Journal of Nonlinear Mathematical Physics Volume 14, Number 3 (2007), 430–442 Article

Linearization of one-dimensional nonautonomous jump-diffusion stochastic differential equations

Gazanfer ÜNAL[‡†] *Ismail IYIGÜNLER*[*] *and C Masood KHALIQUE*[‡]

[‡] *International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, SOUTH AFRICA*
E-mail: Masood.Khalique@nwu.ac.za

[†] *Department of Banking and Finance, and Department of Mathematics, Yeditepe University, Istanbul, Turkey*
E-mail: gunal@yeditepe.edu.tr

[*] *Department of Mathematics, Koc University, Istanbul, Turkey*
E-mail: iigunler@ku.ed.tr

Received December 27, 2006; Accepted in Revised Form March 6, 2007

Abstract

Necessary and sufficient conditions for the linearization of one-dimensional nonautonomous jump-diffusion stochastic differential equations are given. Stochastic integrating factor is introduced to solve the linear jump-diffusion stochastic differential equations. Closed form solutions to certain linearizable jump-diffusion stochastic differential equations are obtained.

1 Introduction

One-dimensional nonlinear stochastic ordinary differential equations (SODEs) arise in science, finance and engineering [6, 2, 11, 10]. However, jump-diffusion stochastic differential equations (JDSDEs) appear to be more realistic in cases where rare events play a prominent role [8, 7]. Closed form solutions of SODEs not only allow us to study the underlying stochastic processes but also provide means to test numerical schemes [5]. Linearizable one-dimensional Itô SODEs may also be useful in the study of two-dimensional problems [12]. Therefore analytical methods are of paramount importance.

Linearization of deterministic ordinary differential equations (ODEs) has been studied by various authors (see for example [1, 3, 4, 13]). Linearization criteria for one-dimensional Itô SODEs have been given by Gard [5]. However, to the best of our knowledge no work has been done on the linearization of one-dimensional JDSDEs. Here we undertake this task and present an analytical method which is based on linearization.

We consider one-dimensional nonlinear nonautonomous SODEs with jump terms of the form

$$dX = f(X,t)dt + g(X,t)dW(t) + r(X,t)dP(t), \qquad X(0) = X_0, \quad t \geq 0, \tag{1.1}$$

where $dW(t)$ is the infinitesimal increment of the Wiener process [11] and $dP(t)$ is the infinitesimal increment of the Poisson process [8, 7]. Here the simple Poisson process $P(t)$ (or P_t) generates the jumps and the Wiener process W_t (or $W(t)$) provides the diffusion. Therefore the stochastic process X_t involves both diffusion and jumps. We assume that $f(X,t)$, $g(X,t)$ and $r(X,t)$ are bounded and integrable functions. We provide necessary and sufficient conditions for the JDSDE given in (1.1) to be linearizable via an invertible transformation; hence, extending Gard's theorem [5] to JDSDEs. Furthermore we introduce the stochastic integrating factors to solve the linear JDSDEs. Finally we provide five examples of nonlinear nonautonomous JDSDEs. We apply our theorem to linearize the equations and then obtain their exact solutions using integrating factors.

2 Linearization

We seek transformations of the form

$$Y = h(X,t) \tag{2.1}$$

which transforms the nonlinear JDSDE given in (1.1) into a linear equation of the type

$$dY = (a_1(t)Y + a_2(t))dt + (b_1(t)Y + b_2(t))dW + (c_1(t)Y + c_2(t))dP(t). \tag{2.2}$$

Itô's lemma [8, 7] for $h(X,t)$ yields

$$\begin{aligned} dY &= \left[\frac{\partial h(X,t)}{\partial t} + f(X,t)\frac{\partial h(X,t)}{\partial X} + \frac{1}{2}g^2(X,t)\frac{\partial^2 h(X,t)}{\partial X^2}\right] dt \\ &\quad + g(X,t)\frac{\partial h(X,t)}{\partial X} dW + \left[h(X + r(X,t),t) - h(X,t)\right] dP. \end{aligned} \tag{2.3}$$

