mathbb{R})$, an integration of this equality with respect to x leads to $\partial_t E = 0$.

Instead of this *conservative* behavior for certain solutions of (2.1), it will be established here that there are also local solutions which can be continuated indefinitely to the right of their breaking time T while the energy functional $E(t)$ is a non-increasing function of t. Such a solution will naturally be called *dissipative*.

3 The formulation of a FDE system of first order

To analyze what happens to a solution of the Camassa-Holm equation after wave breaking, Bressan and Constantin [5] proposed a transformation of the equation to a functional differential system of first order. To adapt their procedure to equation (2.2), we start with a rewriting of the energy formula, namely by introducing three variables (u, v, q) in order to have

$$\int_{\mathbb{R}} \left(u^2 \frac{1}{1+u_x^2} + \frac{u_x^2}{1+u_x^2} \right) (1+u_x^2) dx \equiv \int_{\mathbb{R}} \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2} \right) q d\xi. \tag{3.1}$$

All three new variables are themselves functions of t and ξ. Roughly speaking, from now on when the solution breaks we simply have $v = -\pi$.

The interplay between variables is assured by the use of characteristics [4]. To this end, we introduce the function $y(t,\xi)$ through the equation

$$\frac{\partial}{\partial t} y(t,\xi) = \gamma u(t, y(t,\xi)). \tag{3.2}$$

The starting point of the characteristic, $\overline{y}(\xi)$, satisfies the equation

$$\int_0^{\overline{y}(\xi)} (1 + \overline{u}_x^2) dx = \xi, \qquad \xi \in \mathbb{R}, \quad \overline{u} \in H^1(\mathbb{R}), \tag{3.3}$$

in connection with the initial data $u(0,x) = \overline{u}(x)$. It can be seen easily that $\overline{y}$ is a Lipschitz function of coefficient 1. To derive the differential equations in (u,v,q) that will replace (2.2) in the analysis, let us use the formulas

$$v(t,\xi) = 2\arctan u_x(t, y(t,\xi)), \qquad q(t,\xi) = (1 + u_x^2(t, y(t,\xi)))\frac{\partial y}{\partial \xi}.$$

This is only temporarily, as v shall be given through a differential equation that is invariant to additive multiples of 2π, so that the singularities of arctan will not intervene. The sign problem (u_x goes to $-\infty$ means v is approaching $-\pi$) is solved by putting

$$\frac{u_x}{1+u_x^2} = \frac{1}{2}\sin v,$$

where u_x stands for $u_x(t, y(t,\xi))$. For future use, we have

$$\frac{\partial u}{\partial \xi} = u_x \cdot \frac{\partial y}{\partial \xi} = \frac{u_x}{1+u_x^2} q = \frac{q}{2}\sin v. \tag{3.4}$$

Now, given $\xi_1 < \xi_2$,

$$y(t,\xi_2) - y(t,\xi_1) = \int_{\xi_1}^{\xi_2} \frac{\partial y}{\partial \xi} d\xi = \int_{\xi_1}^{\xi_2} \cos^2 \frac{v(t,\xi)}{2} q(t,\xi) d\xi \tag{3.5}$$

and so

$$\begin{aligned} P(t,\xi) &= \frac{1}{2} e^{-|y(t,\xi)|} * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right) \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y(t,\xi) - y(t,\Xi)|} \left(\frac{3-\gamma}{2} u^2(\Xi) \cos^2 \frac{v(\Xi)}{2} + \frac{\gamma}{2} \sin^2 \frac{v(\Xi)}{2} \right) \\ &\times \ q(t,\Xi) d\Xi \end{aligned}$$

via the change of variables stated in (3.1). Without further mentioning, we have replaced x with $y(t,\xi)$ inside u, u_x. To have a better grasp of this change of variables, let us comment on its possibility: conditions will be provided on both v, q so that the integral from (3.5) will be comparable with $c_3(\xi_2 - \xi_1)$ for a certain constant $c_3 > 0$. This means that, at time t, $y(t,\mathbb{R}) = \mathbb{R}$ and $y(t,\cdot)$ will be a homeomorphism of the real line. Finally, by replacing $y(t,\xi) - y(t,\Xi)$ with its formula (3.5), we obtain a representation for the quantity $P(t,\xi)$

(the former $P(t, y(t, \xi))$) in terms of the new variables (u, v, q). Similar computations lead us to an integral representation of $P_x(t, \xi)$ by means of (u, v, q). We have

$$\partial_t u(t,\xi) = \frac{d}{dt}u(t, y(t,\xi)) = u_t + u_x \cdot \frac{\partial y}{\partial t} = u_t + \gamma u u_x = -P_x.$$

This is our first equation of the first order functional differential (FDE) system, namely

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{2}\left(\int_\xi^{+\infty} - \int_{-\infty}^{\xi}\right)\exp\left(-\left|\int_\xi^{\Xi}\cos^2\frac{v(s)}{2}q(s)ds\right|\right)\\ &\times \left(\frac{3-\gamma}{2}u^2(\Xi)\cos^2\frac{v(\Xi)}{2} + \frac{\gamma}{2}\sin^2\frac{v(\Xi)}{2}\right)q(\Xi)d\Xi.\end{aligned}$$

Next, we have

$$\int_{\xi_1}^{\xi_2} q(t,\xi)d\xi = \int_{y(t,\xi_1)}^{y(t,\xi_2)}(1+u_x^2)dx$$

and so

$$\begin{aligned}\frac{d}{dt}\left[\int_{\xi_1}^{\xi_2} q(t,\xi)d\xi\right] &= \int_{y(t,\xi_1)}^{y(t,\xi_2)}\frac{\partial}{\partial t}(1+u_x^2)dx + [1+u_x^2(t,y(t,\xi_2))]\frac{\partial y}{\partial t}(t,\xi_2)\\ &- [1+u_x^2(t,y(t,\xi_1))]\frac{\partial y}{\partial t}(t,\xi_1).\end{aligned}$$

