Stability of Triangular Equilibrium Points of Exoplanetary System HD 4732c by Including Radiation Effect

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***Abstract*—Exoplanetary system HD 4732 harbors two planets (b and c) in elliptic orbits. The host star (HD 4732) is more energetic than the Sun, while the planets are Jupiter-class. In this work we deal with the triangular equilibrium points with respect to planet HD 4732c because, interestingly, its orbit is located in habitable zone. For this purpose, we set this system to the elliptical restricted three-body problem with primaries (two massive bodies) consisting of the host star and planet HD 4732c. Based on Newtonian gravity in the rotating coordinate frame, we analytically derive the triangular equilibrium points by simply modeling the energetic radiation of the star. We then find that the triangular equilibrium points (Trojans) are stable. This implies an opportunity to discover stable Trojan objects (or a low-mass planet) residing around the triangular points and also in habitable zone.**

Keywords-exoplanetary system HD 4732; elliptic restricted three-body problem; triangular equilibrium points

I. INTRODUCTION

In dynamical astronomy one among many motivating subjects is the Restricted Three-Body Problem (R3BP). The problem consists of three bodies, i.e. two primary bodies having finite masses and the third with a negligible-mass (infinitesimal) body whose motion is influenced by the primaries. There are two types of RTBP regarding the orbit-shape of the primaries focused on their center of mass. It is classified to Circular R3BP (CR3BP) when the orbit is circular and Elliptic R3BP (ER3BP) for that of ellipse. Many analytical and numerical studies undertake CR3BP since it can explain suitably most cases in Solar System dynamics, especially the planetary trojans.

As of July 2014 more than 1100 exoplanetary systems have been discovered, comprising no less than 1800 exoplanets and at least 450 multiple planet system (http://www.exoplanet.eu). Star HD 4732 has been known to have two planets (b and c) [1]. The star belongs to class M spectral type, having smaller mass and temperature than the Sun, but more energetic (eruptions and flares). The planets are Jupiter-class and, intriguingly, the planet c (HD 4732c) is located in habitable zone. Unlike planets in Solar System, many exoplanets have eccentric orbits. Exoplanet HD 4732c is one of them whose orbital eccentricity is 0.23 [1]

It has been known that exoplanet HD 4732b has insignificant influence to the motion of HD 4732c because the planet b has much smaller orbital distance [2]. Hence, orbits of the star (HD 4732) and the exoplanet (HD 4732c) can be assumed to be ellipses with respect to their center of mass. It is appropriate to consider the system as ER3BP to describe motion of the third infinitesimal body. Unlike the classical case that ponders just point-mass primaries without any other effect, we take into account the radiation pressure of HD 4732 because of its energetic radiation.

The classical case has been improved by adding effects of radiation and body-shape of the primaries that lead to the enhanced potential experienced by the third infinitesimal body. Motion and stability of the third body moving around the Lagrangian equilibrium points under the radiation effect have been figured out [3]. Other studies consider (both) radiative primaries suitable for binary star (e.g. [4][5]), as well as (both) oblate primaries case [6][7]. Combined effects of radiation and oblateness or triaxial-shape have also been investigated (e.g. [8][9]).

In this work, it is sufficient to set the ER3BP consisting of a radiative star and a point-mass planet because of unknown oblateness property. We investigate stability of the third infinitesimal body moving around the vicinity of triangular equilibrium points. The equations of motion and locations of the triangular equilibrium points are derived in Sections II and III, respectively. Section IV describes linear stability of triangular points. We discuss the result in Section V and give conclusions in Section VI.

