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New solutions of a higher order wave equation of the KdV type

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Abstract

In this paper we use the Painlevé analysis and study a special case of a water wave equation of the KdV type. More specifically, we use the Pickering algorithm [9] and obtain a new kind of solutions, which constitute of both algebraic and trigonometric (or hyperbolic) functions.

1 Introduction

As it is well known, the KdV equation represents a first order approximation in the study of long wavelength, small amplitude waves of inviscid and incompressible fluids. If one allows the appearance of higher order terms, more complicated wave equations can be obtained. Such an equation, including second order terms, was proposed in [1] and has the form

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) = 0. \tag{1.1}$$

The above equation was examined analytically and numerically in [6, 7, 10, 11] and it was found that, although it is non–integrable in general, it still possesses solitary wave solutions, which, for small values of the parameters α and β, behave like solitons. The equation was further examined in [3, 4, 5] were new wave and periodic solutions were found.

The study of nonlinear integrable and non–integrable partial differential equations (PDEs) regarding the finding of special solutions has been extensive during the last decades. As a result, many new methods have appeared in the bibliography, which have revealed an enormous amount of new solutions for a large class of nonlinear PDEs. One of these methods is the Painlevé analysis for PDEs, introduced by Weiss, Tabor and Carnevale [12] (thus called WTC algorithm), improved by Conte and Musette [8] and further improved by Pickering [9]. It is precisely this analysis that was used in [7, 10] and revealed new solutions for equation (1.1).

In this paper we use the algorithm proposed by Pickering and find new solutions for equation (1.1) for the special case $\rho_1 = 0$ and $\rho_3 = -2\rho_2$. These solutions constitute of both algebraic and trigonometric (or hyperbolic) functions. For simplicity we set $\alpha = \beta = 1$ in (1.1), which is equivalent with applying the transformation

$$u(x,t) = \frac{1}{\alpha} w(X,T) = \frac{1}{\alpha} w\left(\frac{1}{\sqrt{\beta}}x, \frac{1}{\sqrt{\beta}}t\right).$$

Thus (in the above special case) we obtain the equation

$$u_t + u_x + uu_x + u_{xxx} + \rho_2(uu_{xxx} - 2u_x u_{xx}) = 0. \tag{1.2}$$

2 The approach of the Pickering algorithm

Let us consider an n–th order PDE, say $u_t = K[u]$, where $u = u(x,t)$. If the equation passes the Painlevé test, as stated in [12], then the solution admits at least one expansion of the form

$$u = \sum_{k=-p}^{\infty} u_{k+p} \varphi^k, \tag{2.1}$$

where $k \in \mathbb{Z}$, $\varphi = \varphi(x,t)$ is an arbitrary function, and precisely $n-1$ of the functions $u_{k+p} = u_{k+p}(x,t)$ are also arbitrary. According to the WTC algorithm [12], we can then use the truncated expansion

$$u = \sum_{k=-p}^{0} u_{k+p} \varphi^k,$$

and substitute in the equation, in order to find the Lax pair.

The Conte–Musette algorithm [8], as well as the Pickering algorithm [9], constitute improvements of the WTC algorithm, in the sense that one can reveal the Lax Pair and/or some special solutions of the equation in a more systematic way.

According to Pickering approach we consider instead the truncated expansion

$$u = \sum_{k=-p}^{p} u_{k+p} z^k, \tag{2.2}$$

where function $z = z(x,t)$ satisfies the equations

$$z_x = 1 - Az - Bz^2, \tag{2.3a}$$

$$z_t = -C + (AC + C_x)z - (D - BC)z^2, \tag{2.3b}$$

while functions A, B, C and D satisfy the cross–derivative conditions

$$A_t + (AC)_x + C_{xx} - 2D = 0, \tag{2.4a}$$

$$B_t - D_x + 2BC_x + B_x C + AD = 0. \tag{2.4b}$$

Consequently, we substitute relation (2.2) in the equation, using relations (2.3), and, equating to zero the coefficients of z^k, and using cross–derivative conditions (2.4), we calculate A, B, C, D and u_{k+p}. Then, the Lax pair is given by the system

$$\psi_{xx} = A\psi_x + B\psi,$$
$$\psi_t = -C\psi_x + \left(\int D\mathrm{d}x\right)\psi.$$

