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Augmented Strength Reliability of Equipment Under Gamma Distribution

N. Chandra[1] and Subhradev Sen[2]

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Abstract

The strength-reliability of equipment is defined as the probability that the strength of the equipment exceeds its stress. Augmentation of strength-reliability of equipments gives better service protection and longevity. In this paper, the concept of Augmentation Strategy Plan (ASP) is introduced under three possible cases to increase the strength-reliability. Assume that strength and stress of equipment, both follow gamma distribution and strength reliability expressions for all three cases are derived. Possible combinations of parameters are tabulated, for desired level of strength-reliability. Also cost aspects of such augmented situations have been discussed.

Keywords: Stress-Strength set up; Reliability; Gamma Distribution; Augmentation.
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[1] Department of Statistics,
Ramanujan School of Mathematical Sciences,
Pondicherry University,
Puducherry-605 014, India
E-mail: nc.stat@gmail.com

[2] Clinical Research & Biostatistics Division,
Malabar Cancer Centre, Thalassery P.O. Moozhikkara,
Dist.-Kannur,
Kerala-670 103, India
E-mail: subhradev.stat@gmail.com

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1. Introduction

The strength reliability is an important characterization of any component or equipment. The strength-reliability of equipment is defined as the probability that the strength of the equipment exceeds its stress. The gamma distribution has wide applicability in Life testing experiments and reliability engineering. Several authors have studied strength reliability for various combinations of useful life distributions in which gamma model played a vital role. In literature, the estimation of strength reliability for the gamma model has been studied by Constantine & Karson [4]; Ismail, Jayaratnam & Panchapakesan [7]. Constantine, Karson & Tse [5] given a Bootstrap approach also in estimating strength reliability for gamma case. Shawky, Sayed & Nassar [11] have discussed confidence interval estimation of strength reliability using generalized gamma family. Nadarajah [9] derived the mathematical expressions for strength reliability in gamma, compound gamma, log gamma and generalized gamma models. Non-parametric approaches in estimating the strength reliabilities for life distributions, like normal, exponential, gamma and beta, are also been considered by Nadarajah, Mitov & Mitov [10].

We consider equipment which is subjected to fail time to time due to the stresses acting on strength. Stress and strength of equipment are considered to be independently distributed. For better functioning of any equipment it is necessary that strength reliability is sufficiently high. If X (a random variable) denotes the strength and Y (a random variable) denotes the stress, independently distributed to each other, of the equipment, then $R = Pr(X>Y)$ is called strength reliability of the equipment.

The equipment usually fails if R becomes below 0.5. For the purpose of practical utility, an enhancement of strength reliability always adds extra protection to the equipment functioning in a particular mechanical configuration.

A possible enhancement in the strength reliability for the exponential distribution is discussed by Alam and Roohi [1]. This is one of the interesting areas where very few researchers attempted the work, This article will be useful in the contribution of developing the concept of Augmentation Strategy Plan (ASP) in statistical reliability theory. ASP is applied when equipment has an impression of early failure occurs frequently due to weak in strength or poor quality of equipment used in a system; therefore equipment is unreliable. Due to high cost of equipment and time involve in manufacturing the equipment, it may not refuse to reuse, Hence ASP is recommended to enhance the strength of weaker equipment by considering the following possible cases initially suggested by Alam and Roohi [1]:

Case-1: Strength increased by m times of initial expected stress;
Case-2: Strength increased by adding independent and identical components;
Case-3: Strength increased by adding independent components with increased strength.

The applying of ASP on weaker quality of equipment in enhancing its strength is known as Augmented Strength Reliability.

In this paper we proposed the enhancement of strength reliability, R, for a desired higher level, taking three possible cases. We assume two parameter gamma model for both the initial stress and strength distributions, i.e., both strength(X) and stress(Y) follow gamma (α , λ) distribution, independently.

To increase the strength of the equipment to face the common stress, the following three attempts are been made. In the first case, the strength of equipment, having initially Gamma strength, is increased by m times of its initial expected stress. Secondly, a suggestion is made to add ‘n’ independent components, each having Gamma initial strength with the equipment to face the stress. And, thirdly, the strength of the equipment is increased by adding independent components, each having m times of average initial Gamma stress. The strength-reliability expressions for each of the three cases are found and extensive values of the R, for each case, is been also tabulated for given values of the parameters in section 2. Cost aspect is discussed in section 3. An overall remark on the basis three cases and cost considerations is discussed in section 4. The references used for developing the proposed problem are given in section 5.