Since, according to (3.2),

$$\begin{aligned}&[1+u_x^2(t,y(t,\xi_2))]\frac{\partial y}{\partial t}(t,\xi_2) - [1+u_x^2(t,y(t,\xi_1))]\frac{\partial y}{\partial t}(t,\xi_1) =\\ &\gamma\left\{[1+u_x^2(t,y(t,\xi_2))]u(t,y(t,\xi_2)) - [1+u_x^2(t,y(t,\xi_1))]u(t,y(t,\xi_1))\right\}\\ &= \gamma\int_{y(t,\xi_1)}^{y(t,\xi_2)}\frac{\partial}{\partial x}\left[(1+u_x^2)u\right]dx,\end{aligned}$$

we get via (2.4) that

$$\begin{aligned}\frac{d}{dt}\left[\int_{\xi_1}^{\xi_2} q(t,\xi)d\xi\right] &= \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left\{(1+u_x^2)_t + \gamma[u(1+u_x^2)]_x\right\}dx\\ &= \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left[\gamma u_x + (3-\gamma)\partial_x\left(\frac{u^3}{3}\right) - 2Pu_x\right]dx.\end{aligned}$$

Another equation for the FDE system is obtained by differentiating the latter estimate with respect to ξ, that is

$$\begin{aligned}\frac{\partial q}{\partial t} &= (\gamma + (3-\gamma)u^2 - 2P)u_x \cdot \frac{q}{1+u_x^2}\\ &= \frac{1}{2}(\gamma + (3-\gamma)u^2 - 2P)\sin v \cdot q.\end{aligned}$$

The third equation of the FDE system is given by the next computations regarding v, namely

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{d}{dt}[2\arctan u_x(t,y(t,\xi))] = \frac{2}{1+u_x^2}(u_{xt}+u_{xx}\cdot\gamma u)\\ &= \frac{2}{1+u_x^2}\left(\frac{3-\gamma}{2}u^2-\frac{\gamma}{2}u_x^2-P\right) = 2\left(\frac{3-\gamma}{2}u^2-P\right)\cos^2\frac{v}{2}\\ &- \gamma\sin^2\frac{v}{2}.\end{aligned}$$

Given $\overline{u}\in H^1(\mathbb{R})$, a recapitulation of the computations leads to an initial value problem for the system

$$\left\{\begin{array}{lcl}\frac{\partial u}{\partial t} &=& -\frac{1}{2}\left(\int_\xi^{+\infty}-\int_{-\infty}^{\xi}\right)Q(t,\xi,\Xi)d\Xi\\ \frac{\partial v}{\partial t} &=& \left(\frac{3-\gamma}{2}u^2-\frac{1}{2}\int_{\mathbb{R}}Q(t,\xi,\Xi)d\Xi\right)(\cos v+1)-\gamma\sin^2\frac{v}{2}\\ \frac{\partial q}{\partial t} &=& \left(\frac{\gamma}{2}+\frac{3-\gamma}{2}u^2-\frac{1}{2}\int_{\mathbb{R}}Q(t,\xi,\Xi)d\Xi\right)\sin v\cdot q,\end{array}\right. \tag{3.6}$$

where

$$\begin{aligned}Q(t,\xi,\Xi) &= \exp\left(-\left|\int_\xi^\Xi\cos^2\frac{v(t,s)}{2}q(t,s)ds\right|\right)\left(\frac{3-\gamma}{2}u^2(t,\Xi)\cos^2\frac{v(t,\Xi)}{2}\right.\\ &+ \left.\frac{\gamma}{2}\sin^2\frac{v(t,\Xi)}{2}\right)q(t,\Xi),\end{aligned}$$

and the data

$$u(0,\xi)=\overline{u}(\overline{y}(\xi)),\qquad q(0,\xi)=1 \tag{3.7}$$

and

$$v(0,\xi)=\left\{\begin{array}{ll}2\arctan\overline{u}_x(\overline{y}(\xi)) & \text{if } \overline{y}'(\xi)>0,\\ -\pi & \text{if } \overline{y}'(\xi)=0.\end{array}\right. \tag{3.8}$$

The system (3.6) can be regarded as an abstract differential equation in the Banach product space $X=H^1(\mathbb{R})\times(L^2\cap L^\infty)(\mathbb{R})\times L^\infty(\mathbb{R})$ with standard norm. This takes into account the inequalities

$$\left\{\begin{array}{l}\text{meas }(A)\le\text{meas }(\overline{y}(A))+\|\overline{u}_x\|_{L^2}^2\\ \|\arctan\circ\,\overline{u}_x\circ\overline{y}\|_{L^2}^2\le\frac{\pi^2}{4}\text{ meas }(\overline{y}(A))+2\|\overline{u}_x\|_{L^2}^2,\end{array}\right.$$

valid for all $A\subset\mathbb{R}$ measurable, where we take $A=\{\xi\in\mathbb{R}:|\overline{u}_x(\overline{y}(\xi))|\ge 1\}$, see [31]. The right-hand side of (3.6) being locally Lipschitzian, local existence and uniqueness of solution follows by the Cauchy-Lipschitz-Picard theorem [8, p. 104]. This solution extends indefinitely in time since the energy functional, recast as in (3.1), is conserved. See the computations in [31].

To investigate the existence of dissipative solutions to the Camassa-Holm equation, Bressan and Constantin [6, pp. 8] proposed a replacement of the FDE system (3.6) by a switching-type system. We shall adapt here their procedure to prove that the hyperelastic rod equation (2.2) has also dissipative solutions.

By denoting with $\tau(\xi)$ the time of the first wave breaking along the characteristic $y(\cdot,\xi)$, we set $v(t,\xi)=-\pi$ for all $t\geq\tau(\xi)$. This truncation leads to the new FDE system

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial t} & = & -R \\ \frac{\partial v}{\partial t} & = & \left\{\begin{array}{l} \left(\frac{3-\gamma}{2}u^2-P\right)(\cos v+1)-\gamma\sin^2\frac{v}{2} \qquad \text{if } v>-\pi, \\ 0 \qquad \text{if } v\leq-\pi, \end{array}\right. \\ \frac{\partial q}{\partial t} & = & \left\{\begin{array}{l} \left(\frac{\gamma}{2}+\frac{3-\gamma}{2}u^2-P\right)\sin v\cdot q \qquad \text{if } v>-\pi, \\ 0 \qquad \text{if } v\leq-\pi, \end{array}\right. \end{array}\right. \tag{3.9}$$

where

$$\left\{\begin{array}{lll} P(t,\xi) & = & \frac{1}{2}\int_{\{\Xi:\, v(\Xi)>-\pi\}}\tilde{Q}(t,\xi,\Xi)d\Xi, \\ R(t,\xi) & = & \frac{1}{2}\left(\int_{\{\Xi>\xi:\, v(\Xi)>-\pi\}}-\int_{\{\Xi<\xi:\, v(\Xi)>-\pi\}}\right)\tilde{Q}(t,\xi,\Xi)d\Xi \end{array}\right.$$

and

$$\begin{array}{lll} \tilde{Q}(t,\xi,\Xi) & = & \exp\left(-\left|\int_{\{s\in[\xi,\Xi]:\, v(s)>-\pi\}}\cos^2\dfrac{v(t,s)}{2}q(t,s)ds\right|\right) \\ & \times & \left(\dfrac{3-\gamma}{2}u^2(t,\Xi)\cos^2\dfrac{v(t,\Xi)}{2}+\dfrac{\gamma}{2}\sin^2\dfrac{v(t,\Xi)}{2}\right)q(t,\Xi). \end{array}$$

We remark that $v_t\approx\gamma\neq0$ as v approaches the $-\pi$ line. This non-slipping condition ensures the well-posedness of (3.9).