From Equations (2.2) and (2.3) we obtain

$$\frac{\partial h(X,t)}{\partial t} + f(X,t)\frac{\partial h(X,t)}{\partial X} + \frac{1}{2}g^2(X,t)\frac{\partial^2 h(X,t)}{\partial X^2} = a_1(t)h(X,t) + a_2(t), \tag{2.4}$$

$$h(X + r(X,t),t) - h(X,t) = c_1(t)h(X,t) + c_2(t), \tag{2.5}$$

and

$$g(X,t)\frac{\partial h(X,t)}{\partial X} = b_1(t)h(X,t) + b_2(t). \tag{2.6}$$

We now consider Equation (2.6) and look at two cases for $b_1(t)$.

Case 1 $b_1(t) = 0$
In this case the solution of Equation (2.6) is

$$h(X,t) = b_2(t)\int \frac{1}{g(X,t)} dX, \tag{2.7}$$

where we have chosen the arbitrary function of integration to be zero, because in stochastic analysis change of variable is made via Itô's lemma Equation (2.3), therefore, ignoring an arbitrary function of t does not affect the transformation. Substitution of this h into Equation (2.4) yields

$$\dot{b}_2 \int \frac{dX}{g} + b_2 \left(\int \frac{\partial}{\partial t} \left(\frac{1}{g} \right) dX + \frac{f}{g} - \frac{1}{2} \frac{\partial g}{\partial X} \right) = a_1(t)Y + a_2(t), \tag{2.8}$$

where $\dot{b}_2$ denotes the derivative of b_2 with respect to t. Let

$$\int \frac{\partial}{\partial t} \left(\frac{1}{g} \right) dX + \frac{f}{g} - \frac{1}{2} \frac{\partial g}{\partial X} = \alpha(t)Y + \beta(t). \tag{2.9}$$

Then Equation (2.8) becomes

$$\left(\frac{\dot{b}_2}{b_2} + b_2\alpha(t) \right) Y + b_2\beta(t) = a_1(t)Y + a_2(t)$$

which implies that

$$\dot{b}_2 + b_2^2\alpha(t) = b_2 a_1(t), \;\; b_2\beta(t) = a_2(t). \tag{2.10}$$

We choose $b_2(t) = 1$ and so from the above equation we have $a_1(t) = \alpha(t)$ and $a_2(t) = \beta(t)$. Equation (2.2) now becomes

$$dY = (\alpha(t)Y + \beta(t))dt + dW + (c_1(t)Y + c_2(t))dP. \tag{2.11}$$

However, one can obtain a simpler linear JDSDE by setting $a_1(t) = 0$. (Of course the linear JDSDE with $a_1(t) = 0$ is not equivalent to the general linear equation. But this choice leads us to determine some of the coefficient functions of the linear JDSDE beforehand. Most importantly this choice does not affect the linearization conditions at all.) This leads to

$$\dot{b}_2 + b_2^2\alpha(t) = 0,$$

a solution of which is

$$b_2(t) = \left(\int \alpha(t)dt \right)^{-1}. \tag{2.12}$$

Here we have taken the constant of integration to be zero as this choice does not affect the linearization criteria. The linear JDSDE now becomes

$$dY = \beta(t)b_2(t)dt + b_2(t)dW + (c_1(t)Y + c_2(t))dP(t). \tag{2.13}$$

The transformation (2.7) now takes the form

$$Y = h(X, t) = \left(\int \alpha(t)dt \right)^{-1} \int \frac{1}{g(X, t)} dX. \tag{2.14}$$

Differentiation of Equation (2.9) with respect to X yields

$$\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial X}\left(\frac{f}{g}-\frac{1}{2}\frac{\partial g}{\partial X}\right)=\alpha(t)\frac{\partial h}{\partial X}. \tag{2.15}$$

Substitution of

$$\frac{\partial h}{\partial X}=\frac{1}{g\left(\int\alpha(t)dt\right)} \tag{2.16}$$

