Euler-Poincaré Formalism of (Two Component) Degasperis-Procesi and Holm-Staley type Systems

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Abstract

In this paper we propose an Euler-Poincaré formalism of the Degasperis and Procesi (DP) equation. This is a second member of a one-parameter family of partial differential equations, known as b-field equations. This one-parameter family of pdes includes the integrable Camassa-Holm equation as a first member. We show that our Euler-Poincaré formalism exactly coincides with the Degasperis-Holm-Hone (DHH) Hamiltonian framework. We obtain the DHH Hamiltonian structues of the DP equation from our method. Recently this new equation has been generalized by Holm and Staley by adding viscosity term. We also discuss Euler-Poincaré formalism of the Holm-Staley equation. In the second half of the paper we consider a generalization of the Degasperis and Procesi (DP) equation with two dependent variables. we study the Euler-Poincaré framework of the 2–component Degasperis-Procesi equation. We also mention about the b-family equation.

Dedicated to Professor Darryl Holm on his 60th birthday

1 Introduction

We consider the following one parameter equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where b is a constant parameter. This equation is called b-field equations. Equation (1) was introduced in Degasperis, Holm and Hone [10, 11] based on Degasperis and Procesi [9] who singled out the cases $b = 2$ for the Camassa-Holm equation [3, 4] and $b = 3$ for the Degasperis-Procesi (DP) equation, and special attention has been paid by seeking asymptotic symmetries in a more general equation. Later these cases $b = 2$ and $b = 3$ were also shown by different methods in Degasperis, Holm and Hone [10, 11] and in Mikhailov and Novikov [37] to be the only candidates for integrability in the b-family of equations.

Incidentally $b = 2$ case was later recognized as being included in a class of integrable equations derived from hereditary symmetries in Fokas and Fuchssteiner [15].

Equation (1) can also be derived and shown to belong to an asymptotically equivalent family of equations by using Kodama's normal form transformations [30, 31] of the equations that emerge from shallow water asymptotics in Dullin et al. [12, 13]. They have given a clear explanation of how the CH equation arises from asymptotic expansions for shallow water motion.

The parameter b may take any value except -1, for which the asymptotic ordering for shallow water is broken. Incidentally the CH equation was recently rederived as a shallow water equation by using asymptotic methods in three different approaches by Fokas and Liu [16], by Dullin et al. [12, 13] and also by Johnson [26]. These three derivations used different variants of the method of asymptotic expansions for shallow water waves in the absence of surface tension.

The paper of Dullin et al. [13] is very interesting. They have derived an entire family of shallow water wave equations that are asymptotically equivalent to equation (1) at quadratic order in the shallow water expansion parameters. This is one order beyond the linear asymptotic expansion for the KdV equation. The family of asymptotically equivalent shallow water wave equations at quadratic order in this family are related amongst themselves by a continuous group of nonlinear and nonlocal transformations of variables that was first introduced by Kodama [30, 31].

The Degasperis-Procesi (DP) equation was considered for the first time in [9]. Recently Degasperis, Holm and Hone [10] proved the exact integrability of the DP equation by constructing a Lax pair and showed the existence of solutions that are superpositions of multipeakons which are compared with the multipeakon dynamics of the Camassa-Holm equation. Most recently Lundmark and Szmigielski [33, 34] used an inverse scattering approach to determine a completely explicit formula for the general n-peakon solution of the DP equation.

We start our paper by introducing the quantity

$$m = u - u_{xx},$$

which is just the Helmholtz operator action on u. So the one parameter family of PDEs (or b- field equations) may be rewritten in the form

$$m_t + um_x + bu_x m = 0. \tag{1.2}$$

In this b-family of 1D shallow water equations Holm and Staley [23, 24] added viscous dissipation term to study the theoretical and numerical solutions and their behavior. This generalized equation is given by

$$\underbrace{m_t}_{\text{evolution}} + \underbrace{um_x}_{\text{convection}} + \underbrace{bu_x m}_{\text{stretching}} = \underbrace{\nu m_{xx}}_{\text{viscosity}}. \tag{1.3}$$

As shown by Holm and Staley that the solutions of the HS family of b-field equations exhibit stable Burgers ramp/cliff solution structure for the parameter range $-1 < b < 1$. It may be worth to recall that the dispersionless limit of the KdV equation, upon rescaling velocity u, yields the Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0,$$

in which we now add constant viscosity ν. The characteristic of the Burgers solution is classic ramp and cliff solutions.

The ramp/cliff solutions bifurcate to exchange stability with other classes of solutions at $b = \pm 1$. The peakon stability is proved for $b = 2$ in [8] and appears to be true for all $b > 1$. The Holm-Staley b-equation introduce another class of stable leftward moving solutions for $b < -1$. The latter solutions are the "leftons," whose evolution Holm and Staley showed numerically approaches a stationary solution, although the leftons may be moving exponentially slowly at asymptotic times. They have found the solutions of the equation (3) for fluid velocity $u(x,t)$ change their behavior at the special values $b = 0, \pm 1, \pm 2, \pm 3$.

The multicomponent integrable systems have drawn a lot of attention in the last two decades. Recently much attention has also been paid to the 2-component Camassa–Holm equation and its various 2-component generalization [14, 5, 18, 43]. The two component generalization of the Degasperis–Procesi equation is *not so well known* integrable nonlinear equation in $(1 + 1)$–dimension. In fact this is one of the main theme of this paper. Recently, the author and Olver [18] derived the 2-component Camassa–Holm equation and corresponding $N = 1$ super generalization as geodesic flows with respect to the H^1 metric on the extended Bott-Virasoro and superconformal groups respectively. Falqui [14] provided an alternative derivation of the 2-component Camassa–Holm equation based on the theory of Hamiltonian structures on dual of a Lie algebra, and the Lie algebra involved in this method is the same algebra underlying the nonlinear Schrödinger hierarchy.

Most recently, Popowicz [43] presented two different Hamiltonian extensions of the Degasperis - Procesi equation to the two component equations. His construction based on the observation that the second Hamiltonian operator of the Degasperis - Procesi equation could be considered as the Dirac reduced Poisson tensor of the second Hamiltonian operator of the Boussinesq equation.

About ten years ago, Rosenau, [44], introduced a class of solitary waves with compact support as solutions of certain wave equations with nonlinear dispersion. It was found that the solutions of such systems unchanged from collision and were thus called *compactons*. The discovery that solitons may compactify under nonlinear dispersion inspired further investigation of the role of nonlinear dispersion. It has been known for some time that nonlinear dispersion causes wave breaking or lead to the formation of corners or cusps. Beyond compactons, a wide variety of other exotic non-analytic solutions, including peakons, cuspon, mesaons, etc., have been found in to exist in a variety of models that incorporate nonlinear dispersion.

Later, Olver and Rosenau showed [42] that a simple scaling argument shows that most integrable bihamiltonian systems are governed by tri-Hamiltonian structures. They formulated a method of "tri-Hamiltonian duality", in which a recombination of the Hamiltonian

operators leads to integrable hierarchies endowed with nonlinear dispersion that supports compactons or peakons.

We adopt the same technique to develop the EP formalism of the two component Degasperis-Procesi systems, In fact, one of the main reason that we are shifting to Lie derivative approach is that there is no equivalent description of EP flows on the space (matrix) first order differential operators in terms of coadjoint orbit.

1.1 Motivation and organization

Degasperis, Holm and Hone proposed a (bi)Hamiltonian formulation of DP equation. But untill now it is not known whether these Hamiltonian structures are related to any Lie-Poisson structures of any Euler-Poincaré flows. In this paper we discuss the Euler-Poincaré formalism of the Degasperis-Procesi and the Holm-Staley type nonlinear evolution equations and its connection to Degasperis-Holm-Hone Hamiltonian structures.

Infinite-dimensional Lie groups arise naturally in many branches of mathematics and its applications. In the present paper we consider one of the simplest infinite-dimensional Lie group, the group of orientation-preserving diffeomorphisms of the circle. It plays a very important role in conformal field theory, string theory and statistical mechanics. The group $Diff(S^1)$ of smooth orientation-preserving diffeomorphisms of the circle S^1 is endowed with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$. The composition and inverse are both smooth maps so that $Diff(S^1)$ is a Lie group modeled on Fréchet space. Its Lie algebra is the space of smooth vector fields $Vect(S^1)$ on a circle S^1. The Lie algebra of vector fields $Vect(S^1)$ admits an essentially unique central extension, known as the Gelfand-Fuchs cocycle. This extended new algebra is called the Virasoro algebra. This central extension can be lifted to a central extension of the Lie group of orientation-preserving diffeomorphisms of the circle. The corresponding group is called the Bott-Virasoro group[27, 28]. The relationship between geodesic flows on Bott-Virasoro group and integrable systems are known for quite sometimes, for example, the Korteweg-de Vries (KdV) equation appears naturally in the geometry of the Bott-Virasoro group. It is a geodesic flow with respect to $L2$ metric on the Bott-Virasoro group [40]. Similarly the derivation of the Camassa-Holm equation as a geodesic flows was proved by Misiolek [38].