II. FORMULATION AND EQUATIONS OF MOTION

In ER3BP orbits of the primaries share a common value of eccentricity, e. The problem can be expressed in a Cartesian barycentric system with the origin is on the center of mass of the primaries. Let m_1 and m_2 are masses of the star and the planet, respectively. The usual practice chose a system of units equals to the unity, i.e. the gravitational constant and the sum of mass of the primaries. This brings about mass-parameter μ

$$\mu = \frac{m_2}{m_1 + m_2},\ \ 1-\mu = \frac{m_1}{m_1 + m_2},\ \ 0 < \mu \le \frac{1}{2}. \qquad (1)$$

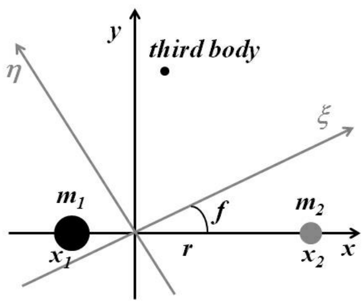

Figure 1. Illustration of inertial and rotating coordinate systems.

The primaries are located in the x-axis of the inertial frame (x, y, z), at the points (x_1, 0, 0) and (x_2, 0, 0), respectively for the star and the planet (Fig. 1). According to Newtonian gravity, the distance (r) between the two primaries becomes

$$r = \frac{1}{1 + e\cos f}, \quad (2)$$

where f is the true anomaly of the system. This implies that

$$x_1 = -\mu r, \quad x_2 = (1-\mu) r\,. \quad (3)$$

Radiation pressure emanating from the star acts on the third infinitesimal body, but it has negligible effect on the planet. The radiation exerts a small acceleration on the third body in the opposite direction to the gravitational acceleration of the star (F_{grav}). A simple model of the net acceleration (F_{net}) from the star working on the third body (e.g. [3][6]) can be expressed by

$$F_{net} = q_1 F_{grav}, \quad (4)$$

where q_1 is the radiation factor of the star that equals to one if no radiation. The more radiation, the smaller value of the factor.

Following the standard formulation based on the rotating pulsating coordinate system (ξ, η, ζ) (Fig. 1), maintaining the primaries in fixed positions and normalizing the unit of length with the instantaneous distance r (see e.g. [3][5] for detailed derivation), the true anomaly f is chosen to be an independent variable rather than the time t which is common in Newtonian gravitation formula. Notice that motion of the third body is kept in planar ER3BP ($\zeta = 0$). Hence, using the relation $df/dt = 1/r^2$ the equations of motion become:

$$\begin{aligned} \frac{d^2\bar{\xi}}{df^2} - 2\frac{d\bar{\eta}}{df} &= \frac{dV}{d\bar{\xi}} \\ \frac{d^2\bar{\eta}}{df^2} + 2\frac{d\bar{\xi}}{df} &= \frac{dV}{d\bar{\eta}} \end{aligned}, \quad (4)$$

where

$$\begin{aligned} V &= \frac{1}{1+e\cos f}\left[\frac{1}{2}\left(\bar{\xi}^2 + \bar{\eta}^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{\mu}{r_2}\right]. \quad (5) \\ r_1^2 &= (\bar{\xi}+\mu)^2 + \bar{\eta}^2, \quad r_2^2 = (\bar{\xi}+\mu-1)^2 + \bar{\eta}^2 \end{aligned}$$

Bars upon (ξ, η) show a transformation to dimensionless system at the instantaneous distance r, so that separation between the primaries is constant and equals to one. Notice that the potential V is not dependent explicitly on f (and t).

III. Locations of Triangular Equilibrium Points

In the classical case the primaries and the triangular equilibrium points configure an equilateral triangle. In this work the configuration may not be an equilateral or isosceles triangle. At the points the third body has zero velocity and zero acceleration in the rotating pulsating coordinate system.