2. Augmented Strength Reliability

The probability density function (p.d.f.) of random variables Y (or X) is

$$\mathrm{f_Y}(y) = \frac{\alpha^{\lambda}}{\Gamma(\lambda)} e^{-\alpha y} y^{\lambda-1} \quad ; y > 0, \alpha, \lambda > 0 \tag{1.1}$$

Then,

$$\mathrm{E(Y)} = \frac{\lambda}{\alpha} \tag{1.2}$$

2.1 *Case-1: Strength increased by m times of initial expected stress*

The expected strength of the equipment is increased m times the expected stress. Hence the new strength X follows gamma ($\frac{\alpha}{m}$, λ), i.e., the p.d.f. of X becomes

$$\mathrm{f}_X(x) = \frac{\left(\alpha/m\right)^{\lambda}}{\Gamma(\lambda)} e^{-\left(\alpha/m\right)x} x^{\lambda-1} \quad ; \ x > 0, \alpha, \lambda > 0 \, , \tag{1.3}$$

Where 'm' is a positive real number.

The strength reliability of the equipment is given by

$$R_1 = \Pr(X > Y)$$

$$= \int_0^{\infty} F_Y(x) f_x(x) dx$$

$$= \frac{\left(\alpha/m\right)^{\lambda}}{(\Gamma(\lambda))^{2}} \int_{0}^{\infty} e^{-\left(\frac{\alpha}{m}\right)x} \gamma(\lambda, \alpha x)\, x^{\lambda - 1}\, dx \;; \tag{1.4}$$

Where $F_Y(.)$ is the distribution function of Y *and* $\gamma(a, x)$ is an incomplete gamma function , defined as

$$\gamma(a, x) = \int_{0}^{x} e^{-t} t^{a-1}\, dt \tag{1.5}$$

Hence,

$$R_1 = \frac{m^{\lambda}\Gamma(2\lambda)}{\lambda(\Gamma(\lambda))^2(1+m)^{2\lambda}}\ {}_2F_1\left(1, 2\lambda; \lambda+1; \frac{m}{m+1}\right) \tag{1.6}$$

Where ${}_pF_q(\alpha_1, \alpha_2, ..., \alpha_p; \beta_1, \beta_2, ..., \beta_q; z)$ is the generalized hypergeometric function. Using 6.455.2 from Gradshteyn and Ryzhik[6] in Eq. (1.6), we have the final expression of strength reliability for case-1 as,

$$R_1 = \frac{m^{\lambda}}{\Gamma(\lambda)(1+m)^{2\lambda}} \sum_{j=0}^{\infty} \frac{\Gamma(2\lambda + j)}{\Gamma(\lambda + 1 + j)} \left(\frac{m}{m+1}\right)^{j} \tag{1.7}$$

Now for an intended value, say δ (>1/2), of Pr(X>Y) for the case-1, we can compute different combinations of the values of m and λ using Eq.(1.7). Actually, for a single intended higher value of the strength-reliability, we can have infinitely many combinations of the values of m and λ. A specific such combination is tabulated in table 1.

The augmentation is smooth for the case-1 as it is clear from the table. In particular, when we fix λ (or m), for increasing values of m (or λ), there is a gradual increase in the values of the strength reliability. Also when m exceeds the value 30, there is a sharp decrease in the reliability values for varying λ (the figures are not shown in the table 1).

2.2 *Case-2: Strength increased by adding independent and identical components*

The strength of the equipment(X) is enhanced by adding n identical equipments, each having strength X_i, where X_i follows Gamma (α , λ) for i=1,2,…,n, independently and the equipment is set to face the stress(Y), where Y follows gamma (α , λ).

Hence, $X = \sum_{i=1}^{n} X_i$ follows gamma (α , $n\lambda$), i.e. X having the p.d.f. as,

$$f_X(x) = \frac{\alpha^{n\lambda}}{\Gamma(n\lambda)} e^{-\alpha x} x^{n\lambda - 1} \;;\; x > 0, \alpha, \lambda > 0 \tag{1.8}$$

Where 'n' is a positive integer.