(from Equation (2.14)) into Equation (2.15)) leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial X}\left(\frac{f}{g}-\frac{1}{2}\frac{\partial g}{\partial X}\right)=\alpha(t)\frac{1}{g\left(\int\alpha(t)dt\right)}. \tag{2.17}$$

After multiplying both sides of (2.17) by $g(X,t)$ we rewrite it as

$$g(X,t)L=\frac{\alpha(t)}{\displaystyle\int\alpha(t)dt}, \tag{2.18}$$

where

$$L=\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial X}\left(\frac{f}{g}-\frac{1}{2}\frac{\partial g}{\partial X}\right). \tag{2.19}$$

Differentiation of Equation (2.18) with respect to X gives

$$\frac{\partial}{\partial X}\left[g(X,t)L\right]=0. \tag{2.20}$$

Now consider Equation (2.5), viz.

$$h(X+r(X,t),t)=(c_1(t)+1)h(X,t)+c_2(t), \tag{2.21}$$

and differentiate it with respect to X. We obtain

$$\left(\frac{\partial}{\partial X}h(X+r(X,t),t)\right)\left(\frac{\partial r(X,t)}{\partial X}+1\right)=\frac{\partial}{\partial X}h(X,t)(c_1(t)+1). \tag{2.22}$$

From Equation (2.7) we have

$$\frac{\partial}{\partial X}h(X,t)=\frac{b_2}{g(X,t)} \tag{2.23}$$

and so Equation (2.22) becomes

$$\frac{b_2}{g(X+r(X,t),t)}\left(\frac{\partial r(X,t)}{\partial X}+1\right)=\frac{b_2}{g(X,t)}(c_1(t)+1).$$

We rewrite the above equation as

$$A=c_1(t)+1, \tag{2.24}$$

where

$$A = \left(\frac{\partial r(X,t)}{\partial X} + 1\right)\frac{g(X,t)}{g(X + r(X,t),t)}.$$

Differentiation of Equation (2.24) with respect to X yields

$$\frac{\partial}{\partial X}A = 0. \tag{2.25}$$

Thus the Equations (2.20) and (2.25) are the linearization conditions.

Case 2 $b_1(t) \neq 0$
The solution of (2.6) in this case becomes

$$h(X,t) = -\frac{b_2(t)}{b_1(t)} + K(t)\exp\left[b_1(t)\int \frac{1}{g(X,t)}dX\right]. \tag{2.26}$$

Without loss of generality we choose $K(t) = 1$ and $b_2(t) = 0$ because it does not affect the linearization conditions. Thus

$$h(X,t) = \exp\left[b_1(t)\int \frac{1}{g(X,t)}dX\right]. \tag{2.27}$$

Substitution of $h(X,t)$ into Equation (2.4) yields

$$\left[\int \frac{\partial}{\partial t}\left(\frac{b_1(t)}{g(X,t)}\right)dX + b_1\left(\frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X}\right) + \frac{b_1^2}{2}\right]\exp\left[b_1(t)\int \frac{1}{g(X,t)}dX\right] = a_1(t)Y + a_2(t). \tag{2.28}$$

We set

$$T\exp\left[b_1(t)\int \frac{1}{g(X,t)}dX\right] = \gamma(t)Y + \delta(t), \tag{2.29}$$

where

$$T = \dot{b}_1(t)\int \frac{1}{g(X,t)}dX + b_1(t)\int \frac{\partial}{\partial t}\left(\frac{1}{g(X,t)}\right)dX + b_1\left(\frac{f}{g} - \frac{1}{2}\frac{\partial g}{\partial X}\right) + \frac{b_1^2}{2}. \tag{2.30}$$

We now have

$$\gamma(t)Y + \delta(t) = a_1(t)Y + a_2(t),$$

which leads to

$$a_1(t) = \gamma(t),\ \ a_2(t) = \delta(t).$$

Differentiation of Equation (2.29) with respect to X yields

$$\frac{\partial T}{\partial X} + \frac{b_1(t)}{g(X,t)}T = \gamma(t)\frac{b_1(t)}{g(X,t)}, \tag{2.31}$$

and from Equation (2.30) we obtain

$$\frac{\partial T}{\partial X} = \frac{\dot{b}_1(t)}{g(X,t)} + b_1(t)L. \tag{2.32}$$