Later in an excellent review Segal [45] presented a clear way the Poisson structure for the KdV equation and its relationship to the family of Hill operators, how KdV is the Euler equation on a specific coadjoint orbit of the central extension of the orientation-preserving diffeomorphisms of the circle and, moreover, how this orbit appears as a symplectic reduction from the space of $PSL_2(\mathbf{R})$-connections in the bundle of osculating projective lines of the circle.

Almost similar thing can be studied for the Burgers equation. One can even show [19, 20] that the Burgers equation also follows from Euler-Poincaré framework under the action of $Vect(S^1)$ on the space of first order differential operators. Unfortunately there is no equivalent description of the action of $Vect(S^1)$ on first order differential operators in terms of coadjoint action. So in this paper we reply heavily on Lie derivative formalism.

In fact, we also derived [21] the KdV-Burgers in this manner as an Euler-Poincaré flow on the combined space of Hill's (second order) and first order differential operators on circle.

1.1.1   Main result

In this paper we turn our attention to a one-parameter family partial differential equation which includes the Camassa-Holm equation. These are all connected to the flows on the space of first and second order differential operators with respect to H^1-Sobolev norm. Degasperis-Holm-Hone (DHH) demonstrated the bihamiltonian formalism of such systems. In this paper we extend their result further to show that these two Hamiltonian operators can be derived from Euler-Poincaré framework. Therefore the DHH bihamiltonian structures follow from the Lie-Poisson structure of the associated Euler-Poincaré flow. We also propose the Euler-Poincaré form of the Holm-Staley equation.

In the second half of the paper we consider a generalization of the Degasperis and Procesi (DP) equation with two independent variables. This also belongs to a two component one-parameter family of partial differential equations, known as two component b–family equation. In this paper we study the Euler-Poincaré framework of the 2–component Degasperis-Procesi equation. We consider two types of 2–component DP equation as studied by [43]. We obtain the Hamiltonian operators for the 2–component Degasperis-Procesi equation.

1.1.2   Organization

There are various ideas involved in this paper. In order to make it self content we give a brief review of all these materials. The paper is organized as follows: In Section 2 we basically put all the basic materials to study this paper. At first we briefly recapitulate basics of Lie-Poisson structure and Euler-Poincaré flows. We also demonstrate the coadjoint flows on the Virasoro orbit with respect to H^1-Sobolev norm. Later we switch to Lie derivative method to interpret the coadjoint flows on orbits. In Section 3 we describe coadjoint action with respect to the right invariant H^1 - metric and reinterpret such flows in terms of Lie derivative actions of $Vect(S^1)$ on the space of second order differential operators. In Section 4 we discuss Euler-Poincaré forms of Burgers and Whitham-Burgers equations. Our main result is content in Section 5. We discuss EP formalism of the Degasperis-Procesi equation. We also show that our formalism is compatible with the theory of Degasperis, Holm and Hone. We exhibit the bihamiltonian nature of DP equation. We also mention the EP formalism for the Holm-Staley equation. Section 6 is devoted to the construction of the Euler-Poincaré framework of two component Degasperis-Procesi systems.

1.2   Acknowledgement

2 Background

The Lie algebra $Vect(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}. \tag{2.1}$$

One parameter family of $Vect(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by [35, 42]

$$\mathcal{L}^{(\mu)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) + \mu f'(x)a(x), \tag{2.2}$$

where

$$\mathcal{L}^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} + \mu f'(x) \tag{2.3}$$

is the Lie derivative with respect to the vector field $f(x)\frac{d}{dx}$. Equation (5) implies a one parameter family of $Vect(S^1)$ action on the space of smooth functions $C^\infty(S^1)$.

Let us denote $\mathcal{F}_\lambda(M)$ the space of tensor-densities of degree λ:

$$\mathcal{F}_\lambda = \{a(x)dx^\lambda \mid a(x) \in C^\infty(S^1)\}.$$

Let Ω be the cotangent bundle of S^1, we say

$$\mathcal{F}_{-\lambda} \in \Gamma(\Omega^{\otimes\lambda}), \quad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

where Γ is the section of the line bundle $\Omega^{\otimes\lambda}$. Here $\mathcal{F}_0(M) = C^\infty(M)$, the space of $\mathcal{F}_1(M)$ and $\mathcal{F}_{-1}(M)$ coincide with the space differential forms and space of vector fields respectively.

Thus the equation (5) can be interpreted as an action of $Vect(S^1)$ on $\mathcal{F}_\mu(S^1)$, a tensor densities on S^1 of degree μ, and equation (4) can be interpreted as an action of $Vect(S^1)$ on $\mathcal{F}_{-1} \in \Gamma(\Omega^{-1})$. In this paper we will mainly consider the action of $Vect(S^1)$ on $\mathcal{F}_{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, square root of the tangent bundle

$$\mathcal{L}_{f(x)\frac{d}{dx}} h(x) = (f(x)h'(x) - \frac{1}{2}f'(x)h(x)), \tag{2.4}$$

where $h(x)\sqrt{\frac{d}{dx}} \in \Gamma(\Omega^{-\frac{1}{2}})$.

2.1   A quick review of coadjoint orbit

Let us briefly go through $Vect(S^1)$. Since the topological dual of the Fréchet space $Vect(S^1)$ is too big, we restrict our attention to the regular dual $\mathfrak{g}^*$, the subspace of $Vect(S^1)^*$ defined by linear functionals of the form

$$f(x)\frac{d}{dx} \longmapsto \int_{S^1} uf\,dx \tag{2.5}$$

for $u \in C^\infty(S^1)$. In fact dual is identified as quadratic differentials $u(x)dx^2 \in Vect(S^1)^*$.

The regular dual $Vect(S^1)^*$ is therefore isomorphic to $C^\infty(S^1)$ by means of the $L2$–innner product

$$< X, Y >= \int_{S^1} XY\,dx.$$

Let F be a smooth real-valued function on $C^\infty(S^1)$. The Fréchet derivative at m, $dF(u)$, is a linear functional on $C^\infty(S^1)$. If F is a regular function, i.e., there exists smooth map $\frac{\delta F}{\delta u} : C^\infty(S^1) \to C^\infty(S^1)$, then we define

$$dF(u) \cdot \mu = \left\langle \frac{\delta F}{\delta u}, \mu \right\rangle \equiv \int_{S^1} \frac{\delta F}{\delta u} \cdot \mu\,dx. \tag{2.6}$$

It is well known that the Virasoro algebra is the unique (upto isomorphism) non-trivial central extension of $Vect(S^1)$. It is given by the Gelfand-Fuchs cocycle

$$\omega(f(x)\frac{d}{dx}, g(x)\frac{d}{dx}) = \int_{S^1} f'(x)g''(x)dx. \tag{2.7}$$

The Virasoro algebra is therefore a Lie algebra on the space $Vect(S^1) \oplus \mathbf{R}$.

$$[(f\frac{d}{dx}, a), (g\frac{d}{dx}, b)] = ([f\frac{d}{dx}, g\frac{d}{dx}]_{Vect(S^1)}, \omega(f,g)), \tag{2.8}$$

where $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ is a two cocycle on the Lie algebra $\mathfrak{g}$. The skew symmetry and the Jacobi identity for the new bracket enforces the cocycle to satisfy

$$\omega([f\frac{d}{dx}, g\frac{d}{dx}], h\frac{d}{dx}) + \omega([h\frac{d}{dx}, f\frac{d}{dx}], g\frac{d}{dx}) + \omega([g\frac{d}{dx}, h\frac{d}{dx}], f\frac{d}{dx}) = 0 \tag{2.9}$$

and antisymmetric condition. Note that ω depends only on X and Y but not on a and b means that the extension is central, that is, the space $\mathbf{R}$ belongs to the centre of the new algebra.

The coadjoint action Ad^* of G on $\mathfrak{g}^*$, the dual of Lie algebra, is dual of the of the adjoint representation. The coadjoint action of the group G on the dual space $\mathfrak{g}^*$ is defined by

$$\left\langle Ad_g^*(\xi), X \right\rangle := \left\langle \xi, Ad_g(X) \right\rangle$$

for all $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. The differential

$$ad^* : \mathfrak{g} \to End(\mathfrak{g}^*)$$

of Ad^* at the identity id is called coadjoint orbit of the Lie algebra $\mathfrak{g}$ and explicitly it is given by

$$\langle ad_Y^*(\xi), X \rangle = \langle \xi, ad_Y X \rangle.$$

The dual of the Virasoro algebra is $C^\infty(S^1) \oplus \mathbf{R}$, and a pairing between this space and Virasoro algebra is given by

$$< (u(x), c), (f(x)\frac{d}{dx}, a) > \quad := \quad \int_{S^1} u(x)f(x)dx + ca. \tag{2.10}$$

Using the following equation

$$< ad^*_{(f(x)\frac{d}{dx},a)}(u(x),c), (g\frac{d}{dx}, b) > \quad = \quad < (u(x),c), ad_{(f(x)\frac{d}{dx},a)}(g\frac{d}{dx}, b) >$$

we obtain

$$ad^*_{(f(x)\frac{d}{dx},a)}(u(x),c) = \frac{1}{2}f''' + 2f'u + fu' = (\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)f.$$

This computaion yields the (second) Hamiltonian operator of the KdV equation

$$\mathcal{O}_{KdV} = \frac{1}{2}\partial 3 + \partial u + u\partial, \tag{2.11}$$

and the Euler–Poincaré flow on the Virasoro orbit yields the KdV equation.