4 Local existence of solutions to the FDE system (3.9)

The local existence of solutions to the FDE system (3.9) is established by following the main steps in the proof of the Bressan-Shen existence theorem for directionally continuous abstract differential equations [7].

We start by operating a further modification of the FDE system. Namely, consider the FDE system

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}U(t,\xi)=F(U(t,\xi))+G(\xi,U(t)), \\ U(0,\xi)=\overline{U}(\xi), \end{array}\right. \qquad \xi\in\mathbb{R}, \tag{4.1}$$

where $U=(u,v,q)$ and

$$F(U)=\left\{\begin{array}{l} \left(0,\frac{3-\gamma}{2}u^2(1+\cos v)-\gamma\sin^2\frac{v}{2},\left(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}\right)\sin v\cdot q\right) \text{ if } v>-\pi, \\ (0,-\gamma,0) \quad \text{if } v\leq-\pi \end{array}\right.$$

and

$$G(\xi,U(t))=\left\{\begin{array}{l} (-R(U),-(1+\cos v)P(U),-\sin v\cdot qP(U)) \text{ if } v>-\pi, \\ (-R(U),0,0) \quad \text{if } v\leq-\pi. \end{array}\right.$$

One can notice that $F:\mathbb{R}^3\to\mathbb{R}^3$ is locally Lipschitz continuous while G becomes discontinuous whenever meas $(\{\xi:\, v(t,\xi)=-\pi\})>0$ at some moment t. We recall that the Lebesgue measure of the set of such t's is 0, see [5], [31, Section 7].

Theorem 1. *Given $\overline{u} \in H^1(\mathbb{R})$, the Cauchy problem (4.1) has a unique solution defined on $[0, T]$ for a small $T > 0$. Here, $\overline{U}$ is collected from the data (3.7), (3.8).*

We remark that the mapping $(t, \xi) \mapsto (u(t, \xi), \max\{v(t, \xi), -\pi\}, q(t, \xi))$, where $U = (u, v, q)$ is given by Theorem 1, is a solution of (3.9).

Proof. (of Theorem 1) Consider the subset X of $L^\infty(\mathbb{R}, \mathbb{R}^3)$ consisting of the functions U that obey the inequalities

$$\|u\|_{L^\infty} \leq C, \quad \frac{1}{C} \leq q(\xi) \leq C \text{ for a.e. } \xi \in \mathbb{R}$$

and

$$\text{meas } \left(\left\{\xi \in \mathbb{R} : \, v(\xi) > -\pi, \, |v(\xi)| \geq \frac{\pi}{2}\right\}\right) \leq C$$

for a constant $C > 2\max\{\|\overline{u}\|_{L^\infty}, 1, \text{meas } (\{\xi \in \mathbb{R} : \, |\overline{u}_x(\xi)| \geq 1\})\}$. Obviously, $\overline{U} \in X$. Here, by $\|U\|_{L^\infty}$ we mean $\max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}, \|q\|_{L^\infty}\}$.

Based on the inferences about the integral operators P, R made at Section 3, one deduces the existence of a constant $k = k(C) > 0$ such that

$$\|P(U)\|_{L^\infty} + \|R(U)\|_{L^\infty} + \|F(U)\|_{L^\infty} + \|G(U)\|_{L^\infty} \leq k, \tag{4.2}$$

and

$$\|F(U_1) - F(U_2)\|_{L^\infty} \leq k\|U_1 - U_2\|_{L^\infty}$$

and

$$\begin{aligned} &\|G(U_1) - G(U_2)\|_{L^\infty} \leq k[\|U_1 - U_2\|_{L^\infty} \\ &+\text{meas } (\{\xi : \, v_1(\xi) > -\pi, \, v_2(\xi) \leq -\pi\}) \\ &+\text{meas } (\{\xi : \, v_1(\xi) \leq -\pi, \, v_2(\xi) > -\pi\})] \end{aligned}$$

for all U, U_1, $U_2 \in X$. Due to certain computational reasons, we impose that $4k > \gamma$. Also, we fix $\delta > 0$ small enough in order that

$$\text{meas } (S_\delta) < \frac{1}{8k}, \quad \text{where } S_\delta = \{\xi \in \mathbb{R} : \, v(0, \xi) \in (-\pi, -\pi + \delta]\}.$$

Introduce now the set $D \subset C([0, T], L^\infty(\mathbb{R}, \mathbb{R}^3))$, where $T > 0$ is small enough, of all the functions $U = U(t)$ with the following properties

$$U(0) = \overline{U}, \qquad \|U(t) - U(s)\|_{L^\infty} \leq 2k \cdot |t - s|, \tag{4.3}$$

and

$$v(t, \xi) - v(s, \xi) \leq -\frac{\gamma}{2}(t - s), \qquad \xi \in S_\delta, \qquad 0 \leq s \leq t \leq T. \tag{4.4}$$

Obviously, D is closed with respect to the topology generated by the standard sup – metric. We also notice that, given $\xi \in \mathbb{R} - S_\delta$, we have

$$v(t, \xi) \geq v(0, \xi) - 2k \cdot t > -\pi, \qquad t \in [0, T], \tag{4.5}$$

provided that $T < \frac{\delta}{2k}$ (recall that the level set $\{\xi : \, v(0, \xi) = -\pi\}$ has measure 0).

Introduce further the integral operator $O : D \to D$ given by the formula

$$O(U)(t,\xi) = \overline{U}(\xi) + \int_0^t [F(U(s,\xi)) + G(\xi, U(s))]ds$$

and notice that $O(U)(t,\cdot) \in X$ for every $t \in [0,T]$. We claim now that O is a contraction in D.