From Equations (2.31) and (2.32) we have

$$\dot{b}_1(t) + b_1(t)g(X,t)L + b_1(t)T = \gamma(t)b_1(t). \tag{2.33}$$

By first differentiating Equation (2.33) with respect to X and then using Equation (2.32), we obtain

$$\dot{b}_1(t) + g(X,t)\frac{\partial}{\partial X}(g(X,t)L) + b_1(t)g(X,t)L = 0. \tag{2.34}$$

Differentiation of the above equation with respect to X gives

$$\frac{\partial}{\partial X}\left(g(X,t)\frac{\partial}{\partial X}(g(X,t)L)\right) + b_1(t)\frac{\partial}{\partial X}(g(X,t)L) = 0.$$

Solving for $b_1(t)$, we obtain

$$b_1(t) = -\frac{\frac{\partial}{\partial X}\left(g(X,t)\frac{\partial}{\partial X}(g(X,t)L)\right)}{\frac{\partial}{\partial X}(g(X,t)L)} \tag{2.35}$$

and differentiation with respect to X yields

$$\frac{\partial}{\partial X}M = 0, \tag{2.36}$$

where

$$M(t) = \frac{\frac{\partial}{\partial X}\left[g(X,t)\frac{\partial}{\partial X}(g(X,t)L)\right]}{\frac{\partial}{\partial X}[g(X,t)L]}.$$

The transformation given in Equation (2.27) is

$$h(X,t) = \exp\left(b_1(t)\int \frac{1}{g(X,t)}dX\right). \tag{2.37}$$

Differentiation of Equation (2.37) with respect to X yields

$$\frac{\partial}{\partial X}h(X,t) = \frac{b_1(t)}{g(X,t)}\ \exp\left(b_1(t)\int \frac{1}{g(X,t)}dX\right). \tag{2.38}$$

Substitution of (2.38) into (2.22) leads to

$$\frac{b_1(t)}{g(X+r(X,t),t)}\exp\left(b_1(t)\int \frac{1}{g(X+r(X,t),t)}dX\right)\left(\frac{\partial r(X,t)}{\partial X}+1\right)$$

$$= \frac{b_1(t)}{g(X,t)} \exp\left(b_1(t) \int \frac{1}{g(X,t)} dX\right) (c_1(t)+1) \tag{2.39}$$

or in terms of A we have

$$A \exp\left(b_1(t)\left(\int \frac{1}{g(X+r(X,t),t)} dX - \int \frac{1}{g(X,t)} dX\right)\right) = (c_1(t)+1)\,. \tag{2.40}$$

Differentiation of Equation (2.40) with respect to X yields

$$\left(\frac{\partial}{\partial X} A\right) + A b_1 \left(\frac{1}{g(X+r(X,t),t)} - \frac{1}{g(X,t)}\right) = 0$$

and solving for $b_1(t)$ we obtain

$$-b_1(t) = \frac{\left(\frac{\partial}{\partial X} A\right)}{A\left(\frac{1}{g(X+r(X,t),t)} - \frac{1}{g(X,t)}\right)}. \tag{2.41}$$

Now differentiation of the above equation with respect to X yields

$$\frac{\partial}{\partial X} B = 0, \tag{2.42}$$

where

$$B(t) = \frac{\left(\frac{\partial}{\partial X} A\right)}{A\left(\frac{1}{g(X+r(X,t),t)} - \frac{1}{g(X,t)}\right)}.$$

Thus Equations (2.36) and (2.42) are the linearization conditions.

We now state the theorem which we have proved above.

Theorem 1. *The nonlinear jump-diffusion stochastic ordinary differential equation*

$$dX = f(X,t)dt + g(X,t)dW(t) + r(X,t)dP(t),$$

is linearizable if and only if the conditions (2.20), (2.25) or (2.36), (2.42) are satisfied.