2.2 Lie-derivative method: a different way of interpreting $Vect(S^1)$ action

By Lazutkin and Penkratova [32], the dual space of the Virasoro algebra can be identified with the space of Hill's operator or the space of projective connections

$$\Delta = \frac{d2}{dx^2} + u(x), \tag{2.12}$$

where u is a periodic potential: $u(x + 2\pi) = u(x) \in C^\infty(\mathbf{R})$. The Hill's operator maps

$$\Delta : \mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{3}{2}}. \tag{2.13}$$

Remark Notice that the operators Δ do not preserve their form under the action of $Diff(S^1)$, $x \to \sigma(x)$, due to the appearance of the first order term $-(\sigma''/\sigma'^3$. Hence we should think the operators are acting on *densities of weight* $-1/2$ rather than on scalar functions, in this case we can always find $u_1 = 0$ as a reparametrization invariant.

The action of $Vect(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta] := \mathcal{L}^{\frac{3}{2}}_{f(x)\frac{d}{dx}} \circ \Delta - \Delta \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{2.14}$$

Thus, right hand side denotes the coadjoint action of $Vect(S^1)$ on its dual Δ with respect to $L2$ norm on the space of algebra.

Lemma 2.1. *The Lie derivative action on Δ yields*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta]_{L2} = (\frac{1}{2}\partial^3 + 2u\partial + u')f, \tag{2.15}$$

and this gives the (second) Hamiltonian structure of the KdV equation

$$\mathcal{O}_{KdV} = (\frac{1}{2}\partial^3 + 2u\partial + u').$$

Proof: By direct computation.
□

Proposition 2.2. *The KdV equation is given by*

$$u_t = 2[\mathcal{L}_{u(x)\frac{d}{dx}}, \frac{d2}{dx^2} + u(x)] = u_{xxx} + 6uu_x. \tag{2.16}$$

2.3   Euler-Poincaré flows and Lie-Poisson structure

Before we are going to embark Euler-Poincaré equation let us quickly recapitulate few things about Lie-Poisson structures.

Lemma 2.3. *The Hamiltonian vector field on $\mu \in \mathfrak{g}^*$ corresponding to a Hamiltonian function f, computed with respect to the Lie-Poisson structure is given by*

$$\frac{d\mu}{dt} = ad^*_{df}\mu \tag{2.17}$$

Proof: It follows from the following identities

$$i_{X_f}dg|_\mu = L_{X_f}g|_\mu = \{f, g\}_{LP}(\mu)$$

$$= < [dg, df], \mu > = < dg, ad^*_{df}\mu > .$$

This implies that $X_f = ad^*_{df}\mu$. Thus the Hamiltonian equation $\frac{d\mu}{dt} = X_f$ yields our result.
□

We write $E(\mu) = \frac{1}{2} < \mu, I\mu >$ for the quadratic energy form on
$\mathfrak{g}$. $E(\mu)$ is used to define the Riemannian metric. We identify
the Lie algebra and its dual with this quadratic form. This identification
is done via an *inertia operator*.
Let I be an inertia operator

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and then $\mu \in \mathfrak{g}^*$ evolve by

$$\frac{d\mu}{dt} = (I^{-1}\mu).\mu, \tag{2.18}$$

where right hand side denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler-Poincaré equation.

Definition 2.4. *The Euler-Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2} < I^{-1}\mu, \mu >$ is given by*

$$\frac{d\mu}{dt} = -ad^*_{I^{-1}\mu}\mu.$$

It characterizes an evolution of a point $\mu \in \mathfrak{g}^*$.

Proposition 2.5. *Let ΩG be infinite dimensional Lie group equipped with a right invariant metric. A curve $t \longrightarrow c(t)$ in ΩG is a geodesic of this metric iff $\mu(t) = d_{c_t} R_{c_t}\dot{c}(t)$ satisfes*

$$\frac{d}{dt}\mu(t) = ad^*_{u(t)}\mu(t), \tag{2.19}$$

where R_{c_t} stands for right translation.

2.3.1 Frozen Lie-Poisson bracket

We also consider the dual of the Lie algebra of $\mathfrak{g}^*$ with a
Poisson structure given by the "frozen" Lie-Poisson structure.
In other words, we fix some point $\mu_0 \in \mathfrak{g}^*$ and define
a Poisson structure given by

$$\{f, g\}_0(\mu) := < [df(\mu), dg(\mu)], \mu_0 > \tag{2.20}$$

It has been shown in [29] that

Proposition 2.6. *The brackets $\{\cdot, \cdot\}_{LP}$ and $\{\cdot, \cdot\}_0$ are compatible for every "freezing" point μ_0.*

Proof: Any linear combination

$$\{\cdot, \cdot\}_\lambda := \{\cdot, \cdot\}_{LP} + \lambda\{\cdot, \cdot\}_0$$

is again a Poisson bracket. Indeed, it is just the translation of the Lie-Poisson bracket from the origin to the point $-\lambda\mu_0$.
$\square$

We can give another interpretation [6, 7] of frozen structure from the definition of cocycle. Given an inertia operator $I : \mathfrak{g} \to \mathfrak{g}^*$ one can define a constant Poisson structure

$$\{f, g\}_0(\mu) = \langle df, I\, dg \rangle \qquad \text{where} \qquad \mu \in \mathfrak{g}^*.$$

A two-cocycle ω is called a coboundary if there is a point $\mu_0 \in \mathfrak{g}^*$ such that

$$\omega(\mathcal{X}, \mathcal{Y}) = \langle [\mathcal{X}, \mathcal{Y}], \mu_0 \rangle, \tag{2.21}$$

where $\mathcal{X}, \mathcal{Y} \in \mathfrak{g}$. A special of Lie-Poisson structure is given by a two cocycle ω which is coboundary. The Lie-Poisson structure generated by a coboundary of $\omega = \partial \mu_0$ is given by

$$\{f, g\}_0(\mu) = \langle [df(\mu), dg(\mu)], \mu_0 \rangle, \qquad \mu_0 \in \mathfrak{g}^*.$$

This behaves like a Lie-Poisson structure frozen at the point $\mu_0 \in \mathfrak{g}^*$ and this coincides with the previous definition of frozen structure. In the case of Virasoro algebra, every two cocycle ω is a coboundary corresponding to a freezing point $\mu_0 \in Vir^*$. With $\mu_0 = (u_0, c_0)$, we get

$$\partial \mu_0 \big((f(x)\frac{d}{dx}, a_0), (g(x)\frac{d}{dx}, b_0) \big) = \big\langle \big[(f(x)\frac{d}{dx}, a_0), (g(x)\frac{d}{dx}, b_0) \big], (u_0, c_0) \big\rangle$$

$$= \int_{S^1} u_0 (fg_x - f_x g)\, dx + c_0 \int_{S^1} f' g''\, dx.$$

This yields an operator $\mathcal{O} : Vir \to Vir^*$, defines a constant Poisson structure given by the formula

$$\{f, g\}_0 = \langle \delta f, \mathcal{O} \delta g \rangle.$$

3   Coadjoint action with respect to H^1 norm and Lie derivative interpretation

Let us first consider standard approach of the coadjoint action of the Virasoro algebra on its dual with respect to H^1 norm.

On the Virasoro algebra we consider the H^1 inner product. Then a pairing between a point $(f(x)\frac{d}{dx}, \nu) \in Vir$ and a point $(udx^{\otimes 2}, \lambda)$ on the dual space is given by

$$< (f(x)\frac{d}{dx}, \nu), (udx^{\otimes 2}, \lambda) >_{H^1} = \lambda \nu + \int_{S^1} f(x) u(x) dx + \int_{S^1} f_x u_x. \qquad (3.1)$$

Let us compute the coadjoint action $ad_{\hat{f}}^* \hat{u}|_{H^1}$ of the vector field $(f(x)\frac{d}{dx}, \nu) \in Vir$ on its dual $(udx^{\otimes 2}, \lambda)$ with respect to H^1 norm.

Lemma 3.1.

$$ad_{\hat{f}}^* \hat{u}|_{H^1} = (1 - \partial^2)^{-1} [f(x)(1 - \partial 2)u_x + 2f'(1 - \partial^2)u + \lambda f'''], \qquad (3.2)$$

where $\hat{u} = (u\, dx^{\otimes 2}, \lambda)$ and $\hat{f} = (f(x)\frac{d}{dx}, \nu)$.