After performing partial derivatives to the potential (5), we obtain

$$\begin{aligned} \frac{dV}{d\bar{\xi}} &= \frac{1}{1+e\cos f}\left[\bar{\xi} - \frac{q_1(1-\mu)(\bar{\xi}+\mu)}{r_1^3} - \frac{\mu(\bar{\xi}+\mu-1)}{r_2^3}\right] \\ \frac{dV}{d\bar{\eta}} &= \frac{1}{1+e\cos f}\left[\bar{\eta} - \frac{q_1(1-\mu)\bar{\eta}}{r_1^3} - \frac{\mu\bar{\eta}}{r_2^3}\right] \end{aligned}. \quad (6)$$

The zero velocity condition leads to (6) equals to zero, from which we obtain

$$r_1 = q_1^{1/3}, \quad r_2 = 1\,. \quad (7)$$

By substituting (7) into (5) we get the coordinate of the triangular points:

$$\bar{\xi}_0 = -\mu + \frac{1}{2}q_1^{1/3}, \quad \bar{\eta}_0 = \pm q_1^{1/3}\sqrt{1 - \frac{1}{4}q_1^{2/3}}\,. \quad (8)$$

The points are written down by subscripts "0" that are symmetry with respect to ξ–axis (Fig. 2). This is in accordance with the result by [3]. It is obvious that the locations depend on q_1. If no radiation ($q_1 = 1$) the points follow the classical case marked by squares in Fig. 2. The more radiative the star (decreasing q_1, marked by triangles in Fig. 2), the closer the triangular points to the star.

IV. Stability of Triangular Points

Suppose that the third body gets a small displacement from the triangular equilibrium points by small quantities of u and v, respectively, such that

$$u = \xi - \bar{\xi}_0, \quad v = \eta - \bar{\eta}_0\,. \quad (9)$$

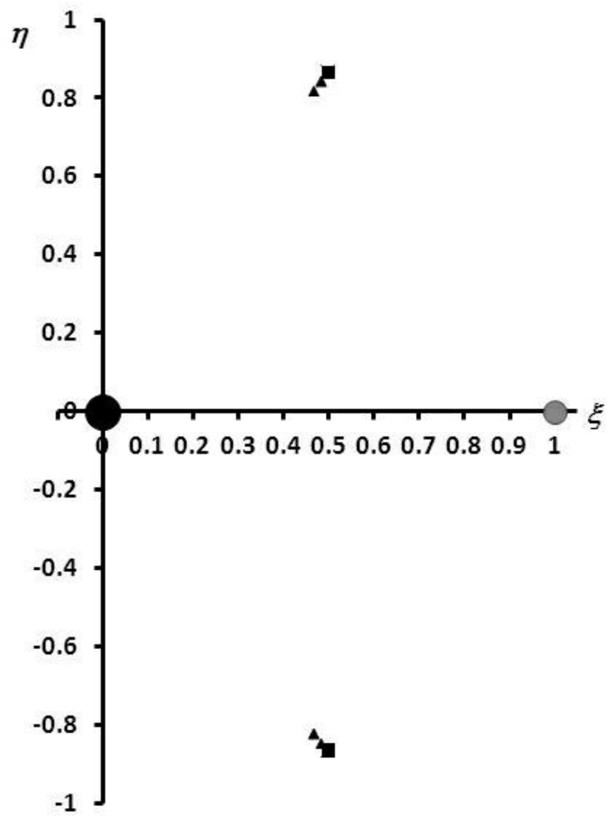

Figure 2. Locations of triangular points for q_1 = 0.9 and 0.8 (triangles). Large black circle denotes the star, and grey circle is HD 4732c. The classical case (q_1 = 1) marks by squares.

The variation equations can be obtained by substituting (9) into (4) and then expanding in a Taylor series about the equilibrium points. By taking only the linear terms we yield

$$\begin{aligned}\frac{d^2u}{df^2}-2\frac{dv}{df}&=\frac{d^2V}{d\bar{\xi}^2}u+\frac{d^2V}{d\bar{\xi}d\bar{\eta}}v=V_{\bar{\xi}\bar{\xi}}u+V_{\bar{\xi}\bar{\eta}}v\\ \frac{d^2v}{df^2}+2\frac{du}{df}&=\frac{d^2V}{d\bar{\xi}d\bar{\eta}}u+\frac{d^2V}{d\bar{\eta}^2}v=V_{\bar{\xi}\bar{\eta}}u+V_{\bar{\eta}\bar{\eta}}v\end{aligned}. \quad (10)$$

Note that the partial derivatives are evaluated at the equilibrium points. Analysing stability of motion of the third body around of the triangular points needs the characteristic equation of the system.