3 Solution to linear JDSDEs

Although there are other methods such as variation of parameters available (see for instance [11]) for the integration of one-dimensional Itô SODEs, here we make use of stochastic integrating factors for linear JDSDEs.

Definition 1. The function $\mu = \mu(t, W, P)$ with the property

$$d(\mu Y) = D_1(t)dt + D_2(t)dW + D_3(t)dP$$

is called an integrating factor for the one-dimensional linear JDSDE (2.2).

We now consider the chain rule [8]

$$d(\mu Y) = \mu dY + Y d\mu + dY d\mu. \tag{3.1}$$

Here $d\mu$ and $d\mu dY$ are given by [7]

$$d\mu = \left(\frac{\partial \mu}{\partial t} + \frac{1}{2}\frac{\partial^2 \mu}{\partial W^2}\right) dt + \frac{\partial \mu}{\partial W} dW + (\mu(t, W, P+1) - \mu(t, W, P))dP \tag{3.2}$$

and

$$d\mu dY = \frac{\partial \mu}{\partial W}(b_1 Y + b_2)dt + (\mu(P+1) - \mu(P))(c_1 Y + c_2)dP, \tag{3.3}$$

respectively, where

$$\mu(t, W, P) = \mu(P) \qquad \text{and} \qquad \mu(t, W, P+1) = \mu(P+1).$$

Using the multiplication rules [7] we obtain

$$dtdP(t) = 0, \quad dP(t)dW(t) = 0, \quad dP^m(t) = dP,$$

$$dtdW(t) = 0, \quad dW(t)dW(t) = dt \quad \text{and} \quad dW^m(t) = 0.$$

Equation (3.1) can now be written as

$$\begin{aligned} d(\mu Y) \;\; = \;\; & \left\{\left[a_1(t)\mu + \frac{\partial \mu}{\partial t} + \frac{1}{2}\frac{\partial^2 \mu}{\partial W^2} + \frac{\partial \mu}{\partial W} b_1(t)\right] Y + (a_2(t)\mu + \frac{\partial \mu}{\partial W} b_2(t))\right\} dt \\ & + \left\{\left[b_1(t)\mu + \frac{\partial \mu}{\partial W}\right] Y + b_2(t)\mu\right\} dW \\ & + \{[c_1(t)\mu + (\mu(P+1) - \mu(P)) + c_1(t)(\mu(P+1) - \mu(P))]\, Y \\ & + [c_2(t)\mu + c_2(t)(\mu(P+1) - \mu(P))]\} dP. \end{aligned} \tag{3.4}$$

The right hand side of the above equation should not involve the variable Y to comply with the definition of the integrating factor. Thus we must have

$$a_1(t)\mu + \frac{\partial \mu}{\partial t} + \frac{1}{2}\frac{\partial^2 \mu}{\partial W^2} + \frac{\partial \mu}{\partial W} b_1(t) = 0, \tag{3.5}$$

$$b_1(t)\mu + \frac{\partial \mu}{\partial W} = 0 \text{ and} \tag{3.6}$$

$$(1 + c_1(t))\mu(P+1) = \mu(P). \tag{3.7}$$

We now solve the Equations (3.5)−(3.7). To achieve this goal we let

$$\mu(t, W, P) = M_1(t, W) M_2(t, P).$$

Equation (3.7) now is

$$(1 + c_1(t)) M_2(P+1) = M_2(P). \tag{3.8}$$

By letting $M_2 = \alpha^P$ (no extra constant of integration is needed in integrating factor) in Equation (3.8) we obtain

$$\alpha = \frac{1}{1 + c_1(t)},$$

which leads to

$$M_2(t, P) = \frac{1}{(1 + c_1(t))^P}. \tag{3.9}$$

The solution to Equation (3.6) is

$$\mu(t, W, P) = \frac{\exp[-b_1(t)W_t + q(t)]}{(1 + c_1(t))^P}. \tag{3.10}$$

Substitution of (3.10) into Equation (3.5) leads to

$$\frac{dq}{dt} - \frac{b_1^2(t)}{2} + a_1(t) - \frac{P\dot{c}_1(t)}{(1 + c_1(t))} = 0$$

the solution of which is

$$q(t) = \frac{1}{2}\int_0^t b_1^2(s)ds - \int_0^t a_1(s)ds + \int_0^t \frac{P(s)\dot{c}_1(s)}{1 + c_1(s)}ds. \tag{3.11}$$