Proof: We know

$$< ad_{\hat{f}}^* \hat{u}, \hat{g} >_{H^1} = < \hat{u}, [\hat{f}, \hat{g}] >_{H^1} = < \hat{u}, -(fg' - f'g), \int_{S^1} f'g'' >_{H^1} \qquad (3.3)$$

Here we follow the sign convention of Marsden and Ratiu [36].

$$\text{R.H.S.} \quad = \quad - \int_{S^1} (ufg' - uf'g)dx - \int_{S^1} u'(fg' - f'g)'dx + \lambda \int_{S^1} f'g''dx$$

$$= \int_{S^1} [f(1 - \partial 2)u' + 2f'(1 - \partial 2)q + \lambda f''']g.$$

Let us compute now the L.H.S. of equation (27)

$$L.H.S. = \int_{S^1} (ad^*_{\hat{f}}\hat{u})g dx + \int_{S^1} (ad^*_{\hat{f}}\hat{u})' g' dx$$

$$= \int_{S^1} [(1 - \partial 2)ad^*_{\hat{f}}\hat{u}]g dx.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.
□

Corollary 3.2. *Using the Helmholtz function $m = u - u_{xx}$ Equation (26) can rewritten as*

$$ad^*_{\hat{f}}\hat{u}|_{H^1} = (1 - \partial^2)^{-1}[fm_x + 2f_x m + \lambda f_{xxx}], \tag{3.4}$$

and corresponding Hamiltonian operator is given by

$$\mathcal{O}^{H^1} = (1 - \partial^2)^{-1}(\lambda \partial^3 + \partial m + m\partial). \tag{3.5}$$

It is clear that for coadjoint action with respect to H^1 norm the field u is replaced by the Helmholtz operator action on u, i.e., $m = u - u_{xx}$.

3.1 Lie derivative action and coadjoint flow

In this paper we mainly study the evolution equation in terms Lie derivative action, hence, we briefly [for details, [29]] describe how to express the coadjoint action of $Vect(M)$ by the negative of the Lie derivative.

Let us recall that in the adjoint representation, the group $Diff(M)$ acts on its Lie algebra $Vect(M)$ by the transformation of coordinates

$$Ad_\phi(\eta) = \phi_* \eta \circ \phi^{-1},$$

where $\phi \in Diff(M)$ is a volume preserving diffeomorphism.

Let u be an element of the dual of $Vect(M)$. This is presumably one form modulo exact one form. Therefore, we find

$$< \eta, Ad^*_\phi(u) > = < Ad^*_{\phi^{-1}}(\eta), u > = < \eta, \phi^{-1*}(u) >,$$

where we have tacitly applied the volume preserving condition.

Hence, we obtain $Ad^*_\phi(u) = \phi^{-1*}(u)$, that is, the diffeomorphism maps u to $\phi^{-1*}(u)$.

Thus, from the definition of the Lie derivative of a differential form u in the direction of a vector field η at a point $x \in M$ we obtain

$$ad^*_\eta(u) = -L_\eta u. \tag{3.6}$$

In fact for this reason one obtains the same Hamiltonian operator from two different computations.

3.2   Camassa–Holm in Lie derivative method

Let us apply the above result to express the coadjoint action of $Vect(S^1)$ in terms of Lie derivative. In otherwords, let us state coadjoint action in Lie derivative language. It is clear from equation (18) that the coadjoint action with respect to H^1 norm must be expressed in terms m, i.e., Helmholtz operator acting on u.

Definition 3.3. *The $Vect(S^1)$ action on the space of Hill's operator Δ with respect to H^1–metric is defined as*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta]_{H^1} := \mathcal{L}^{\frac{3}{2}}_{f(x)\frac{d}{dx}} \circ \tilde{\Delta} - \tilde{\Delta} \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}, \tag{3.7}$$

where

$$\tilde{\Delta} = \lambda \frac{d2}{dx^2} + m(x) \qquad m = (u - u_{xx}).$$

Therefore Lie derivative $\mathcal{L}_{f(x)\frac{d}{dx}}$ action yields the following scalar operator, i.e. the operator of multiplication by a function.

Proposition 3.4.

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta]_{H^1} = (\frac{1}{2}\lambda\partial^3 + 2m\partial + m')f. \tag{3.8}$$

Proof: By direct computation.
□

The L.H.S. of equation denotes the coadjoint action evaluated with respect to H^1 norm. Thus we obtain the R.H.S. of Equation (32).

Lemma 3.5. *The coadjoint action of vector field $f(x)\frac{d}{dx}$ on its dual with respect to the right invariant H^1 metric can be realized as*

$$ad^*_{\hat{f}} \hat{u}|_{H^1} = (1 - \partial^2)^{-1}[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta]_{H^1}$$

This yields the Hamiltonian structure of the Camassa-Holm equation

$$\mathcal{O}^{H^1} = (1 - \partial^2)^{-1}(\frac{1}{2}\lambda\partial^3 + 2m\partial + m'). \tag{3.9}$$

At this stage we assume $\lambda = 0$, since we do not require the cocycle term to compute the Camassa–Holm equation. It is clear that the term $\frac{1}{2}\lambda\partial^3$ manufactures from the cocycle term.

Therefore, the Euler-Poincaré equation

$$u_t = -\mathcal{O}^{H^1}_{CH}\frac{\delta H}{\delta u} \qquad \text{where } \mathcal{O}^{H^1}_{CH} = 2m\partial + m'$$

$$= -(1 - \partial2)^{-1}(2m\partial + m')\frac{\delta H}{\delta u}] \qquad \text{with } H = \frac{1}{2}\int_{S^1} u2 \, dx$$

$$\implies \qquad m_t + 2mu' + m'u = 0$$

yields the Camassa-Holm equation.

4 Euler-Poincaré formalism of Burgers and Whitham-Burgers equations

At this stage let us focus our goal once again. Our aim is to give an Euler-Poincaré formalism of the Degasperis-Procesi and the Holm-Staley equation. We divide this programme into several pieces. In this section, we first discuss the formulation of the Burgers and the Whitham Burgers equations. The H^1 analogue (in our notation) of the Burgers equation is called the Whitham Burgers equation.

4.1 Burgers Equation

Let us consider a first order differential operator of the following form

$$\Delta_1 = \frac{d}{dx} + u(x), \tag{4.1}$$

acting on the space of tensor densities of degree $\frac{-1}{2}$, i.e., $\mathcal{F}_{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$.

This Δ_1 maps

$$\Delta_1 = \frac{d}{dx} + u(x) \ : \ \mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{1}{2}}. \tag{4.2}$$

Definition 4.1. *The $Vect(S^1)$- action on Δ_1 is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ := \ \mathcal{L}^{\frac{1}{2}}_{f(x)\frac{d}{dx}} \circ \Delta_1 - \Delta_1 \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{4.3}$$

The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

Lemma 4.2. *The action of vector field $f(x)\frac{d}{dx}$ on the space first oder operator*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ = \ \frac{1}{2}f''(x) + uf'(x) + u'f(x) \tag{4.4}$$

yields the operator

$$\mathcal{O}_B = \frac{1}{2}\frac{d2}{dx^2} + u\frac{d}{dx} + u'(x). \tag{4.5}$$

Proof: By direct computation.
$\square$

Remark The operator (38) is not a Poisson operator, since it does not satisfy the skew symmetric condition. When a vector field $Vect(S^1)$ acts on the space of Hill's operator, it generates a Poisson flow, that is, operator involves in this flow is Poisson operator. But when $Vect(S^1)$ acts on the space of first order differential operators, it does not generate a Poisson flow. Thus we obtain an almost Poisson operator.

Lemma 4.3. *The Euler-Poincaré flow on the space of first order differential*

$$u_t = [L_{u(x)\frac{d}{dx}}, \Delta_1]$$

yields the Burgers equation

$$u_t = 4uu_x + u_{xx}, \tag{4.6}$$

for $H = \int_{S^1} u2\, dx$.

Proof: We get the Burgers equation from the Hamiltonian equation

$$u_t = \mathcal{O}_B \frac{\delta H}{\delta u}, \qquad H[u] = \int_{S^1} u2\ dx. \tag{4.7}$$

□

4.1.1 Leibniz–Poisson structure

It has been noticed recently that a different type of Poisson bracket is sometimes necessary to incorporate dissipative type systems. A well known example is almost Poisson brackets, the brackets do not satisfy Jacobi identity, are employed to study non-holonomic constrained system. It has been proposed [1, 2, 39] that the modeling of certain dissipative phenomena by adding a symmetric bracket to a known antisymmetric one. This new bracket is called Leibniz bracket, given as

$$[.,.]_{Leibniz} = \{.,.\}_{skew} + \{\{.,.\}\}_{sym},$$

where the bracket $\{.,.\}_{skew}$ is skewsymmetric, $\{\{.,.\}\}_{sym}$ is symmetric, and the sum is a Leibniz bracket. In the infinite-dimensional case, this bracket captures the modeling of a surprising number of physical examples.

Lemma 4.4. *The Leibniz-Poisson bracket associated to the Burgers operator is given us Leibniz bracket*

$$[f, g]_{Leib} := \int_{S^1} \frac{\delta f}{\delta u} \mathcal{O}_{Burgers} \frac{\delta g}{\delta u}\ dx, \tag{4.8}$$

where

$$\mathcal{O}_{Burgers} = \partial u + \frac{1}{2}\partial^2.$$

Proof: It is easy to see that

$$\mathcal{O}_{Burgers}^{skew} = \partial u, \qquad \mathcal{O}_{Burgers}^{sym} = \frac{1}{2}\partial^2.$$

So the diffusion part is coming from the symmetric part of $\mathcal{O}_{Burgers}$.
□

It is known that plenty of dissipative systems are connected to such structure. The Euler–Poincaré flows of this class of systems is considered in [22].