In order to look for special solutions we assume that A, B and C are constants and $D = 0$. Then system (2.3) implies

$$z(x,t) = \frac{-A + \sqrt{A^2+4B}\tanh\left[\sqrt{A^2+4B}(x - Ct - c_0)/2\right]}{2B}, \tag{2.5}$$

where c_0 is an integration constant, while conditions (2.4) are identically satisfied. Consequently, we again substitute relation (2.2) in the equation and, equating to zero the coefficients of z^k, we calculate u_{k+p}.

We should finally mention that the above procedure can lead to special solutions even when the solution of the equation does not admit an expansion of the form (2.1) with the necessary amount of arbitrary functions, i.e. when the equation does not pass the Painlevé test.

3 The new solutions

In [6] it was shown that, although equation (1.1) does not pass the Painlevé test, it always admits an expansion of the form (2.1), where $p = 2$. In the special $\rho_1 = 0$ and $\rho_3 = -2\rho_2$, u_0 remains arbitrary. Thus, in order to find special solutions for the equation, we can substitute expansion

$$u = \frac{u_0}{z^2} + \frac{u_1}{z} + u_2 + u_3 z + u_4 z^2 \tag{3.1}$$

in (1.2) and evaluate u_n, as was done in [10]. However, in contrast with the procedure followed in [10], u_n are now considered as functions of x and t, and not constants. Because of the fact that u_0 remains arbitrary, this will lead, at some stage, to a highly nonlinear equation, which seems impossible to be solved. In order to avoid this, we consider the truncated expansion

$$u = \frac{u_0}{z^2} + \frac{u_1}{z} + u_2, \tag{3.2}$$

instead of (3.1). Consequently, we substitute this expansion in (1.2), (using relations (2.3)), and equate to zero the coefficients A_n of z^n, $n = -6, \ldots, 0$.

Relation $A_{-6} = 0$ yields

$$u_1 = -Au_0 - u_{0,x},$$

while relation $A_{-5} = 0$ yields

$$u_2 = \frac{1}{12\rho_2}\left[\left(\rho_2 A^2 - 8\rho_2 B - 1\right)u_0 + 6\rho_2\left(Au_{0,x} + u_{0,xx}\right) - 12\right].$$

Then, relation $A_{-4} = 0$ implies

$$3u_0\left[(\rho_2 A^2 + 4\rho_2 B + 1)u_{0,x} - 2\rho_2 u_{0,xxx}\right] = 0. \tag{3.3}$$

Thus, we have two different case:

(I) $\rho_2 A^2 + 4\rho_2 B + 1 \neq 0$**.**

In this case relation (3.3) implies

$$u_0 = f_1(t)\exp\left[-x\sqrt{\frac{\rho_2 A^2 + 4\rho_2 B + 1}{2\rho_2}}\right] + f_2(t) + f_3(t)\exp\left[x\sqrt{\frac{\rho_2 A^2 + 4\rho_2 B + 1}{2\rho_2}}\right],$$

while relation $A_{-3} = 0$ yields

$$\begin{aligned}
&3({\rho_2}^2(A^2+4B)^2-1)f_3(t)\exp\left[x\sqrt{\frac{2(\rho_2 A^2 + 4\rho_2 B + 1)}{\rho_2}}\right]\\
&\quad -2\left[({\rho_2}^2(A^2+4B)^2-1)f_2(t) - 12(\rho_2(C-1)+1)\right]\exp\left[x\sqrt{\frac{\rho_2 A^2 + 4\rho_2 B + 1}{2\rho_2}}\right]\\
&\quad +3({\rho_2}^2(A^2+4B)^2-1)f_1(t) = 0,
\end{aligned}$$

which implies

$$B = \frac{1-\rho_2 A^2}{4\rho_2}, \quad C = \frac{\rho_2 - 1}{\rho_2}.$$

Then relation $A_{-2} = 0$ implies

$$\begin{aligned}
&\exp\left[-\frac{x}{\sqrt{\rho_2}}\right]\left[(1-\rho_2)f_1(t) + {\rho_2}^{3/2}f_1'(t)\right] + {\rho_2}^{3/2}f_2'(t)\\
&\quad +\exp\left[\frac{x}{\sqrt{\rho_2}}\right]\left[(\rho_2-1)f_3(t) + {\rho_2}^{3/2}f_3'(t)\right] = 0.
\end{aligned}$$