Following e.g. [3][5], we derive a characteristic equation (11) with λs are its roots,

$$\lambda^4+\left(4-V_{\bar{\xi}\bar{\xi}}-V_{\bar{\eta}\bar{\eta}}\right)\lambda^2+V_{\bar{\xi}\bar{\xi}}V_{\bar{\eta}\bar{\eta}}-V_{\bar{\xi}\bar{\eta}}^2=0. \quad (11)$$

After doing some algebra to (6) for second partial derivatives we obtain

$$\begin{aligned}V_{\bar{\xi}\bar{\xi}}&=r\left[\begin{array}{l}1-\frac{q_1(1-\mu)}{r_1^3}+\frac{3q_1(1-\mu)(\bar{\xi}+\mu)^2}{r_1^5}\\ -\frac{\mu}{r_2^3}+\frac{3\mu(\bar{\xi}+\mu-1)^2}{r_2^5}\end{array}\right]\\ V_{\bar{\eta}\bar{\eta}}&=r\left[1-\frac{q_1(1-\mu)}{r_1^3}+\frac{3q_1(1-\mu)\bar{\eta}^2}{r_1^5}-\frac{\mu}{r_2^3}+\frac{3\mu\bar{\eta}^2}{r_2^5}\right], \quad (12)\\ V_{\bar{\xi}\bar{\eta}}&=r\left[\frac{3q_1(1-\mu)(\bar{\xi}+\mu)\bar{\eta}}{r_1^5}+\frac{3\mu(\bar{\xi}+\mu-1)\bar{\eta}}{r_2^5}\right]\end{aligned}$$

and after evaluating (12) at the equilibrium points (8) (superscript "0") yields

$$\begin{aligned}V_{\bar{\xi}\bar{\xi}}^0&=\frac{3/4}{1+e\cos f}\left[q_1^{2/3}+\mu\left(4-q_1^{2/3}\right)\left(1-q_1^{2/3}\right)\right]\\ V_{\bar{\eta}\bar{\eta}}^0&=\frac{3/4}{1+e\cos f}\left[4-q_1^{2/3}-\mu\left(4-q_1^{2/3}\right)\left(1-q_1^{2/3}\right)\right]. \quad (13)\\ V_{\bar{\xi}\bar{\eta}}^0&=\frac{3/4}{1+e\cos f}q_1^{1/3}\sqrt{4-q_1^{2/3}}\left[1+\mu q_1^{2/3}-3\mu\right]\end{aligned}$$

Following [8], along a cycle (f: 0–2π) the factor $(1+e\cos f)^{-1}$ turns out to be $(1-e^2)^{-1/2}$.

The roots of (11) at the equilibrium points will be

$$\begin{aligned}\lambda_{12}^2&=-\frac{1}{2}\left(4-V_{\bar{\xi}\bar{\xi}}^0-V_{\bar{\eta}\bar{\eta}}^0\right)\pm\frac{1}{2}\sqrt{\Delta}\\ \Delta&=\left(4-V_{\bar{\xi}\bar{\xi}}^0-V_{\bar{\eta}\bar{\eta}}^0\right)^2-4\left(V_{\bar{\xi}\bar{\xi}}^0V_{\bar{\eta}\bar{\eta}}^0-V_{\bar{\xi}\bar{\eta}}^{0\,2}\right)\end{aligned}. \quad (14)$$