Take U_1, $U_2 \in D$ such that $d(U_1,U_2) = \sup_{t\in[0,T]} \|U_1(t) - U_2(t)\|_{L^\infty} = \eta$. The "wave breaking" for U_i happens at the moment $\tau_i(\xi) = \sup\{t \in [0,T] : v(t,\xi) > -\pi\}$. We claim that $|\tau_1(\xi) - \tau_2(\xi)| \leq 2\eta$ for every $\xi \in S_\delta$. To see this, let us assume that $\tau_1(\xi) \geq t \geq \tau_2(\xi)$. The formulas (4.4), (4.3) yield

$$\begin{aligned}
&\frac{\gamma}{2}[\tau_1(\xi) - \tau_2(\xi)] \\
&\leq \frac{\gamma}{2}[\tau_1(\xi) - \tau_2(\xi)] + \left(2k - \frac{\gamma}{2}\right)[t - \tau_2(\xi)] = \frac{\gamma}{2}[\tau_1(\xi) - t] - 2k[\tau_2(\xi) - t] \\
&\leq |v_1(\tau_1(\xi),\xi) - v_1(t,\xi)| - |v_2(\tau_2(\xi),\xi) - v_2(t,\xi)| \\
&\leq |v_1(t,\xi) - v_2(t,\xi)| \leq \|U_1 - U_2\|_{L^\infty} = \eta,
\end{aligned}$$

where $v_i(\tau_i(\xi),\xi) = -\pi$. The claim regarding τ's is confirmed.

We claim further that $V_{ij}(t) = \{\xi \in \mathbb{R} : v_i(t,\xi) > -\pi,\ v_j(t,\xi) \leq -\pi\} \subseteq S_\delta$ for all $t \in [0,T]$, where $i \neq j \in \{1,2\}$. This follows plainly from noticing that $V_{ij}(t) \subseteq \{\xi : v_j(t,\xi) \leq -\pi\}$ and estimate (4.5).

To conclude that O is a contraction, we perform the next computations

$$\begin{aligned}
&\|O(U_1)(t) - O(U_2)(t)\|_{L^\infty} \leq \int_0^t \|F(U_1(s)) - F(U_2(s))\|_{L^\infty} ds \\
&+ \int_0^t \|G(U_1(s)) - G(U_2(s))\|_{L^\infty} ds \\
&\leq 2k \int_0^t \|U_1(s) - U_2(s)\|_{L^\infty} ds + k \int_0^t [\text{meas } (V_{12}) + \text{meas } (V_{21})]ds \\
&\leq 2kT\eta + 2k \int_{S_\delta} |\tau_1(\xi) - \tau_2(\xi)| d\xi \leq 2kT\eta + 4k \cdot \text{meas } (S_\delta) \cdot \eta \leq \frac{\eta}{2},
\end{aligned}$$

where T is eventually diminished. The claim being confirmed, the proof is complete.

■

5 Global existence of solutions to the FDE system (3.9)

The local solution to (4.1) whose existence was proved in Theorem 1 will now be extended throughout $[0,+\infty)$ by adapting the energy procedure from [5, 31]. In fact, we claim that the mathematical energy with the formula

$$\mathcal{E}(t) = \int_{\mathbb{R}} \left(u^2(t,\xi) \cos^2 \frac{v(t,\xi)}{2} + \sin^2 \frac{v(t,\xi)}{2} \right) q(t,\xi) d\xi$$

remains constant for as long as the solution exists. This will be established by effectively computing $\partial_t \mathcal{E}$.

We start by deducing, via the Lebesgue-Besicovitch differentiation theorem [24, p. 43], that

$$\begin{cases} \partial_\xi R &= \begin{cases} -q\left(\frac{3-\gamma}{2}u^2\cos^2\frac{v}{2}+\frac{\gamma}{2}\sin^2\frac{v}{2}-P\cos^2\frac{v}{2}\right) & \text{when } v(t,\xi)>-\pi \\ 0 \text{ when } v(t,\xi)=0, \end{cases} \\ \partial_\xi P &= qR\cos^2\frac{v}{2}. \end{cases}$$

Further, we have

$$\begin{aligned} u_\xi(t,\xi) &= \overline{u}_x\cdot\overline{y}'(\xi)-\int_0^t \partial_\xi R(s,\xi)ds=\frac{\overline{u}_x}{1+\overline{u}_x^2} \\ &+ \int_0^t q\left(\frac{3-\gamma}{2}u^2\cos^2\frac{v}{2}+\frac{\gamma}{2}\sin^2\frac{v}{2}-P\cos^2\frac{v}{2}\right)ds. \end{aligned}$$

Since

$$\begin{aligned} & q\left(\frac{3-\gamma}{2}u^2\cos^2\frac{v}{2}+\frac{\gamma}{2}\sin^2\frac{v}{2}-P\cos^2\frac{v}{2}\right) \\ &= \frac{q}{2}\left(2\cos^2\frac{v}{2}-1\right)\left(\frac{3-\gamma}{2}u^2-P\right)+\frac{q}{2}\left(\frac{3-\gamma}{2}u^2-P\right) \\ &+\frac{q\gamma}{4}\left(\sin^2 v-2\cos v\sin^2\frac{v}{2}\right) \\ &= \frac{q}{2}\cos v\left(\frac{3-\gamma}{2}u^2-P\right)+\frac{q}{2}\left(\frac{3-\gamma}{2}u^2-P\right)(\cos^2 v+\sin^2 v) \\ &+\frac{q\gamma}{4}\left(\sin^2 v-2\cos v\sin^2\frac{v}{2}\right) \\ &= \left(\frac{q}{2}\cos v\left(\frac{3-\gamma}{2}u^2-P\right)(1+\cos v)-\frac{q\gamma}{2}\cos v\sin^2\frac{v}{2}\right) \\ &+\left(\frac{q\gamma}{4}\sin^2 v+\frac{q}{2}\sin^2 v\left(\frac{3-\gamma}{2}u^2-P\right)\right) \\ &= \frac{q}{2}\cos v\cdot v_t+q_t\cdot\frac{1}{2}\sin v \\ &= \partial_t\left(\frac{q}{2}\sin v\right), \end{aligned}$$

we get

$$u_\xi(t,\xi)=\frac{q(t,\xi)}{2}\sin v(t,\xi)+\left(\frac{\overline{u}_x}{1+\overline{u}_x^2}-\frac{1}{2}\sin(2\arctan\overline{u}_x)\right) \qquad \text{a.e. in } \mathbb{R}$$

and the conclusion is reached via the elementary identity

$$\sin\phi=2\frac{\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}},$$

that is, $u_\xi=\frac{q}{2}\sin v$ for as long as the solution exists.