4.2 The Whitham-Burgers equation

Let us study the H^1 analogue of our previous Lemma (4.2). It is clear that the formula (37) is valid for $L2$ norm. We are interested to get the same type of formula for the H^1 case. Let normalize the first order differential operator as $\Delta_1 = 2\frac{d}{dx} + u(x)$.

Thus we conclude from the argument of our previous Section and the identification

Lemma 4.5.

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, 2\frac{d}{dx} + u(x)]_{H^1} = f''(x) + mf'(x) + m'f(x), \qquad m = u - u_{xx} \qquad (4.9)$$

Again, we can interpret this equation as an action of the vector field $\mathcal{L}_{f(x)\frac{d}{dx}}$ on the space of modified first order scalar differential operator $2\frac{d}{dx} + m$. The factor "2" is just the normalization constant.

The L.H.S. denotes coadjoint action with respect to H^1 norm. Once again we convert this to $L2$ action, given as

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, 2\frac{d}{dx} + u(x)]_{L2} = (1 - \partial^2)^{-1}[\mathcal{L}_{f(x)\frac{d}{dx}}, 2\frac{d}{dx} + u(x)]_{H^1}.$$

Therefore, the Hamiltonian operator of the Whitham-Burgers equation becomes

$$\mathcal{O}_{WB}^{H^1} = (1 - \partial 2)^{-1}(\partial 2 + \partial m). \qquad (4.10)$$

Lemma 4.6. *The Euler–Poincaré flow on the space with respect to H^1 norm on the first order differential operators yields the Whitham Burgers equation*

$$m_t = u_{xx} + (mu)_x. \qquad (4.11)$$

Proof:
$$u_t = \mathcal{O}_{WB}^{H^1}\frac{\delta H}{\delta u}]$$

$$= (1 - \partial 2)^{-1}(\partial 2 + \partial m)\frac{\delta H}{\delta u}$$

for Hamiltonian $H = \frac{1}{2}\int_{S^1} u2 \, dx$.
$\square$

5 Euler-Poincaré formalism of the Degasperis-Procesi

In this section our goal is to construct the Euler-Poincaré framework for the the Degasperis-Procesi equation

$$m_t + um_x + 3mu_x = 0, \qquad \text{with } m = u - u_{xx}. \qquad (5.1)$$

5.1   EP flow and Hamiltonian structures

Interestingly the Degasperis-Procesi equation can not be derived from the action of $Vect(S^1)$ on the spaces of second or first order differential operators. We need to combine the $Vect(S^1)$ action on both second and first order differential operators, Δ_2 and Δ_1 respectively, with respect to H^1 norm.

Definition 5.1. *The $Vect(S^1)$ action the pencil of operators $\Delta^{\lambda,\mu} := \lambda\Delta_2 + \mu\Delta_1$ is given by*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{\lambda,\mu}]_{H^1} = \lambda[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_2]_{H^1} + \mu[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1]_{H^1}, \tag{5.2}$$

where

$$\Delta_2 = k_1\frac{d2}{dx^2} + m(x), \qquad \Delta_1 = 2k_2\frac{d}{dx} + m(x).$$

The pencil of Hamiltonian structures corresponding to $Vect(S^1)$ action on $\Delta^{\lambda,\mu}$ is given by

$$\mathcal{O}_{\lambda,\mu}^{H^1} = (1 - \partial^2)^{-1}\lambda(\frac{1}{2}k_1\partial 3 + \partial m + m\partial) + \mu(k_2\partial 2 + \partial m), \tag{5.3}$$

where $m = u - u_{xx}$.

If we assume $k_1 = k_2 = 0$ and $\lambda = 2$ and $\mu = -1$, we obtain the operator of Degasperis-Procesi equation

$$\mathcal{O}_{DP} = (1 - \partial^2)^{-1}(\partial m + 2m\partial). \tag{5.4}$$

Thus we obtain:

Theorem 5.2. *The Degasperis-Procesi equation is the Euler-Poincaré flows of H^1 norm on the combined space of Hill's and first order differential operators.*

Proof: It follows directly from

$$u_t = -(1 - \partial 2)^{-1}(2m\partial + \partial m)\frac{\delta H}{\delta u},$$

for $H = \frac{1}{2}\int_{S^1} u2\, dx$.
$\square$

5.1.1   Degasperis-Holm-Hone (DHH) Hamiltonian structures

Recently Degasperis, Holm and Hone showed that the Degasperis-Procesi system exhibits bihamiltonian features. They expressed the Degasperis-Procesi equation as

$$m_t = B_i\frac{\delta H_i}{\delta m} \qquad i = 0, 1, \tag{5.5}$$

where $m = u - u_{xx}$. Thus they studied the flow of Helmholtz function. They showed that there is only one local Hamiltonian structure

$$B_0 = \partial_x(1 - \partial_x^2)(4 - \partial_x^2), \tag{5.6}$$

and the second Hamiltonian structure is given by

$$B_1 = m^{2/3}\partial_x m^{1/3}(\partial_x - \partial_x 3)^{-1}m^{1/3}\partial_x m^{2/3}. \tag{5.7}$$

The first Hamiltonian structure can be rewritten as

$$B_0 = \mathcal{O}_0(1 - \partial_x^2), \tag{5.8}$$

where $\mathcal{O}_0 = \partial_x(4 - \partial_x^2)$. In view of the chain rule formula for variational derivatives we obtain

$$\frac{\delta H_0}{\delta m} = (1 - \partial_x^2)^{-1}\frac{\delta H_0}{\delta u}. \tag{5.9}$$

Therefore Hamiltonian equation for B_0 becomes

$$m_t = B_0\frac{\delta H_0}{\delta m} = \mathcal{O}_0(1 - \partial_x^2)\frac{\delta H_0}{\delta m} = \mathcal{O}_0\frac{\delta H_0}{\delta u}.$$

It is fairly easy to see that the second Hamiltonian indeed yields the Degasperis–Procesi equation from a sheet computation

$$m_t = B_1\frac{\delta H}{\delta m},$$
$$\implies \quad = 3(m^{2/3}\partial_x m^{1/3}(1 - \partial_x^2)^{-1}m$$
$$\implies \quad = m_x u + 3mu_x,$$

where the Hamiltonian is given by

$$H = -\frac{9}{2}\int m \, dx.$$

Since the first operator B_0 is constant coefficient and hence the Jacobi identity is trivial but Degasperis et. al. said in their papers [10, 11] that for the non-local operator B_1 failed to satisfy the Jacobi identity. The proof of the Jacobi identity for this operator was resolved by Hone and Wang [25], using the trihamiltonian formalism of Olver and Rosenau [42].

Remark (a) Degasperis et al. studied the flow of Helmholtz function $m = u - u_{xx}$ of independent variable u. They obtained the evolution from the variation of derivatives of H with respect to m. Thus there is an extra factor of $(1 - \partial^2)$ appearing in the constant Hamiltonian operator (41). Moreover we are studing the variation of H with respect to u. These two variational derivatives are connected by chain rules

$$\frac{\delta H}{\delta u} = (1 - \partial^2)\frac{\delta H}{\delta m}.$$

Of couse there is an extra $(1 - \partial^2)^{-1}$ factor in our Hamiltonian operator $\mathcal{O}_{DP}$. But this factor acts like Helmholtz operator on u to change the time derivative from u_t to m_t. This makes the difference between DHH method and that of ours. Thus the quintic operator is actually a cubic operator in our case.

5.2 Second DHH Hamiltonian structure and Euler-Poincaré flow

In this section we show the connection between the Degasperis-Holm-Hone Hamiltonian equation and our Euler-Poincaré formalism.

In [10, 11] Degasperis, Holm and Hone presented a Lagrangian formulation for the entire b family of equation, with a Legendre transformation leading to the (second) Hamiltonian structure

$$m_t = \hat{B}\frac{\delta H}{\delta m}, \qquad \text{for} \ \ b \neq 1 \tag{5.10}$$

$$\hat{B} = (bm\partial + m_x)(\partial - \partial^3)^{-1}(bm\partial + (b-1)m_x), \qquad H = \frac{1}{b-1}\int m \, dx. \tag{5.11}$$

This second Hamiltonian structure yields the Hamiltonian structure of the DP equation for $b = 3$.

Proposition 5.3. *The Degasperis-Holm-Hone equation*

$$m_t = \hat{B}\frac{\delta H}{\delta m}, \qquad \hat{B} = (3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x)\frac{\delta H}{\delta m} \tag{5.12}$$

is equivalent to the Euler-Poincaré flow on the combined space of Hill's operator and the space of first order differential operators with respect to H^1-metric.