The strength reliability becomes

$$R_2 = \Pr(X > Y)$$

$$= \frac{\alpha^{n\lambda}}{\Gamma(\lambda)\Gamma(n\lambda)} \int_0^{\infty} e^{-\alpha x} \gamma(\lambda, \alpha x) x^{n\lambda - 1}\, dx$$

$$= \frac{\Gamma(n\lambda + \lambda)}{2^{(n\lambda+\lambda)}\, \lambda\, \Gamma(\lambda)\Gamma(n\lambda)}\ {}_2F_1\left(1, \mathrm{n}\lambda + \lambda\, ; \lambda + 1\, ; \frac{1}{2}\right) \tag{1.9}$$

The strength reliability in this case is obtained as,

$$R_2 = \frac{1}{2^{(n\lambda+\lambda)}\Gamma(n\lambda)} \sum_{\mathrm{j}=0}^{\infty} \frac{\Gamma(\mathrm{n}\lambda + \lambda + \mathrm{j})}{\Gamma(\lambda + 1 + \mathrm{j})} \left(\frac{1}{2}\right)^{j} \tag{1.10}$$

The possible combinations of n and λ for a desired value of Pr (X>Y) for case-2 has been tabulated in table 2, using Eq.(1.10).

The values clearly show that augmentation is effective for possible high range values of λ and adding new components with original strength gives a boost to the component strength reliability. However, much higher values of n (the number of components to be added) will also help to obtain around 99% reliability, though such addition is not suggestive practically.

2.3 *Case-3: Strength increased by adding independent components with increased strength*

The strength of the equipment(X) is increased by adding n identical components each having strength (X_i), which is m times the expected stress of the equipment, is set to face the stress of the equipment(Y) which follows gamma(α , λ).
Mathematically,

$$X = \sum_{i=1}^{n} X_i \text{ follows gamma } (\alpha/\mathrm{m}, n\lambda)$$

Where each X_i follows gamma (α /m, λ), for i=1, 2, …, n, independently.

The p.d.f. of X becomes

$$\mathrm{f}_X(x) = \frac{\left(\alpha/m\right)^{n\lambda}}{\Gamma(n\lambda)} e^{-\left(\alpha/m\right)x} x^{n\lambda - 1}\ ;\ x > 0, \alpha, \lambda > 0 \tag{1.11}$$

Where 'm' is a positive real number and 'n' is a positive integers.

The strength reliability in this case is given by

$$R_3 = \Pr(X > Y)$$

$$= \frac{\left(\alpha/m\right)^{n\lambda}}{\Gamma(\lambda)\Gamma(n\lambda)} \int_0^\infty e^{-\left(\frac{\alpha}{m}\right)x} \gamma(\lambda, \alpha x)\, x^{n\lambda - 1}\, dx$$

$$= \frac{m^\lambda\, \Gamma(n\lambda + \lambda)}{\lambda\Gamma(\lambda)\Gamma(n\lambda)\, (1+m)^{n\lambda+\lambda}}\ {}_2F_1\left(1, \mathrm{n}\lambda + \lambda; \lambda + 1; \frac{m}{m+1}\right) \tag{1.12}$$

The strength reliability in this case has the form

$$R_3 = \frac{m^\lambda}{\Gamma(n\lambda)(1+m)^{n\lambda + \lambda}} \sum_{\mathrm{j}=0}^{\infty} \frac{\Gamma(\mathrm{n}\lambda + \lambda + \mathrm{j})}{\Gamma(\lambda + 1 + \mathrm{j})} \left(\frac{m}{m+1}\right)^j \tag{1.13}$$

Note that, when we put n=1 in the above expression R_3 equals R_1 and putting m=1 in R_3 we have the expression as in for R_2. The strength reliability of the equipment for different combinations of m and n for fixed value of λ is tabulated in the table 3(a) to 3(d). The values are obtained using Eq.(1.13). It is observed from the tables below that when the value of λ increases, the values of m do not give possible connection with it in the same way, for the same range of values of n. It is also studied that the addition of the components with increased strength for higher value of λ may harm the strength reliability of the equipment.