Thus

$$f_1(t) = c_1\exp\left[\frac{(\rho_2-1)t}{{\rho_2}^{3/2}}\right], \quad f_2(t) = c_2, \quad f_3(t) = c_3\exp\left[\frac{(1-\rho_2)t}{{\rho_2}^{3/2}}\right],$$

where c_i, $i = 1, 2, 3$ are integration constants. Finally, $A_{-1} \equiv A_0 \equiv 0$.

Thus, relation (3.2), together with (2.5) and all the above results, yields the following solution:

$$u = \frac{1}{4\rho_2}\left(A_1 - A_2 + A_3\right), \tag{3.4}$$

where

$$\begin{aligned}
A_1 &= c_1(A\sqrt{\rho_2}-1)^2 e^{-\xi} + c_2(\rho_2 A^2 - 1) - 4 + c_3(A\sqrt{\rho_2}+1)^2 e^{\xi},\\
A_2 &= \frac{2(\rho_2 A^2 - 1)(c_1(A\sqrt{\rho_2}-1)e^{-\xi} + c_2 A\sqrt{\rho_2} + c_3(A\sqrt{\rho_2}+1)e^{\xi})}{A\sqrt{\rho_2} - \tanh\left[\frac{1}{2}(\xi + 4c_0\sqrt{\rho_2})\right]},\\
A_3 &= \frac{(\rho_2 A^2 - 1)^2(c_1 e^{-\xi} + c_2 + c_3 e^{\xi})}{\left(A\sqrt{\rho_2} - \tanh\left[\frac{1}{2}(\xi + 4c_0\sqrt{\rho_2})\right]\right)^2},
\end{aligned}$$

$$\xi = \frac{1}{{\rho_2}^{3/2}}(\rho_2 x + (1-\rho_2)t),$$

and ρ_2, A, c_0, c_1, c_2, c_3, remain arbitrary.

(II) $\rho_2 A^2 + 4\rho_2 B + 1 = 0$**.**

In this case

$$B = -\frac{\rho_2 A^2 + 1}{4\rho_2},$$

and relation (3.3) implies

$$u_0 = f_1(t) + f_2(t)x + f_3(t)x^2.$$

Then, relation $A_{-3} = 0$ implies

$$f_3(t) = \frac{3(\rho_2(C-1)+1)}{5\rho_2}, \tag{3.5}$$

while relation $A_{-2} = 0$ yields

$$\begin{aligned} &25{\rho_2}^2\left[(6\rho_2 C - f_1(t))f_2(t) + 6\rho_2 f_1'(t)\right] \\ &\quad +5\rho_2\left[6(\rho_2(1-C)-1)f_1(t) + \rho_2(36C(\rho_2(C-1)+1) - 5f_2(t)^2 + 30\rho_2 f_2'(t))\right]x \\ &\quad -45\rho_2(\rho_2(C-1)+1)f_2(t)x^2 - 18(\rho_2(C-1)+1)^2x^3 = 0. \end{aligned}$$

Equating to 0 the coefficients of x^k, $k = 3, 2, 1, 0$, we find

$$C = \frac{\rho_2 - 1}{\rho_2},$$

(thus, relation (3.5) implies $f_3(t) = 0$) and

$$f_2(t) = -\frac{6\rho_2}{t + 6\rho_2 c_2}, \quad f_1(t) = \frac{6(\rho_2-1)t + c_1}{t + 6\rho_2 c_2},$$

where c_1, c_2 are integration constants. Finally, $A_{-1} \equiv A_0 \equiv 0$.