Thus from Equations (3.10) and (3.11) we obtain

$$\mu = (1 + c_1(t))^{-P} \exp\left[\frac{1}{2}\int_0^t b_1^2(s)ds - \int_0^t a_1(s)ds - b_1(t)W_t + \int_0^t \frac{P(s)\dot{c}_1(s)}{1 + c_1(s)}ds\right]. \tag{3.12}$$

Invocation of (3.12) in (3.4) gives

$$d(\mu Y) = (a_2(t) - b_1(t)b_2(t))\mu(P)dt + b_2(t)\mu(P)dW + c_2(t)\mu(P + 1)dP \tag{3.13}$$

and integration yields the solution

$$Y = \frac{1}{\mu}\left[\int_0^t (a_2(s) - b_1(s)b_2(s))\mu(P)ds + \int_0^t b_2(s)\mu(P)dW + \int_0^t c_2(s)\mu(P + 1)dP\right]. \tag{3.14}$$

4 Applications

We now consider some linearizable nonlinear nonautonomous jump-diffusion equations. We give calculations in detail for the first example. The other examples are partially discussed and the solutions are given in Table 1. Our first example is from [9] and is

$$dX = \frac{1}{3}X^{\frac{1}{3}}dt + X^{\frac{2}{3}}dW + \left(\frac{3}{2}tX^{\frac{2}{3}} + \frac{3}{4}t^2X^{\frac{1}{3}} + \frac{1}{8}t^3\right)dP, \quad X(0) = X_0. \tag{4.1}$$

As it is seen below, Equation (4.1) satisfies the linearization criteria (2.20) and (2.25).

We have

$$\frac{\partial}{\partial X}\left[g(X,t)L\right]=\frac{\partial}{\partial X}\left[X^{\frac{2}{3}}\left(\frac{\partial}{\partial t}\left(X^{-\frac{2}{3}}\right)+\frac{\partial}{\partial X}\left(\frac{1}{3}X^{-\frac{1}{3}}-\frac{1}{2}\frac{\partial}{\partial X}X^{\frac{2}{3}}\right)\right)\right]=0$$

and

$$\frac{\partial}{\partial X}A=\frac{\partial}{\partial X}\left[\left(\frac{3}{2}tX^{-\frac{1}{3}}+\frac{1}{4}t^{2}X^{-\frac{2}{3}}+1\right)\frac{X^{\frac{2}{3}}}{\left(X^{\frac{1}{3}}+\frac{1}{2}t\right)^{2}}\right]=0.$$

Hence the transformation

$$Y=h(X)=\int_{X_0}^{X}\frac{1}{X^{\frac{2}{3}}}dX=3X^{\frac{1}{3}}-3X_0^{\frac{1}{3}}$$

linearizes the Equation (4.1) and we obtain

$$dY=dW+\frac{3}{2}tdP.$$

Integration gives

$$Y=W_t+\frac{3}{2}\int_0^t sdP(s).$$

Hence the solution X_t is given by

$$X_t=\left[X_0^{\frac{1}{3}}+\frac{1}{3}W_t+\frac{1}{2}\int_0^t sdP(s)\right]^3.$$

Our second example is taken from [2] but with an additional jump term and is

$$dX=\alpha\left(\beta-x\right)dt+\sigma X^{\frac{1}{2}}dW+\left(\sigma X^{\frac{1}{2}}t+\alpha\beta t^{2}\right)dP,\ \ X(0)=X_{0.} \tag{4.2}$$