3. Discussion: Real Life Examples and Cost Aspects

Equipment with a desired expected strength is well assumed in this article and it is also clear that a possibility is always there to allow more than one components to work together. But limitations of both the methods are inevitable, for a mechanical configuration of equipment which is made of adding identical components with increased expected strength may not keep always the other specifications of the product's quality characteristics. For example, an aero plane with several wheels working together has always a limitation for the number of wheels attached for serving the purpose. Keeping the form and the other engineering aspects in a particular shape, it always needs to keep a control over the cost.
An equipment with high desired reliability, following any one of the three methods suggested above, may not give the optimal utility unless the cost associated with such a configuration is suitable minimized.

With all these constraints, the crux to find out desired strength reliability can be formulated as below:

(a) If the case –1 is adopted to increase the strength reliability, we can use the table -1 for searching a suitable combination of values. For a desired value, say δ, of the expression in Eq. (1.7), it is possible to find more than one values of m and λ.

If C_1 is the cost of production of an item of expected strength λ/α, C_2 is the cost of increasing per unit; then the cost of production of an item of expected strength $m\lambda/\alpha$ can be taken as a linear function in 'm' as $C_1+(m-1)\frac{\lambda}{\alpha}C_2$.

For a fixed cost C_0, given values of λ and α, we can find out the value of 'm' such that $C_1+(m-1)\frac{\lambda}{\alpha}C_2=C_0$.

(b) When equipment with increased strength is not feasible, we may find out the number of components with original strength to be joined together, for a specified value of λ, using Eq. (1.10). In this case cost will be proportional to the number of equipments joined to face the common stress.

(c) In case both the case -1 and case-2 are possible, we may fix an upper limit for the number of components with increased strength attached to face the common stress. Suppose at most n_0 components can be joined together and the expected strength can be increased up to $m_0\,\lambda/\alpha$ for each component, then for specified values of λ and α, we maximize: $\dfrac{m^{\lambda}}{\Gamma(n\lambda)(1+m)^{n\lambda+\lambda}}\sum_{j=0}^{\infty}\dfrac{\Gamma(n\lambda+\lambda+j)}{\Gamma(\lambda+1+j)}\left(\dfrac{m}{m+1}\right)^{j}$

Subject to the constraints,

$$n\left\{C_1+(m-1)\frac{\lambda}{\alpha}C_2\right\}\le C_0\ \text{(given)}$$

$$n\le n_0\ ;\ \text{for}\ n\ge m_0$$

Since n is a positive integer, it is just an integer programming problem. An analytical solution can be obtained for sufficiently small value of n_0.

4. Concluding Remarks

The paper reveals further scope to work with the mechanical configuration for adding the components, as per suggestion, to increase the strength. It is clear that the strength reliability expressions change depending on the components added in a series connection or in a parallel connection. Though the situation of such configurations is beyond the scope this article, it is essential to incorporate them.

Augmentation Strategy Plan (ASP) may be purposeful when the weaker quality of equipment needs to reuse due to the constraints of time and cost involved in manufacturing also some time unavailable.

Further, there may be possibilities that when the components are added in series or parallel connection, the equipment may not be subjected to face a common stress. The estimation aspects of the studied cases as well as other possible considerations are yet to be searched. The studies in these directions are left for future research.

Table 1. Strength reliability for case-1

m→ λ↓	1.5	2.0	2.5	3.0	5.0	8.0	10.0	15.00
0.10	0.5179	0.5205	0.5402	0.5480	0.5696	0.5889	0.5978	0.6136
0.75	0.5837	0.6404	0.6818	0.7135	0.7908	0.8466	0.8684	0.9009
1.00	0.6000	0.6667	0.7143	0.7500	0.8334	0.8889	0.9091	0.9375
1.50	0.6264	0.7082	0.7642	0.8045	0.8904	0.9393	0.9548	0.9739
2.50	0.6664	0.7675	0.8313	0.8734	0.9490	0.9802	0.9878	0.9945
3.50	0.6970	0.8096	0.8751	0.9148	0.9751	0.9932	0.9964	0.9963
4.50	0.7223	0.8418	0.9058	0.9413	0.9875	0.9985	0.9984	0.9980
6.00	0.7525	0.8779	0.9369	0.9658	0.9925	0.9996	0.9992	0.9980
8.00	0.7869	0.9118	0.9620	0.9827	0.9981	0.9999	0.9998	0.9999
15.00	0.8637	0.9687	0.9928	0.9982	0.9999	0.9999	0.9999	0.9999
20.00	0.8979	0.9845	0.9986	0.9989	0.9999	0.9999	0.9999	0.9999
25.00	0.9224	0.9912	0.9992	0.9999	0.9999	0.9999	0.9999	0.9999