We also have

$$\begin{aligned} &\partial_t\left(q\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)\right) \\ &= q_t\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)+q\left(2uu_t\cos^2\frac{v}{2}+(1-u^2)\sin\frac{v}{2}\cos\frac{v}{2}\cdot v_t\right). \end{aligned}$$

We introduce the expressions of u_t, v_t, q_t as collected from the FDE system under investigation here and recast the formula obtained by focusing on the coefficients of three quantities, namely P, $\frac{3-\gamma}{2}u^2$ and $\frac{\gamma}{2}$. Now, the sum of all monomials containing P reads as $-qP\sin v$, the sum of monomials that contain $\frac{3-\gamma}{2}u^2$ is $\frac{3-\gamma}{2}u^2 q\sin v$ and, at last, the sum of monomials containing $\frac{\gamma}{2}$ is $\frac{\gamma}{2}u^2 q\sin v$. A fourth term is given by u_t, that is $-2quR\cos^2\frac{v}{2} = -2uP_\xi$.

Finally, we have

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}}\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)qd\xi\right) &= \int_{\mathbb{R}}\left(-qP\sin v+\frac{3-\gamma}{2}u^2q\sin v\right.\\ &+ \left.\frac{\gamma}{2}u^2q\sin v-2uP_\xi\right)d\xi\end{aligned}$$

and, according to (3.4),

$$= \int_{\mathbb{R}}\partial_\xi(u^3-2Pu)d\xi = 0$$

since $\lim_{x\to\pm\infty}u(\xi)=0$ [8, p. 130] and $P\in L^\infty(\mathbb{R})$. The claim has been confirmed.

Given $\xi\in\mathbb{R}$, we estimate that

$$\begin{aligned}u^2(t,\xi) &= \int_{-\infty}^{\xi}\partial_\Xi\left(u^2(t,\Xi)\right)d\Xi \le 2\int_{\mathbb{R}}|u|\cdot\left|\frac{q}{2}\sin v\right|d\Xi\\ &\le \int_{\mathbb{R}}|q|\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)d\Xi = \mathcal{E}(0).\end{aligned}$$

Further, there exists $k>0$ such that

$$\|P(t)\|_{L^\infty}+\|R(t)\|_{L^\infty}\le k, \qquad \text{where } k=\frac{1}{2}\max\left\{\frac{|3-\gamma|}{2},\frac{|\gamma|}{2}\right\}\cdot\mathcal{E}(0),$$

for as long as the solution exists.

It follows directly from the expression of q_t in the FDE system that

$$\begin{aligned}&\exp\left(-\left(\frac{|\gamma|}{2}+\frac{|3-\gamma|}{2}\mathcal{E}(0)+k\right)t\right)\\ &\le q(t,\xi)\le\exp\left(\left(\frac{|\gamma|}{2}+\frac{|3-\gamma|}{2}\mathcal{E}(0)+k\right)t\right) \qquad \text{a.e. in } \mathbb{R}\end{aligned} \tag{5.1}$$

for as long as the solution exists.

Finally, from these estimates we deduce that, given $\eta>0$, there exists $C=C(\eta,k,\mathcal{E}(0),T)>0$ such that

$$\text{meas}\left(\{\xi\in\mathbb{R}: |v(t,\xi)|\ge\eta\}\right)\le C, \qquad t\in[0,T].$$

It can be concluded from this "non-explosive" behavior of (u,v,q) over a compact interval that the solution exists throughout the nonnegative half-axis, cf. [8, p. 104].

6 Continuous dependence of the initial data

The local solution of problem (4.1) obtained in Theorem 1 is the fixed point of the contraction O. This implies also the continuous dependence on the initial L^∞ data. We are interested here in establishing the continuous dependence of the solution on the initial data in H^1 norm.

Theorem 2. *Given $(\overline{u}_n)_{n\geq 1}$ and $\overline{u}$ in $H^1(\mathbb{R})$ such that $\overline{u}_n$ converges strongly to $\overline{u}$ when $n \to +\infty$, let u_n, u be the solutions of the corresponding problems (3.6) – (3.8). Then, for any $T > 0$, u_n converges uniformly to u throughout $[0,T] \times \mathbb{R}$ as $n \to +\infty$.*

Proof. Let $U_i = (u_i, v_i, q_i)$ be solutions of the equation (3.6) with the initial data given by (3.7), (3.8) for $\overline{u}_i \in H^1$, where $i \in \{1, 2\}$.

For $T > 0$ fixed, we consider the set $Y = \{\xi \in \mathbb{R} : v_1(T,\xi) = -\pi\} \cup \{\xi \in \mathbb{R} : v_2(T,\xi) = -\pi\}$ of measure $\mathcal{Y}$. The formula of the mathematical energy $\mathcal{E}$ allow us to conclude that $\mathcal{Y}$ is uniformly bounded by a constant depending on T and on an upper bound of the energies of U_i.

For $\xi \in Y$ fixed, we introduce the "wave-breaking time" given by $\tau(\xi) = \inf\{t \in [0,T] : \min\{v_1(t,\xi), v_2(t,\xi)\} = -\pi\}$. The application $\xi \mapsto \tau(\xi)$ being measurable, there exists, see [5, p. 15], a measure-preserving measurable mapping $[0,\mathcal{Y}] \ni s \mapsto \xi(s) \in Y$ such that $s \leq s'$ if and only if $\tau(\xi(s)) \geq \tau(\xi(s'))$. Its inverse will be denoted $\xi \mapsto s(\xi)$.