Proof: Our goal is to reduce equation (56) into EP form. We compute the right hand side of (56).

$$\hat{B}\frac{\delta H}{\delta m} = (3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x)\frac{\delta H}{\delta m}$$

$$\implies \quad = (3m\partial + m_x)(\partial - \partial^3)^{-1}m_x \qquad \text{where} \ \ \frac{\delta H}{\delta m} = 1$$

$$\implies \quad = (3m\partial + m_x)u, \qquad \text{where} \ \ u = (1 - \partial^2)^{-1}m.$$

Thus we obtain

$$m_t = (3m\partial + m_x)\frac{\delta H_1}{\delta u}$$

where $H_1 = \frac{1}{2}\int_{S^1} u2 \, dx$.

Since $m = (1 - \partial^2)u$, hence the above Degasperis-Holm-Hone form is equivalent to

$$u_t = (1 - \partial^2)^{-1}(3m\partial + m_x)\frac{\delta H_1}{\delta u}$$

Therefore the Degasperis-Holm-Hone form of Hamiltonian structure coincides with the Euler-Poincaré framework.

$\square$

Hence the second Hamiltonian structure of Degasperis–Procesi equation $\mathcal{O}_{DP} = (1 - \partial^2)^{-1}(\partial m + 2m\partial)$. is equivalent to the second DHH Hamiltonian structure.

Our *next goal* is to show that the first Hamiltonian structure of Degasperis Holm and Hone also leads to Euler-Poincaré flow. For this we need to use" frozen Lie-Poisson " structure described in section 2. Therefore, we interpret the Degasperis-Procesi equation as an Euler-Poincaré flow on the space of first and second order differential operators.

5.3 Bihamiltonian feature of the Degasperis-Procesi equation

We carry out the coadjoint orbit calculation when the dual of the vector field is $\frac{d2}{dx^2} + u + \Lambda$, for $\Lambda \in \mathbf{R}$. Since we are interested to compute the orbit for H^1 norm, hence the Helmholtz operator action on $u + \Lambda$ yields

$$m = (1 - \partial^2)(u + \Lambda) \equiv u - u_{xx} + \Lambda.$$

The action of $Vect(S^1)$ on $\bar{\Delta}_2 = \frac{d2}{dx^2} + m(x)$ operators is defined by the action of Lie derivative

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta]_{H^1} := (\frac{1}{2}\partial^3 + \Lambda\partial + 2m\partial + m_x)f \qquad m = (u - u_{xx}). \tag{5.13}$$

It is a coadjoint action of $Vect(S^1)$ with respect to H^1 norm, where the following "modified" Gelfands-Fuchs cocycle on $Vect(S^1)$

$$\omega_{mGF}(f(x)\frac{d}{dx}, g(x)\frac{d}{dx}) = \int_{S^1} (f'g'' + \Lambda f'g)dx \tag{5.14}$$

is employed.

This cocycle is cohomologues to the Gelfand-Fuchs cocycle, hence, the corresponding central-extension is isomorphic to the Virasoro algebra. The additional term in (58) is a coboundary term. It is easy to check that the functional

$$\int_{S^1} f'g \, dx = \frac{1}{2} \int_{S^1} (f'g - fg')dx$$

depends on the commutator of $f\frac{d}{dx}$ and $g\frac{d}{dx}$.

The pencil of Hamiltonian structures associated to $Vect(S^1)$ on

$$\bar{\Delta}^{\lambda,\mu} = \lambda\bar{\Delta}_2 + \mu\bar{\Delta}_1$$

is given by

$$\tilde{\mathcal{O}}^{H^1}_{\lambda,\mu} = (1 - \partial^2)^{-1}[\lambda(\frac{1}{2}k_1\partial3 + \Lambda\partial + \partial m + m\partial) + \mu(k_2\partial2 + \partial m)]. \tag{5.15}$$

5.3.1 First DHH Hamiltonian structure and Euler-Poincaré form

Let us assume $k_1 = k_2 = 0$ and $\lambda = 1$ and $\mu = 0$. Thus, by all practical purposes we are confining to Virasoro orbits.

Let us use the concept of "frozen Lie-Poisson bracket". We choose the Hamiltonian operator at $u(x) = 0$. We also assume that the only cocycle term is $\int_S^1 f'g$. Freezing at the point $\mu_0 \equiv (u(x)dx^2, \Lambda) = (0,1)$ yields a Poisson structure induced by a coboundary $\partial\mu_0$. Let us evaluate the operator $\mathcal{O}_0$ defines a constant Poisson bracket $\{f,g\}_0 = \langle \delta f, \mathcal{O}_0\delta g \rangle$. The differential 2-cocycle on Vir corresponding to the freezing point $(0,1)$ is reprsented by

$$\mathcal{O}_0 = (1 - \partial^2)^{-1}\partial. \tag{5.16}$$

This leads us to the first Hamiltonian operator of the Degasperis-Procesi equation.

Proposition 5.4. *The Degasperis-Procesi equation with respect to first Hamiltonian structure of Degasperis-Holm-Hone exactly coincides with the Euler–Poincaré flow on the combined space of Hill's operator and first order differential operators with respect to Hamiltonian*

$$\frac{\delta H_2}{\delta u} = 2u2 - u_x^2 - uu_{xx}. \tag{5.17}$$

Proof: Let us start with Euler-Poincaré flow corresponding to the Hamiltonian operator (52)

$$u_t = (1 - \partial^2)^{-1}\partial\frac{\delta H_2}{\delta u}$$

$$\implies \quad = (4 - \partial^2)\frac{\delta H_0}{\delta u}, \qquad \text{where} \qquad H_0 = \frac{1}{6}\int_{S^1} u3 \, dx.$$

Thus we obtain

$$u_t = (1 - \partial^2)^{-1}\partial(4 - \partial^2)\frac{\delta H_0}{\delta u}.$$

$$\implies \quad m_t = \partial(4 - \partial^2)\frac{\delta H_0}{\delta u}.$$

Once again the chain rule formula for variational derivatives

$$\frac{\delta H_0}{\delta u} = (1 - \partial^2)\frac{\delta H_0}{\delta m}$$

yields

$$m_t = \partial(4 - \partial^2)(1 - \partial^2)\frac{\delta H_0}{\delta m}.$$

Hence we obtain the first Hamiltonian structure B_0 of Degasperis, Holm and Hone from the Euler-Poincaré flow with respect to frozen Lie Poisson structure.

□

Thus we establish our claim that the Degasperis, Holm and Hone Hamiltonian structures can be obtained from the Euler-Poincaré flows.

5.4   The Holm-Staley Equation

Let us study Euler-Poincaré formalism of the Holm-Staley equation

$$m_t + \kappa m_{xx} + m_x u + 3m_x u = 0.$$

There is a dispersive term added to the Degasperis-Procesi equation. In this Section we show that Euler-Poincaré framework of the Holm-Staley equation can be done via pencil of Hamiltonian flows.

Let us recall the Hamiltonian operator associated to the action $Vect(S^1)$ on the pencil of operators $\lambda\Delta_2 + \mu\Delta_1$, given as

$$\mathcal{O}_{\lambda,\mu}^{H^1} = (1 - \partial^2)^{-1}[\lambda(\frac{1}{2}k_1\partial 3 + \partial m + m\partial) + \mu(k_2\partial 2 + \partial m)].$$

At first we set $\lambda = 0$ and $\mu = 1$. Thus, we only consider the action of $Vect(S^1)$ on the space of first order differential operators Δ_1. In particular, this reduction allows us to consider only the Hamiltonian structure associated to $Vect(S^1)$ action on Δ_1.

Let us consider *frozen Hamiltonian structure* on the space of Δ_1, i.e. $u = 0$ and $k_2 = 1$, this yields

$$\mathcal{O}_{frozen} = (1 - \partial^2)^{-1}\partial 2. \tag{5.18}$$

Proposition 5.5. *The following Euler-Poincaré flow generated by the action of $Vect(S^1)$ on the combined space of differential operators Δ_2 and Δ_1*

$$u_t = -\mathcal{O}_{DP}\frac{\delta H}{\delta u} - \mathcal{O}_{frozen}\frac{\delta \tilde{H}}{\delta u}, \tag{5.19}$$

where $H = \frac{1}{2}\int_{S^1} u2\, dx$ and

$$\tilde{H} = \int_{S^1} (u2 + u_x^2)\, dx, \tag{5.20}$$

yields the Holm–Staley equation.

The second Hamiltonian $\tilde{H}$ takes values on H^1 norm of u. Thus, we can think the Holm-Staley equation as a deformation of the Degasperis-Procesi system.

6   Two component Degasperis-Procesi equation

Let us focus on two component Degasperis-Procesi equation. This should be the immediate generalization of the two component Camassa–Holm equation, defined as

$$u_t = um_x + 2mu_x + vv_x\,,$$
$$v_t = (uv)_x\,. \tag{6.1}$$

The two component Degasperis-Procesi equation can be viewed as the second member of the two component b–field equation.

Consider the following matrix linear differential operators on $C^\infty(S^1) \oplus C^\infty(S^1)$ [35]:

$$\Delta = \begin{pmatrix} \frac{c_1 d2}{dx^2} + u(x) & c_2\frac{d}{dx} + v(x) \\ c_3\frac{d}{dx} + v(x) & c_4 \end{pmatrix}$$

where $c_i \in \mathbf{R}$, $u(x) = u(x + 2\pi)$ and $v(x) = v(x + 2\pi)$.