Table 2. Strength reliability for case-2

n→ λ↓	2	3	5	8	10	15	20
0.10	0.6706	0.7580	0.8480	0.9101	0.9330	0.9645	0.9797
0.75	0.7297	0.8497	0.9516	0.9908	0.9969	0.9998	0.9999
1.00	0.7500	0.8750	0.9688	0.9961	0.9985	0.9999	0.9999
1.50	0.7844	0.9123	0.9866	0.9993	0.9999	0.9999	0.9999
2.00	0.8125	0.9375	0.9941	0.9999	0.9999	0.9999	0.9999
3.00	0.8555	0.9673	0.9988	0.9999	0.9999	0.9999	0.9999
4.00	0.8867	0.9824	0.9997	0.9999	0.9999	0.9999	0.9999
5.00	0.9102	0.9904	0.9999	0.9999	0.9999	0.9999	0.9999
8.00	0.9534	0.9983	0.9999	0.9999	0.9999	0.9999	0.9999
10.00	0.9693	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Table 3(a). Strength reliability for case-3 (when λ= 0.10)

m→ n↓	**λ= 0.10**								
	1.5	**2.0**	**2.5**	**3.0**	**5.0**	**8.0**	**10.0**	**15.0**	**20.0**
2	0.6929	0.7083	0.7199	0.7292	0.7541	0.7754	0.7850	0.8014	0.8130
3	0.7813	0.7972	0.8090	0.8182	0.8424	0.8623	0.8710	0.8854	0.8947
5	0.8706	0.8853	0.8954	0.9039	0.9239	0.9390	0.9452	0.9550	0.9608
8	0.9296	0.9414	0.9496	0.9554	0.9691	0.9782	0.9816	0.9865	0.9891
10	0.9502	0.9603	0.9669	0.9716	0.9819	0.9883	0.9905	0.9935	0.9948
15	0.9768	0.9832	0.9871	0.9897	0.9947	0.9972	0.9980	0.9988	0.9982
20	0.9882	0.9923	0.9945	0.9959	0.9983	0.9993	0.9995	0.9995	0.9971

Table 3(b). Strength reliability for case-3 (when λ= 0.50)

m→ n↓	**λ= 0.50**							
	1.5	**2.0**	**2.5**	**3.0**	**5.0**	**8.0**	**10.0**	**15.0**
2	0.7746	0.8165	0.8452	0.8660	0.9130	0.9428	0.9535	0.9682
3	0.8760	0.9083	0.9286	0.9423	0.9695	0.9837	0.9880	0.9932
5	0.9591	0.9750	0.9834	0.9883	0.9959	0.9985	0.9991	0.9990
8	0.9915	0.9961	0.9979	0.9988	0.9998	0.9999	0.9999	0.9942
10	0.9969	0.9988	0.9995	0.9997	0.9999	0.9999	0.9999	0.9828
15	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.8219
20	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.5421

Table 3(c). Strength reliability for case-3 (when λ= 1.0)

m→ n↓	λ= 1.0				
	1.5	**2.0**	**2.5**	**3.0**	**5.0**
2	0.8400	0.8889	0.9183	0.9375	0.9722
3	0.9360	0.9630	0.9768	0.9844	0.9954
5	0.9898	0.9959	0.9981	0.9990	0.9999
8	0.9993	0.9998	0.9999	0.9999	0.9999
10	0.9999	0.9999	0.9999	0.9999	0.9999
15	0.9999	0.9999	0.9999	0.9999	0.9982
20	0.9999	0.9999	0.9999	0.9999	0.9700

Table 3(d). Strength reliability for case-3 (when λ= 1.5)

m→ n↓	λ= 1.5			
	1.5	**2.0**	**2.5**	**3.0**
2	0.8830	0.9299	0.9548	0.9692
3	0.9657	0.9843	0.9913	0.9955
5	0.9973	0.9993	0.9998	0.9999
8	0.9999	0.9999	0.9999	0.9999
10	0.9999	0.9999	0.9999	0.9999
15	0.9999	0.9999	0.9968	0.9693
20	0.9999	0.9962	0.9969	0.7327

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