Thus, relation (3.2), together with (2.5) and all the above results, yields the following solutions: For $\rho_2 > 0$,

$$u = -\frac{A_1 + A_2 + A_3}{12\rho_2(t + 6\rho_2 c_2)\left(A\sqrt{\rho_2}\cos\frac{\xi}{2} + \sin\frac{\xi}{2}\right)^2}, \tag{3.6}$$

where

$$\begin{aligned} A_1 &= 2(\rho_2 A^2 + 1)(6\rho_2(x - t + 3c_2) + 9t - c_1), \\ A_2 &= \left[6\rho_2\left(\rho_2 A(A(t-x) - 6) + x - t + 6c_2(\rho_2 A^2 - 1)\right) + c_1(\rho_2 A^2 - 1)\right]\cos\xi, \\ A_3 &= 2\sqrt{\rho_2}\left[A\left(3\rho_2(2(t-x) + 3\rho_2 A + 12c_2)\right) - 9\rho_2\right]\sin\xi, \end{aligned}$$

$$\xi = \frac{1}{{\rho_2}^{3/2}}(\rho_2 x + (1-\rho_2)t + 4{\rho_2}^2 c_0),$$

and ρ_2, A, c_0, c_1, c_2, remain arbitrary. On the other hand, for $\rho_2 < 0$,

$$u = \frac{A_1 + A_2 - A_3}{12\rho_2(t + 6\rho_2 c_2)\left(A\sqrt{-\rho_2}\cosh\frac{\xi}{2} + \sinh\frac{\xi}{2}\right)^2}, \tag{3.7}$$

where

$$\begin{aligned}
A_1 &= 2(\rho_2 A^2 + 1)(6\rho_2(x - t + 3c_2) + 9t - c_1),\\
A_2 &= \left[6\rho_2\left(\rho_2 A(A(t-x) - 6) + x - t + 6c_2(\rho_2 A^2 - 1)\right) + c_1(\rho_2 A^2 - 1)\right]\cosh\xi,\\
A_3 &= 2\sqrt{-\rho_2}\left[A\left(3\rho_2(2(t-x) + 3\rho_2 A + 12c_2)\right) - 9\rho_2\right]\sinh\xi,
\end{aligned}$$

$$\xi = \frac{1}{(-\rho_2)^{3/2}}(\rho_2 x + (1-\rho_2)t + 4{\rho_2}^2 c_0),$$

and ρ_2, A, c_0, c_1, c_2, remain again arbitrary.

4 Concluding remarks

In this paper we have used the Pickering algorithm and have obtained special solutions for a water wave equation of the KdV type. Solution (3.4) will not be considered as new, since (as can be easily verified) it embeds in the general form

$$u = \frac{b_0 + b_1 e^{\xi} + b_2 e^{2\xi}}{(a_0 + a_1 e^{\xi})^2},$$

where ξ has the form $b(x - ct)$. Thus, it could be obtained by applying the reduction $u = f(\xi)$ in equation (1.2), or by using a method which assumes a priori a specific wave form for the solution, as for example the exp–function method (see [2]).

On the other hand, solutions (3.6) and (3.7) appear to be new, since they constitute of both algebraic and trigonometric (or hyperbolic) functions. As far as we know, all the solutions that have been found for equation (1.1) are pure wave, periodic, or algebraic. Moreover, note that solutions (3.6) and (3.7) have a quite complicated form and cannot be obtained simply by applying the reduction $u = f(\xi)$ in equation (1.2).

Finally, we should mention the following: As is explained in [9], expansion (3.2) can be rewritten as a standard truncated WTC expansion, since it does not contain positive powers of z. However, we believe that it is still better to use the Pickering algorithm, since it simplifies considerably the calculations.

Obviously, the Pickering algorithm, as applied in this paper, has lead to new solutions, due to the arbitrariness of function u_0. We do not know yet if this situation is more general, i.e. if such arbitrariness can lead to new solutions for other PDEs as well. Results in this direction will be presented elsewhere.

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