We define now the distance functional

$$\begin{aligned}
J(t) &= J(U_1(t), U_2(t)) \\
&= \|u_1(t) - u_2(t)\|_{L^\infty} + \|v_1(t) - v_2(t)\|_{L^2} + \|q_1(t) - q_2(t)\|_{L^2} \\
&+ c_0 \int_0^{\mathcal{Y}} e^{c_1 s} |v_1(t,\xi(s)) - v_2(t,\xi(s))| ds \\
&= J_1(U_1(t), U_2(t)) + c_0 J_2(U_1(t), U_2(t)), \\
&= J_1(t) + c_0 J_2(t), \qquad \text{where } c_0,\ c_1 > 0,
\end{aligned}$$

and claim that there exist c_i's, where $i \in \overline{0,4}$, large enough positive numbers depending only on T, $\mathcal{E}(0)$ and γ such that

$$\frac{d}{dt} J(t) \leq c_3 J(t) + c_4 \qquad \text{in } [0,T].$$

It is obvious that this estimate provides an a priori bound on $\|u_1(t) - u_2(t)\|_{L^\infty}$ when $t \in [0,T]$:

$$\|u_1(t) - u_2(t)\|_{L^\infty} \leq J(t) \leq \left(J(0) + \frac{c_4}{c_3} \right) e^{c_3 T}.$$

We start by splitting the set Y into $Y_1(t)$ given by

$$\begin{aligned}
Y_1(t) &= \{\xi \in Y : v_1(t,\xi) > -\pi,\ v_2(t,\xi) = -\pi\} \\
&\cup \{\xi \in Y : v_1(t,\xi) = -\pi,\ v_2(t,\xi) > -\pi\},
\end{aligned}$$

$Y_2(t)$ given by

$$Y_2(t) = \{\xi \in Y : v_1(t,\xi) = v_2(t,\xi) = -\pi\},$$

and $Y_3(t)$ given by

$$Y_3(t) \quad = \quad \{\xi \in Y : v_1(t,\xi) > -\pi,\, v_2(t,\xi) > -\pi\}.$$

Additionally, $Y_3(t) = \{\xi(s) : s \in [0, m(t)]\}$, where $m(t) = \text{meas }(Y_3(t))$ for $t \in [0,T]$, and $s(\xi) \geq m(t)$ for $\xi \in Y_1(t)$.

Repeating the computations at page 103 in the proof of Theorem 1, we get that

$$\frac{d}{dt}J_1(t) \leq k[J_1(t) + \text{meas }(Y_1(t))] \leq k[J_1(t) + \mathcal{Y}], \qquad k > 0.$$

This follows from the fact that, given $t \in [0,T]$ such that either $v_1(t,\xi) = -\pi$ or $v_2(t,\xi) = -\pi$, we have either $v_1(\tau,\xi) = -\pi$ for all $\tau \in [t,T]$ or $v_2(\tau,\xi) = -\pi$ for all $\tau \in [t,T]$. In particular,

$$\frac{\partial}{\partial t}|v_1(t,\xi) - v_2(t,\xi)| \leq k[J_1(t) + \mathcal{Y} + |v_1(t,\xi) - v_2(t,\xi)|]$$

for all $\xi \in Y_1(t) \cup Y_3(t)$.

We also notice that

$$J_2(t) = \int_Y e^{c_1 s(\xi)}|v_1(t,\xi) - v_2(t,\xi)|d\xi \quad \text{and} \quad \int_Y = \int_{Y_1(t)} + \int_{Y_3(t)}$$

by taking into account the definition of $Y_2(t)$.

Now,

$$\begin{aligned}
\frac{d}{dt}J_2(t) &= \int_{Y_1(t)\cup Y_3(t)} e^{c_1 s(\xi)}\frac{\partial}{\partial t}(|v_1(t,\xi) - v_2(t,\xi)|)d\xi \\
&\leq k[J_1(t) + \mathcal{Y}]\int_Y e^{c_1 s(\xi)}d\xi + kJ_2(t) \\
&= k[J_1(t) + \mathcal{Y}]\int_0^{\mathcal{Y}} e^{c_1 s}ds + kJ_2(t) \leq k\left[\left(J_1(0) + \frac{\mathcal{Y}}{k}\right)e^{kT} + \mathcal{Y}\right] \\
&\times \frac{e^{c_1\mathcal{Y}} - 1}{c_1} + kJ_2(t) = kJ_2(t) + K, \qquad t \in [0,T].
\end{aligned}$$

We conclude by

$$J(t) \leq \left(J(0) + \frac{\mathcal{Y} + K}{k}\right)e^{kT}, \qquad t \in [0,T].$$

The proof is complete.

■

7 Global dissipative solutions of the hyperelastic rod equation

The global solution (u, v, q) to (3.9) obtained before will lead us to a global dissipative solution of equation (2.2).

We shall introduce the solution U of (2.2). In fact, we can attach to (u, v, q) from (3.9) the quantity $y(t, \xi)$ in accordance with (3.2), namely

$$y(t,\xi) = \overline{y}(\xi) + \gamma \int_0^t u(s,\xi)ds.$$

Now, we have

$$U(t,x) = u(t,\xi) \qquad \text{whenever} \quad y(t,\xi) = x.$$

Since $\overline{u}_x \in L^2(\mathbb{R})$, we see that

$$\lim_{\xi\to\pm\infty} \overline{y}(\xi) = \pm\infty.$$

As $\|u(s)\|_\infty \le \sqrt{\mathcal{E}(0)}$ at every time s, it is clear that $y(t,\xi)$ covers the entire $\mathbb{R}$ when ξ ranges $\mathbb{R}$ at all moments $t \ge 0$.

It might happen that $y(t,\xi) = y(t,\xi')$ in some cases. In order to conclude that U is well-defined, meaning that $u(t,\xi) = u(t,\xi')$, we shall need to verify that the quantity $y(t,\xi)$ agrees with (3.5).

In this respect, a straightforward computation yields

$$\left(q\cos^2\frac{v}{2}\right)_t = \frac{\gamma q}{2}\sin v = \gamma u_\xi.$$

Again, by definition of $\overline{y}(\xi)$ and using the Lebesgue-Besicovitch differentiation theorem, we have

$$\begin{aligned} y_\xi(t,\xi) &= \overline{y}'(\xi) + \gamma \int_0^t u_\xi(s,\xi)ds \\ &= q(0,\xi)\cos^2\frac{v(0,\xi)}{2} + \int_0^t \partial_s\left(q\cos^2\frac{v}{2}\right)ds \\ &= q(t,\xi)\cos^2\frac{v(t,\xi)}{2} \qquad \text{a.e. in } \mathbb{R}. \end{aligned}$$

Returning to the definition of U, we see that if

$$0 = y(t,\xi') - y(t,\xi) = \int_\xi^{\xi'} q(t,\Xi)\cos^2\frac{v(t,\Xi)}{2}d\Xi \tag{7.1}$$

then, as q lies between two positive bounds, one gets $\cos\frac{v}{2} = 0$ and, consequently,

$$u(t,\xi') - u(t,\xi) = \int_\xi^{\xi'} \frac{q(t,\Xi)}{2}\sin v(t,\Xi)d\Xi = 0.$$

So, U is well-defined.