We consider Δ as an $Vect(S^1)$ modules, defined as

$$\Delta \; : \; \mathcal{F}_{-\frac{1}{2}} \oplus \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_{\frac{1}{2}}. \tag{6.2}$$

We will study the Euler–Poincaré flows on space of operators (66). Let us first consider the action of $Vect(S^1)$ on the $\boldsymbol{\Delta}$–operators which is defined as

$$T^\lambda_{(f(x)\frac{d}{dx})} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} L^\lambda_{f(x)\frac{d}{dx}} m \\ L^{\lambda-1}_{f(x)\frac{d}{dx}} n \end{pmatrix} \tag{6.3}$$

where $m(x), n(x) \in C^\infty(S^1)$.

Definition 6.1. *The $Vect(S^1)$ action on the space of $\boldsymbol{\Delta}$ operators is given by*

$$[T_{f(x)\frac{d}{dx}}, \boldsymbol{\Delta}] = T^{\frac{1}{2}}_{f(x)\frac{d}{dx}} \circ \boldsymbol{\Delta} - \boldsymbol{\Delta} \circ T^{-\frac{1}{2}}_{(f(x)\frac{d}{dx}, a(x))}. \tag{6.4}$$

The (generalized) Lie derivative $T^\lambda_{(f(x)\frac{d}{dx})}$ satisfies

$$[T^\lambda_{f(x)\frac{d}{dx}}, T^\lambda_{f(x)\frac{d}{dx}}] = T^\lambda_{((fg'-f'g)\frac{d}{dx})}. \tag{6.5}$$

Let us compute the action of Lie derivative on the space of $\boldsymbol{\Delta}$.

Lemma 6.2.

$$[T_{f(x)\frac{d}{dx}}, \boldsymbol{\Delta}] = \begin{pmatrix} fu' + 2f'u + c_1 f''' & fv' + f'v + c_2 f'' \\ fv' + f'v + c_3 f'' & 0 \end{pmatrix}. \tag{6.6}$$

Proof: Our proof follows from the direct computation.
□

Our aim is to derive 2–component Degasperis–Procesi equation, and this is related to Euler–Poincaré flows on space of differential operators with respect to H^1 metric.

Remark   It must be worth to note that the two component Camassa–Holm equation is an Euler–Poincaré flow on the dual semi–direct product algebra $Vect(S^1) \ltimes C^\infty(S^1)$ with respect to H^1 metric

$$\langle \widehat{f}, \widehat{g} \rangle_{H^1} = \int_{S^1} f(x)g(x) + a(x)b(x) + \partial_x f(x)\partial_x g(x)\, dx \tag{6.7}$$

where

$$\widehat{f} = \left(f\frac{d}{dx}, a \right), \qquad \widehat{g} = \left(g\frac{d}{dx}, b \right).$$

Corollary 6.3. *The $Vect(S^1)$-action restricted to hyperplanes $c_1 = c_2 = c_3 = 0$ yields*

$$[T_{f(x)\frac{d}{dx}}, \boldsymbol{\Delta}] = \begin{pmatrix} fu' + 2f'u & fv' + f'v \\ fv' + f'v & 0 \end{pmatrix}. \tag{6.8}$$

It is clear that this action is obtained respect to $L2$ metric. Let us now consider the action of $Vect(S^1)$ with respect to H^1–metric.

Definition 6.4. *The $Vect(S^1)$ action on the space of Δ operators with respect to H^1–metric is given by*

$$[T_{f(x)\frac{d}{dx}}, \Delta]_{H^1} = T^{\frac{1}{2}}_{f(x)\frac{d}{dx}} \circ \tilde{\Delta} - \tilde{\Delta} \circ T^{-\frac{1}{2}}_{(f(x)\frac{d}{dx}, a(x))}, \tag{6.9}$$

where

$$\tilde{\Delta} = \begin{pmatrix} \frac{c_1 d2}{dx^2} + m(x) & c_2 \frac{d}{dx} + v(x) \\ c_3 \frac{d}{dx} + v(x) & c_4 \end{pmatrix} \qquad m = u - \alpha 2 u_{xx}.$$

Lemma 6.5. *The $Vect(S^1)$-action with respect to H^1 metric on the space of matrix Sturm–Liouville operators restricted to hyperplanes $c_1 = c_2 = c_3 = c_4 = 0$ yields*

$$[T_{f(x)\frac{d}{dx}}, \Delta]_{H^1} = \begin{pmatrix} fm' + 2f'm & fv' + f'v \\ fv' + f'v & 0 \end{pmatrix}. \tag{6.10}$$

Proof: It is obvious from equation (73).
□

Still we do not touch the $[T_{f(x)\frac{d}{dx}}, \Delta]_{H^1}$ factor. Let us now *compute the left hand side of Eqn. (74)*. We assign $[T_{f(x)\frac{d}{dx}}, \Delta]$ to be a coadjoint action with respect to H^1 metric, and given by

$$\left\langle [T_{f(x)\frac{d}{dx}}, \Delta]_{H^1} \begin{pmatrix} u \\ v \end{pmatrix}, (g\ b) \right\rangle = \int_{S^1} \left[([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u})g + ([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u})'g' + ([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u})b \right] dx$$

$$= \int_{S^1} \left[[(1 - \partial 2)([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u})g] + ([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u})b \right] dx$$

$$= \left\langle ((1 - \partial 2)([T_{f(x)\frac{d}{dx}}, \Delta]\hat{u}), ([T_{f(x)\frac{d}{dx}}, \Delta])\hat{u}), (g\ b) \right\rangle$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \hat{u}.$$

Thus by equating the the right and left hand sides, we obtain the desired formula.
□

Lemma 6.6. *The coadjoint action with respect to H^1 inner product is given by*

$$[T_{f(x)\frac{d}{dx}}, \Delta] = \begin{pmatrix} (1 - \partial 2)^{-1}(fm' + 2f'm) & (1 - \partial 2)^{-1}(fv' + f'v) \\ fv' + f'v & 0 \end{pmatrix}, \tag{6.11}$$

Proposition 6.7. *The Hamiltonian operator corresponding to the $Vect(S^1)$-action with respect to H^1 metric on the space of Sturm–Liouville operators is given by*

$$\mathcal{O}_1 = \begin{pmatrix} (1-\partial 2)^{-1}(\partial m + m\partial) & (1-\partial 2)^{-1}\partial v \\ \partial v & 0 \end{pmatrix} \tag{6.12}$$

We conclude that

Theorem 6.8. *The Euler–Poincare flows on the space of operators $\boldsymbol{\Delta}$ in the H^1 metric satisfies the 2–component Camassa–Holm equation*

$$\begin{aligned} m_t &= um_x + 2u_x m + vv_x \ , \\ v_t &= (uv)_x \ , \end{aligned} \tag{6.13}$$

for $H = \frac{1}{2}\int_{S^1}(u2 + v2)\, dx$.

6.1   2–component Degasperis–Procesi equation

Our goal is to give an Euler–Poincaré framework to the 2–component Degasperis–Procesi equation. We have already seen that ordinary Degasperis–Procesi equation is a combination of Camassa–Holm flow and Whitham–Burgers flow. Our target is to find a two component analogue of the Whitham–Burgers equation.

Let us consider another following matrix linear differential operators on $C^\infty(S^1)\bigoplus C^\infty(S^1)$:

$$\boldsymbol{\Delta}_1 = \begin{pmatrix} c_1 \frac{d}{dx} + u(x) & c_2 \\ c_3 \frac{d}{dx} + v(x) & c_4 \end{pmatrix} \tag{6.14}$$

where $c_i \in \mathbf{R}$, $u(x) = u(x + 2\pi)$ and $v(x) = v(x + 2\pi)$.

We consider $\boldsymbol{\Delta}_1$ as an $Vect(S^1)$ modules, defined as

$$\boldsymbol{\Delta}_1 \ : \ \mathcal{F}_{-\frac{1}{2}} \oplus \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_{\frac{1}{2}}. \tag{6.15}$$

We will show that there also exists a reacher structure on the space of operators $\boldsymbol{\Delta}_1$. Let us define an action of $Vect(S^1)$.

Lemma 6.9. *The $Vect(S^1)$–action with respect to $L2$ metric on the space of $\boldsymbol{\Delta}_1$ operators*

$$[T_{f(x)\frac{d}{dx},a(x)}, \boldsymbol{\Delta}_1] = \begin{pmatrix} fu' + f'u + c_1 f''' + va' & 0 \\ fv' + f'v + c_3 f'' + \frac{1}{2}c_4 a' & 0 \end{pmatrix}. \tag{6.16}$$

Proof: By direct calculation.
□

Corollary 6.10. *The $Vect(S^1)$-action on the space of $\mathbf{\Delta}_1$ operators restricted to hyperplanes $c_1 = c_3 = 0$ yields*

$$[T_{f(x)\frac{d}{dx}}, \mathbf{\Delta}_1] = \begin{pmatrix} fu' + f'u & 0 \\ fv' + f'v & 0 \end{pmatrix}. \tag{6.17}$$

Now we will define an action of $Vect(S^1)$ with respect to H^1 metric. We follow the same procedure. We conclude then

Lemma 6.11. *The $Vect(S^1)$-action with respect to H^1 metric on the space of first order matrix differential operators (78) restricted to hyperplanes for $c_1 = c_3 = c_4 = 0$ yields*

$$[T_{f(x)\frac{d}{dx}}, \mathbf{\Delta}_1]_{H^1} = \begin{pmatrix} fm' + f'm & 0 \\ fv' + f'v & 0 \end{pmatrix} \qquad m = u - u_{xx}. \tag{6.18}$$

Once again after computing the L.H.S. of Eqn. (82) in the previous method we obtain the Hamiltonian operator.