Using the transformation

$$Y=\frac{1}{\sqrt{\alpha\beta}}\left(X^{\frac{1}{2}}-X_0^{\frac{1}{2}}\right),\ \ \left(\sigma=2\sqrt{\alpha\beta}\right), \tag{4.3}$$

we see that Equation (4.2) is transformed into

$$dY=-\frac{\alpha}{2}Ydt+dW+tdP. \tag{4.4}$$

Integration of (4.4) and use of (4.3) lead to the solution given in Table 1. The third example which we present is from [11]. It is a mean-reverting Ornstein-Uhlenbeck equation with an additional jump term. We have

$$dX=\mu X\left(\theta\left(t\right)-\ln X\right)dt+\rho X^{\frac{1}{2}}dW+\zeta\left(t\right)XdP,\ \ X(0)=X_{0.} \tag{4.5}$$

Table 1: 1

Linearizable Equations and Solutions

Equation	$dX=\alpha\left(\beta-X\right)dt+\sigma X^{\frac{1}{2}}dW+\left(\sigma X^{\frac{1}{2}}t+\alpha\beta t^{2}\right)dP$
Criteria (2.20,2.25)	Solution $X_t=\left(X_0^{\frac{1}{2}}+\sqrt{\alpha\beta}e^{\frac{-\alpha t}{2}}\left[\int\limits_0^t e^{\frac{\alpha}{2}s}dW_s+\int\limits_0^t se^{\frac{\alpha}{2}s}dP_s\right]\right)^2$
Equation	$dX=\mu X\left(\theta\left(t\right)-\ln X\right)dt+\rho X^{\frac{1}{2}}dW+\zeta\left(t\right)XdP$
Criteria (2.20,2.25)	Solution $X_t=X_0\exp\left[\rho e^{-\mu t}\left(\int\limits_0^t(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho})e^{\mu s}ds+\int\limits_0^t e^{\mu s}dW_s+\frac{1}{\rho}\int\limits_0^t\ln\left(1+\zeta\left(s\right)\right)e^{\mu s}dP_s\right)\right]$
Equation	$dX=\xi\left(t\right)X\left(\eta\left(t\right)-X\right)dt+\delta XdW+\lambda\left(t\right)XdP$
Criteria (2.36,2.42)	Solution $X_t=X_0\Psi\left(t\right)\left(\int\limits_0^t\xi\left(s\right)\Psi\left(s\right)ds\right)^{-1}$, $\Psi\left(t\right)=(\frac{1}{\lambda(t)+1})^{-P_t}\exp\left[\int\limits_0^t\left(\xi\left(s\right)\eta\left(s\right)+\frac{\dot{\lambda}(s)}{\lambda(s)+1}P_s\right)ds-\frac{1}{2}\delta^2 t-\delta W_t\right]$
Equation	$dX=\left(a\left(t\right)X^{\frac{3}{4}}+\frac{3}{8}b^2X^{\frac{1}{2}}\right)dt+bX^{\frac{3}{4}}dW+\left(\left(X^{\frac{1}{4}}+bt^{\frac{1}{3}}\right)^4-X\right)dP$
Criteria (2.20,2.25)	Solution $X_t=\left[X_0^{\frac{1}{4}}+\frac{1}{4}\int\limits_0^t a\left(s\right)ds+\frac{b}{4}W_t+b\int\limits_0^t s^{\frac{1}{3}}dP_s\right]^4$

The transformation given by

$$Y=\frac{1}{\rho}\ln\frac{X}{X_0} \tag{4.6}$$

linearizes Equation (4.5) into

$$dY=\left(-\mu Y+\frac{\rho}{2}+\frac{\mu\theta\left(t\right)}{\rho}\right)dt+dW+\frac{1}{\rho}\ln\left(1+\zeta\left(t\right)\right)dP. \tag{4.7}$$

Integration of Equation (4.7) can be done by the aid of an integrating factor described above. Finally the use of (4.6) leads to the solution which is given in Table 1. Our next example is the population growth model in a noisy environment with a jump term [5]. Here

$$dX=\xi\left(t\right)X\left(\eta\left(t\right)-X\right)dt+\delta XdW+\lambda\left(t\right)XdP,\ \ X(0)=X_{0.} \tag{4.8}$$