Consider the sets

$$\mathcal{M}(t) = \{\xi \in \mathbb{R} : 1 + \cos v(t,\xi) = 0\} \quad \text{and} \quad \mathcal{N} = \{t \ge 0 : \text{meas}\,(\mathcal{M}(t)) > 0\}$$

when $t \ge 0$. We recall, see [31, Section 7], that

$$\text{meas}\,(\mathcal{N}) = 0.$$

It is clear that we have, via the coarea formula [24, p. 117],

$$\begin{aligned}\mathcal{E}(0) &= \int_{\mathbb{R}} \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2}\right) q d\xi \geq \int_{\mathbb{R}-\mathcal{M}(t)} \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2}\right) q d\xi \\ &= \int_{\mathbb{R}-\mathcal{M}(t)} \left(u^2 + \tan^2 \frac{v}{2}\right) \left(q \cos^2 \frac{v}{2}\right) d\xi = \int_{\mathbb{R}-\mathcal{M}(t)} \left(u^2 + \tan^2 \frac{v}{2}\right) \frac{\partial y}{\partial \xi} d\xi \\ &= \int_{\{y\in\mathbb{R}:\ \text{the set } \{\xi\in\mathbb{R}: y(t,\xi)=y\} \text{ is singleton}\}} [U^2(t,y) + U_x^2(t,y)] dy\end{aligned}$$

because of $u_\xi(t,\xi) = U_x(t, y(t,\xi)) \cdot y_\xi(t,\xi)$. We see that, via (7.1), if the set $\{\xi \in \mathbb{R} : y(t,\xi) = y\}$ is not singleton then it is a subset of $\mathcal{M}(t)$. In this case, however, the set $\{y(t,\xi) : \xi \in \mathcal{M}(t)\}$ is itself a singleton. This means that

$$\int_{\mathbb{R}} [U^2(t,x) + U_x^2(t,x)] dx \leq \mathcal{E}(0) < +\infty.$$

The equality holds, evidently, when $t \in \mathbb{R}_+ - \mathcal{N}$. In particular, we have the following identity

$$\begin{aligned}&\int_{\mathbb{R}} [U^2(t,x) + U_x^2(t,x)] dx \\ &= \int_{\{\xi:\, v(t,\xi) > -\pi\}} \left(u^2(t,\xi) \cos^2 \frac{v(t,\xi)}{2} + \sin^2 \frac{v(t,\xi)}{2}\right) q(t,\xi) d\xi. \qquad (7.2)\end{aligned}$$

Morrey's inequality [24, p. 143] implies that U is Hölder continuous with exponent $\frac{1}{2}$ as a function of x. A closer look at Morrey's inequality allows us to say that U is locally Hölder continuous in $\mathbb{R}_+ \times \mathbb{R}$. This will follow as a by-product from the next computations.

We shall establish now that the map $t \mapsto U(t)$, from $\mathbb{R}_+$ to $L^2(\mathbb{R})$, is Lipschitz continuous. Let $\tau \geq 0$, $h \in (0,1)$ be given and assume that $y(\tau,\xi) = x$ for a certain $\xi \in \mathbb{R}$. Then, via the triangle inequality,

$$\begin{aligned}|U(\tau+h,x) - U(\tau,x)| &\leq |U(\tau+h, y(\tau,\xi)) - U(\tau+h, y(\tau+h,\xi))| \\ &+ |U(\tau+h, y(\tau+h,\xi)) - U(\tau, y(\tau,\xi))|.\end{aligned}$$

Since

$$\begin{aligned}|y(\tau,\xi) - y(\tau+h,\xi)| &\leq \sup_{t\in[\tau,\tau+h]} \{|y_t(t,\xi)|\} \cdot |\tau+h-\tau| \\ &\leq \sup_{t\in[\tau,\tau+h]} \{\|u(t)\|_{L^\infty}\} \cdot h \leq \sqrt{\mathcal{E}(0)} h,\end{aligned}$$

we get that

$$\begin{aligned}&|U(\tau+h, y(\tau,\xi)) - U(\tau+h, y(\tau+h,\xi))| \\ &\leq \sup_{|z-x|\leq\sqrt{\mathcal{E}(0)}h} |U(\tau+h,x) - U(\tau+h,z)| \\ &\leq \sup_{|z-x|\leq\sqrt{\mathcal{E}(0)}h} \left|\int_x^z |U_x(\tau+h,w)| dw\right| \leq \int_{x-\sqrt{\mathcal{E}(0)}h}^{x+\sqrt{\mathcal{E}(0)}h} |U_x(\tau+h,w)| dw \\ &= \int_{-\sqrt{\mathcal{E}(0)}h}^{\sqrt{\mathcal{E}(0)}h} |U_x(\tau+h,x+w)| dw.\end{aligned}$$

With regard to U's to-be-proved Hölder continuity, we remark also that, via Morrey's inequality, we have a different estimate, namely

$$\begin{aligned} |U(\tau+h,z)-U(\tau+h,x)| &\leq cR\cdot\left(\frac{1}{2R}\int_{x-R}^{x+R}|U_x(\tau+h,w)|^2dw\right)^{1/2} \\ &\leq c\sqrt{\frac{\mathcal{E}(0)R}{2}}=O(\sqrt{R}),\quad |z-x|\leq R, \end{aligned}$$

where $R=\sqrt{\mathcal{E}(0)h}$ and $c>0$.

Further,

$$\begin{aligned} &|U(\tau+h,y(\tau+h,\xi))-U(\tau,y(\tau,\xi))| = |u(\tau+h,\xi)-u(\tau,\xi)| \\ &\leq \int_\tau^{\tau+h}|u_t(t,\xi)|dt = \int_\tau^{\tau+h}|P_x(t,\xi)|dt \leq \|P_x(t)\|_{L^2}\sqrt{h}. \end{aligned}$$

Again, to discuss the Hölder continuity of U, we notice that

$$|U(\tau+h,y(\tau+h,\xi))-U(\tau,y(\tau,\xi))| = O(\sqrt{h}).$$

To sum the facts, we have $|U(\tau+h,x)-U(\tau,x)|=O(\sqrt{h})$ and, consequently, U is locally Hölder continuous with exponent $\frac{1}{2}$ as a function of t. This clarifies the issue of Hölder continuity regarding U.