Lemma 6.12. *The Hamiltonian operator corresponding to the $Vect(S^1)$-action with respect to H^1 metric on the space of operators (78) is given by*

$$\mathcal{O}_2 = \begin{pmatrix} (1 - \partial^2)^{-1}\partial m & 0 \\ \partial v & 0 \end{pmatrix} \tag{6.19}$$

Proof: We follow the previous method.
$\square$

Proposition 6.13. *The Euler–Poincare flows on the space of operators $\mathbf{\Delta}_1$ in the H^1 metric satisfies*

$$\begin{aligned} m_t &= um_x + u_x m, \\ v_t &= (uv)_x, \end{aligned} \tag{6.20}$$

for $H = \frac{1}{2}\int_{S^1}(u2 + v2)\,dx$.

Let us define a pencil of Hamiltonian opertors

$$\mathcal{O}_3 = \lambda \begin{pmatrix} (1 - \partial 2)^{-1}(\partial m + m\partial) & (1 - \partial 2)^{-1}\partial v \\ \partial v & 0 \end{pmatrix} + \mu \begin{pmatrix} (1 - \partial 2)^{-1}(\partial m + m\partial) & 0 \\ \partial v & 0 \end{pmatrix}. \tag{6.21}$$

If we assume $\lambda = 2$ and $\mu = -1$, we obtain the Hamiltonian operator for two component Degasperis–Procesi equation

$$\mathcal{O}_{2DP} = \begin{pmatrix} (1 - \partial 2)^{-1}(\partial m + 2m\partial) & 2(1 - \partial 2)^{-1}\partial v \\ \partial v & 0 \end{pmatrix}. \tag{6.22}$$

Hence we conclude

Theorem 6.14. *The Euler–Poincare flows on the combined space of operators $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_1$ in the H^1 metric satisfies*

$$
\begin{aligned}
m_t &= um_x + 3u_x m + 4vv_x \ , \\
v_t &= (uv)_x \ ,
\end{aligned}
\tag{6.23}
$$

for $H = \frac{1}{2}\int_{S^1}(u2 + v2)\,dx$.

Corollary 6.15. *Let us assume $\lambda = b$ and $\mu = 1 - b$. Then the Hamiltonian operator of the b–field equation is defined by*

$$
\mathcal{O}_b = \lambda\mathcal{O}_1 + \mu\mathcal{O}_2,
$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are Hamiltonian operators of the Camassa–Holm and Whitham–Burgers equation respectively. The Euler–Poincare flows on the combined space of operators $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_1$ yields the 2–component b–field equation

$$
\begin{aligned}
m_t &= um_x + bu_x m + 2bvv_x \ , \\
v_t &= (uv)_x \ ,
\end{aligned}
\tag{6.24}
$$

for $H = \frac{1}{2}\int_{S^1}(u2 + v2)\,dx$.

6.2   Second type of 2–component Degasperis–Procesi equation

We can derive second type of two component Degasperis–Procesi equation if we consider following matrix linear differential operators on $C^\infty(S^1)\bigoplus C^\infty(S^1)$:

$$
\boldsymbol{\Delta}_2 = \begin{pmatrix} c_1\frac{d}{dx} + u(x) & c_3\frac{d}{dx} + v(x) \\ c_2 & c_4 \end{pmatrix}.
\tag{6.25}
$$

Once again we consider $\boldsymbol{\Delta}_2$ as an operator on $Vect(S^1)$–modules:

$$
\boldsymbol{\Delta}_2 \ : \ \mathcal{F}_{-\frac{1}{2}}\oplus\mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{1}{2}}\oplus\mathcal{F}_{-\frac{1}{2}}.
\tag{6.26}
$$

When we carry out the same calculation for the above operator, we obtain the Hamiltonian operator corresponding to the $Vect(S^1)$-action with respect to H^1 metric on the space of operators (89) is given by

$$
\mathcal{O}_4 = \begin{pmatrix} (1 - \partial^2)^{-1}\partial m & (1 - \partial^2)^{-1}\partial v \\ 0 & 0 \end{pmatrix}
\tag{6.27}
$$

Therefore from the pencil of Hamiltonian opertors

$$
\mathcal{O}_5 = \lambda\mathcal{O}_1 + \mu\mathcal{O}_4
$$

for $\lambda = 2$ and $\mu = -1$ we obtain Hamiltonian operator for another type of two component Degasperis–Procesi equation

$$\mathcal{O}_{2DP'} = \begin{pmatrix} (1 - \partial 2)^{-1}(\partial m + 2m\partial) & (1 - \partial 2)^{-1}\partial v \\ 2\partial v & 0 \end{pmatrix}, \tag{6.28}$$

and this yields

$$\begin{aligned} m_t &= um_x + 3u_x m + 2vv_x , \\ v_t &= 2(uv)_x , \end{aligned} \tag{6.29}$$

for $H = \frac{1}{2}\int_{S^1}(u2 + v2)\, dx$.

Corollary 6.16. *The second form of b–field equation from the Hamiltonian operator*

$$\mathcal{O}_{2b'} = b\mathcal{O}_1 + (1 - b)\mathcal{O}_4$$

is given by

$$\begin{aligned} m_t &= um_x + (b + 1)u_x m + 2vv_x , \\ v_t &= b(uv)_x , \end{aligned} \tag{6.30}$$

6.3 Construction of 2-component Holm–Staley Equation

In this Section we consider a generalized 2-component Degasperis–Procesi equation, known as the Holm–Staley equation. This is the b-family of 1D shallow water equations, to which Holm and Staley [23, 24] added viscous dissipation in their theoretical and numerical investigations of its solution behavior. It is defined as

$$\begin{aligned} m_t &= \nu m_{xx} + um_x + (b + 1)u_x m + 2vv_x , \\ v_t &= b(uv)_x , \end{aligned} \tag{6.31}$$

Let us recall the Hamiltonian operator associated to the action of $Vect(S^1)$ on the pencil of operators $\lambda\boldsymbol{\Delta} + \mu\boldsymbol{\Delta}_1$. At first we set $\lambda = 1$ and $\mu = 0$. Thus we consider the action of $Vect(S^1)$ on the space of second order matrix differential operators $\boldsymbol{\Delta}_2$ at some fixed point. Let us compute the Lie-Poisson structure corresponding the *freezing point* at $\mu_0 \equiv (u, v, c_1, c_2, c_3) = (0, 0, 0, 1, 0)$. Again we follow the same pattern of calculation and construct the frozen Poisson structure

$$\{f, g\} = \langle \delta f, \mathcal{O}_6 \delta g \rangle$$

from the operator $\mathcal{O}_6$. The operator $\mathcal{O}_6$ defining the constant Poisson structure corresponding to the freezing point μ_0 is given by

$$\mathcal{O}_6 = \begin{pmatrix} 0 & (1 - \partial^2)^{-1}\partial^2 \\ 0 & 0 \end{pmatrix}. \tag{6.32}$$

Thus we obtain the frozen Hamiltonian structure.

The following Euler–Poincaré flow on the combined space of first and second order differential operators

$$u_t = \mathcal{O}_{2b'}\nabla H(u, v) + \mathcal{O}_6\nabla \tilde{H}(u), \tag{6.33}$$

where $H(u, v) = \frac{1}{2}\int_{S^1}(u2 + v2)\, dx$ and $\tilde{H}(u) = \frac{1}{2}\int_{S^1}(u2 + u_x^2)\, dx$, yields the two component Holm–Staley equation.

7   Conclusion and Outlook

In this paper we have given the Euler-Poincaré formalism of the Degasperis-Procesi equation. We show that this is a superposition of two flows, on the space of Hill's operatora and the first order differential operators. Thus the Poisson operators of the Degasperis-Procesi flow is the pencil of two operators. We have also considered the newly formulated Holm-Staley equation. We have also proposed the Euler-Poincaré formalism for the Holm-Staley equation.

All these systems we have considered in this paper belong to a one-parameter family of non-evolutionary partial differential equations which include the Camassa-Holm equation and a new Degasperis-Procesi equation. There are plenty of open problems related to such system. One of the straight forward problem is to study the supersymmetric version of two component analogue of such one-parameter family of systems. It will be really exciting to construct the Hamiltonian structure of the super analogues of such one parameter family of partial differential equations. One can even try the multicomponent extension of such b–field equation.

It has been noticed that the algebraic structure of the b-field equation is quite different from the Camassa–Holm equation. It would be really nice to unveil the structures of these equations. In particular very little is known for $b > 2$ equations.

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