To guarantee the stable motion, the roots should be pure imaginary. This can be achieved by two restrictions, i.e. the first term of (14) should be negative; and Δ in the second term must be zero or real positive:

$$\begin{aligned}&-\frac{1}{2}\left(4-V_{\bar{\xi}\bar{\xi}}^0-V_{\bar{\eta}\bar{\eta}}^0\right)<0\ \Rightarrow\ V_{\bar{\xi}\bar{\xi}}^0+V_{\bar{\eta}\bar{\eta}}^0-4<0\\ &\Delta=\left(4-V_{\bar{\xi}\bar{\xi}}^0-V_{\bar{\eta}\bar{\eta}}^0\right)^2-4\left(V_{\bar{\xi}\bar{\xi}}^0V_{\bar{\eta}\bar{\eta}}^0-V_{\bar{\xi}\bar{\eta}}^{0\,2}\right)\geq 0\end{aligned}. \quad (15)$$

Both restrictions in (15) can eventually provide negative values of λ^2, and yielding pure imaginary roots.

After substituting (13) into (15) we obtain relation (16) that provides critical values of eccentricity of the system (e_c) and mass-parameter (μ_c). Smaller than these values, motion of the third object around the triangular points are stable. This is illustrated as a circle in Fig. 3 for stable motion of the third object around the triangular points of HD 4732c.

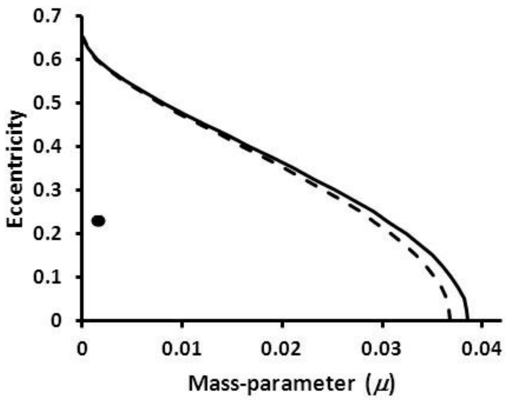

Figure 3. Relation beween e and μ in (16). Below the lines are area of stable motion. Unbroken and broken lines denote $q_1 = 1$ and 0.8, respectively. The circle stands for HD 4732c.

Negative sign of the second term of μ_c in (16) is chosen because of the boundary in (1). Equation (16) is consistent with the one given in [3] but (16) seizes a cycle. Fig. 3 shows the relation in (16) for $q_1 = 1$ and 0.8.

$$e_c = \frac{1}{4}\sqrt{7}$$
$$\mu_c = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\left(3 - 4\sqrt{1-e^2}\right)^2}{9\left(4 - q_1^{2/3}\right)}} \quad . \tag{16}$$

V. Application to HD 4732c and Discussion

The radiation factor (q_1) is actually close to the unity even for the energetic star such as HD 4732. However, this may displace slightly the locations of triangular equilibrium points compared to the classical case (see Fig. 2). Tab. 1 shows values of q_1 and the corresponding μ_c for HD 4732c whose e is 0.23. Because the minimum mass of HD 4732c is 2.36 Jupiter-mass, and the upper limit can be ~2.95 Jupiter-mass, values of μ are 0.0013–0.0016, which stay much smaller than μ_c given in Tab. 1 for several values of q_1.

TABLE I. The values of μ_c against q_1 for HD 4732c ($e = 0.23$)

q_1	μ_c
1.0	0.0304
0.9	0.0298
0.8	0.0291

Because motion of the third body around the triangular points (or known as Trojans) of HD 4732c is stable, it makes possible to be populated by many negligible-mass objects such as asteroids or a single object whose mass is much smaller than Jupiter, e.g. Earth-mass (~1/1000 Jupiter-mass). Because HD 4732c is located in habitable zone, the Trojan object is also in habitable zone because it has the same orbital distance with the planet from the host star.

VI. Conclusions

Analytical study about stability of triangular equilibrium points (Trojans) of HD 4732c has been described. This study include radiation effect of the energetic host star HD 4732. Conclusions of this study are:

- Stability of motion around the triangular points is dependent on mass-parameter, eccentricity of the system, and radiation factor of the star.
- Radiation pressure of the star affects the triangular points to shift closer to the star.
- Motion of the Trojan(s) of HD 4732c can be stable. This imply that the Trojan(s) reside in habitable zone.

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