Now, using the Schwarz inequality and the Fubini-Tonelli technique, we have

$$\begin{aligned} &\int_{\mathbb{R}}|U(\tau+h,x)-U(\tau,x)|^2dx \leq 2\int_{\mathbb{R}}\left(\int_{-\sqrt{\mathcal{E}(0)h}}^{\sqrt{\mathcal{E}(0)h}}|U_x(\tau+h,x+w)|dw\right)^2dx \\ &+2\int_{\mathbb{R}}\left(\int_\tau^{\tau+h}|P_x(t,\xi)|dt\right)^2 q(\tau,\xi)\cos^2\frac{v(\tau,\xi)}{2}d\xi \\ &\leq 2\int_{\mathbb{R}}\left(2\sqrt{\mathcal{E}(0)}h\int_{-\sqrt{\mathcal{E}(0)h}}^{\sqrt{\mathcal{E}(0)h}}|U_x(\tau+h,x+w)|^2dw\right)dx \\ &+2\int_{\mathbb{R}}\left(h\int_\tau^{\tau+h}|P_x(t,\xi)|^2dt\right)Q_+d\xi \\ &\leq 8\mathcal{E}(0)h^2\int_{\mathbb{R}}|U_x(\tau+h,r)|^2dr + 2hQ_+\int_\tau^{\tau+h}\|P_x(t)\|_{L^2}dt \leq c'h^2, \end{aligned}$$

where $c'=c'(\tau)$ and Q_+ is un upper bound of q (recall (5.1)). In conclusion, U is locally Lipschitz continuous and so, by means of an infinite dimensional version of Rademacher's theorem [1], it is differentiable almost everywhere with $U_t(t)\in L^2(\mathbb{R})$.

Given $t\in\mathbb{R}_+-\mathcal{N}$, since $y_\xi(t,\cdot)$ has constant non-null sign throughout $\mathbb{R}$, the application $\xi\mapsto y(t,\xi)$ is a homeomorphism of the real line and so it can be used as a change of variables leading to

$$\begin{aligned} R(t,y(t,\xi)) &= \frac{1}{2}\left(\int_{y(t,\xi)}^{+\infty}-\int_{-\infty}^{y(t,\xi)}\right)\exp(-|y(t,\xi)-x|) \\ &\times \left(\frac{3-\gamma}{2}U^2(t,x)+\frac{\gamma}{2}U_x^2(t,x)\right)dx. \end{aligned}$$

To arrive to (2.2), which reads as (recall that meas $(\mathcal{N})=0$)

$$U_t+\gamma UU_x=-R \text{ in } L^2(\mathbb{R}) \qquad \text{for a.e. } t\geq 0$$

by taking into account the total derivative $\frac{d}{dt}U(t,y(t,\xi))=U_t+U_xy_t=U_t+\gamma UU_x$, it suffices to establish that

$$\begin{aligned}&\frac{d}{dt}\left[\int_{\mathbb{R}}U(t,x)\Phi(x)dx\right]\\&=\int_{\mathbb{R}}[\gamma[U^2(t,x)\Phi'(x)+U(t,x)U_x(t,x)\Phi(x)]-R(t,x)\Phi(x)]dx\end{aligned}$$

for all $\Phi\in C_c^\infty(\mathbb{R},\mathbb{R})$. This reduces to

$$\begin{aligned}&\frac{d}{dt}\int_{\{\xi:\,v(t,\xi)>-\pi\}}u(t,\xi)\Phi(y(t,\xi))q(t,\xi)\cos^2\frac{v(t,\xi)}{2}d\xi\\&=\int_{\{\xi:\,v(t,\xi)>-\pi\}}\left[-R\Phi+\gamma\left(u^2\Phi'+u\frac{u_\xi}{y_\xi}\Phi\right)\right]q\cos^2\frac{v}{2}d\xi\end{aligned}$$

and it can be established in straightforward manner by using the equations (3.9).

The final feature of U as solution to the hyperelastic rod equation (2.2) is given by the fact that

$$\{\xi\in\mathbb{R}:\,v(t,\xi)=-\pi\}\subseteq\{\xi\in\mathbb{R}:\,v(t',\xi)=-\pi\},\qquad \text{where } t\leq t'.$$

This leads to

$$\{\xi\in\mathbb{R}:\,v(t',\xi)>-\pi\}\subseteq\{\xi\in\mathbb{R}:\,v(t,\xi)>-\pi\}$$

and so, by means of (7.2), we get

$$\int_{\mathbb{R}}[U^2(t,x)+U_x^2(t,x)]dx\geq\int_{\mathbb{R}}[U^2(t',x)+U_x^2(t',x)]dx,\qquad t\leq t'.$$

8 A semigroup of dissipative solutions for the hyperelastic rod equation

Due to the uniqueness of solution to the Cauchy problem (4.1), the global dissipative solutions of (2.2) can be organized as semigroup $\{S(t)\}_{t\geq 0}$ of applications, namely

$$S(t)(\overline{u})=u(t),\qquad t\geq 0,\,\overline{u}\in H^1(\mathbb{R}).$$

As main features of this semigroup, we have

$$\lim_{n\to+\infty}\|u_n-u\|_{L^\infty([0,T]\times B,\mathbb{R})}=0\qquad\text{when }\lim_{n\to+\infty}\|\overline{u}_n-\overline{u}\|_{H^1(\mathbb{R})}=0$$

for any bounded $B\subset\mathbb{R}$, a consequence of the considerations in Section 6, and

$$u_x(t,x)\leq C(1+t^{-1}),\qquad t>0,\qquad C=C(\|\overline{u}\|_{H^1(\mathbb{R})}),$$

a fact that follows from the estimate (4.4).

The semigroup property, that is $S(t) \circ S(\tau) = S(t+\tau)$ for any t, $\tau \geq 0$, follows from the fact that the functions

$$\tilde{u}(t,\alpha) = u(\tau + t, \xi(\alpha)), \quad \tilde{v}(t,\alpha) = v(\tau + t, \xi(\alpha))$$

and

$$\tilde{q}(t,\alpha) = \frac{q(\tau + t, \xi(\alpha))}{q(\tau,\alpha)},$$

where $t \geq 0$ and τ is fixed, provide a global dissipative solution of (2.2) that starts from $S(\tau)(\overline{u})$. Here, the application $\alpha \mapsto \xi(\alpha)$ is the inverse of application $\xi \mapsto \alpha(\xi)$ with the formula

$$\int_0^{y(\tau,\xi)} [1 + u_x^2(\tau,x)]dx = \alpha(\xi), \qquad \xi \in \mathbb{R}.$$

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