The transformation

$$Y=\left(\frac{X}{X_0}\right)^{-1} \tag{4.9}$$

linearizes Equation (4.8) into

$$dY=\left[\left(\delta^2-\xi\left(t\right)\eta\left(t\right)\right)Y+\xi\left(t\right)\right]dt+\delta YdW+\left(-\frac{\lambda\left(t\right)}{\lambda\left(t\right)+1}\right)YdP. \tag{4.10}$$

Integration of (4.10) via integrating factors and use of (4.9) lead to the solution given in Table 1. The final example we give is from [7] and in this example we have

$$dX = \left(a\left(t\right)X^{\frac{3}{4}} + \frac{3}{8}b^2X^{\frac{1}{2}}\right)dt + bX^{\frac{3}{4}}dW + \left(\left(X^{\frac{1}{4}} + bt^{\frac{1}{3}}\right)^4 - X\right)dP, \;\; X(0) = X_{0.} \quad (4.11)$$

The transformation

$$Y = \frac{4}{b}\left(X^{\frac{1}{4}} - X_0^{\frac{1}{4}}\right) \quad (4.12)$$

linearizes Equation (4.11) into

$$dY = \frac{a\left(t\right)}{b}dt + dW + 4t^{\frac{1}{3}}dP. \quad (4.13)$$

As before integration of (4.13) and use of the transformation (4.12) lead to the solution given in Table 1.

5 Conclusions

Linearization conditions of the one-dimensional nonautonomous JDSDEs are given. An integrating factor has been introduced to integrate (solve) the linear JDSDEs. Several examples of linearizable nonlinear nonautonomous JDSDEs are given and their closed form solutions are obtained.

Acknowledgements The authors thank the referee for valuable comments and suggestions that helped improve the paper. G. Ünal would like to thank C. M. Khalique and M. McPherson for their kind hospitality during his visit to IISAMM, North-West University, Mafikeng Campus, South Africa, where most of this work was done.

References

[1] Berkovich, L M, The method of an exact linearization of n-order ordinary differential equations, *Nonlinear Mathematical Physics* **3** (1996), 341-350.

[2] Cox J C, Ingersoll J E and Ross S A, A Theory of the Term Structure of Interest Rates, *Econometrica* **53** (1985), 385-408.

[3] Euler N and Euler M, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, *Journal of Nonlinear Mathematical Physics*, **11** (2004), 399-421.

[4] Euler N, Wolf T, Leach P G and Euler M, Linearizable third-order ordinary differential equations and generalized Sundman transformations: The Case $X''' = 0$, *Acta Applicandae Mathematicae*, **6** (2003), 89-115.

[5] Gard T C, Introduction to Stochastic Differential Equations, Marcel Dekker, New York, 1988.

[6] Gardiner C W, Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences, Springer-Verlag, Berlin, 1985.

[7] Hanson F B, Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation, http://www.math.uic.edu/˜hanson/pub/SIAMbook, 2006.

[8] Ikeda N and Watanabe S, Stochastic Differential Equations and Diffusion Processes, Second Edition, North-Holland, Amsterdam, 1989.

[9] Maghsoodi Y, Mean Square Efficient Numerical Solution of Jump-Diffusion Stochastic Differential Equations, *The Indian Journal of Statistics*, **A 58** (1996), 25-47.

[10] Mikosch T, Elementary Stochastic Calculus with Finance in View, World Scientific Publication, Singapore, 1998.

[11] Oksendal B, Stochastic Differential Equations: An Introduction with Applications, Sixth Edition, Springer -Verlag, Berlin, 2003.

[12] Wafo Soh C and Mahomed F M, Integration of Stochastic Ordinary Differential Equations From a Symmetry Standpoint, *Journal of Physics A: Mathematics and General.* **34** (2001), 177-192.

[13] Wafo Soh C and Mahomed F M, Linearization criteria for a system of second-order ordinary differential equations, *International Journal of Non-Linear Mechanics* **36** (2001), 671-677.