A Simple Numerical Method For Van der Pol-Duffing Oscillator Equation

Chengli Zhang

Faculty of Metallurgical and Energy Engineering
Kunming University of Science and Technology
Kunming, China
e-mail: Zhangchengli1118@163.com

Yun Zeng

Department of Engineering Mechanics
Kunming University of Science and Technology
Kunming, China
e-mail: zengyun001@163.com

Abstract—The famous Van der Pol-Duffing oscillator is a typical nonlinear oscillator, its accurate analytical solution can not yet directly be got now. The segmenting recursion method was put forward in this paper, the oscillator equation was reasonably linearized within small segment, then the approximate analytical solutions within small segment were got, and the recursion formulas were obtained, finally all numerical solutions were got by the recursion formulas. The numerical solutions of several Van der Pol-Duffing oscillator equation were got by means of the segmenting recursion method, and the numerical solutions were compared with Runge-Kutta method or Gear linear multistep method. The comparative results show that the segmenting recursion method is very effective to solve Van der Pol-Duffing oscillator equation, not only method is simple, programming is easy, but result is accurate, this method is a universal new method to solve similar oscillator.

Keywords- Van der Pol-Duffing; nonlinear; linearization; segmenting method; recursion method

I. INTRODUCTION

The famous Van der Pol-Duffing oscillator is a typical nonlinear oscillator, which has rich dynamical behaviors, widely exists in electronics, circuit, mechanics, finance and other models, has important research value and wide application background, is often used to verify the effectiveness and accuracy of new method in [1] - [5].

The accurate analytical solution of Van der Pol-Duffing oscillator equation can not yet be obtained now in [6] - [8], thus judgment method and asymptotic method were widely applied in [9] - [12]. Judgment method is mainly used for nonlinear control system to avoid directly solving equations in [13]. Asymptotic method is mainly used for mathematics and mechanics fields, such as phase plane method, describing function method, approximate analytical method and numerical method, and so on in [14]. Approximate analytical method mainly includes linearization method, perturbation method, homotopy analysis method, variational iteration method, etc. in [15]. Numerical method is applied widely, the traditional method includes Euler method, Runge-Kutta method and linear multistep method, etc. the newer method includes differential geometry theory, geometric integration theory, differential algebraic method, etc. in [16].

Linearization method mainly includes Taylor series method, small signal analysis method, least square method,

neglecting higher order small quantity method, segmenting method, and so on in [17]. Segmenting method divides working area into many small segments, and the curves within small segment are approximately replaced by straight lines in [18], then the approximate analytical solutions within small segments are obtained. In principle, segmenting method is suitable for all nonlinear system.

The segmenting recursion method put forward in this paper is a simple new method combined segmenting method with recursion method. First, the equation within small segment is linearized by segmenting, and then the recursion formulas were obtained by solving the approximate analytical solution within small segment, finally, all numerical solutions are got based on the recursion formulas. The numerical solutions of several Van der Pol-Duffing oscillator were got by means of the segmenting recursion method, and the numerical solutions were compared with Runge-Kutta method or Gear linear multistep method to verify the effectiveness solving Van der Pol-Duffing oscillator, then step length selection is analyzed.

II. SEGMENTING RECURSION METHOD

Van der Pol-Duffing forced oscillator equation is as follows:

$$\begin{cases} \frac{d^2 y}{dt^2} + (\varepsilon + \mu y^2) \frac{dy}{dt} + \alpha y + \beta y^3 = b \cos(\omega t) \\ y(0) = y_0, \frac{dy}{dt}(0) = v_0 \end{cases} \quad (1)$$

where ε is linear damping coefficient, μ is nonlinear damping coefficient, α is linear item coefficient, β is nonlinear stiffness coefficient, b is excitation amplitude, ω is excitation frequency, y_0 is the initial value of y at $t = 0$, v_0 is the initial value of dy/dt at $t = 0$.

When $b = 0$, (1) degrades into self-excitation oscillator equation, when $\beta = 0$, (1) degrades into Van der Pol oscillator equation; When $\mu = 0$, (1) degrades into Duffing oscillator equation.

Equation (1) is organized into the following form:

$$\frac{d^2 y}{dt^2} + A \frac{dy}{dt} = B, \quad y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0 \quad (2)$$

where $A = \varepsilon + \mu y^2$, $B = b \cos(\omega t) - \alpha y - \beta y^3$.

The specific steps solving (2) by segmenting recursive method (or SRM for short) are as follows:

Segmenting linearization: according to the principle of segmenting method, t is divided into many small segments, when step length Δt is small enough, because A and B changed very little within small segment $[t_i, t_i + \Delta t]$, A and B can be regarded as fixed values, and takes A and B of $t = t_i + 0.5\Delta t$ as the fixed values of A and B within $[t_i, t_i + \Delta t]$, namely (2) can be regarded as linear second-order ordinary differential equation within $[t_i, t_i + \Delta t]$.

Solving linear second-order ordinary differential equation: within $[t_i, t_i + \Delta t]$, based on Laplace transform, (2) is transformed to:

$$Y(s) = \frac{y_i s^2 + (y_i A + v_i) s + B}{s^2(s + A)} \quad (3)$$

where y_i is the initial value of y within $[t_i, t_i + \Delta t]$, v_i is the initial value of dy/dt within $[t_i, t_i + \Delta t]$, namely $(dy/dt)_i$.

The analytical solution of (3) within $[t_i, t_i + \Delta t]$ is obtained by means of inverse Laplace transform:

$$\begin{cases} \frac{d^2 y}{dt^2} = (B - Av_i)e^{-At} \\ \frac{dy}{dt} = \frac{B}{A} + (v_i - \frac{B}{A})e^{-At} \\ y = y_i + \frac{v_i}{A}t + \frac{B}{A^2} - \frac{1}{A}(\frac{B}{A} - v_i)e^{-At} \end{cases} \quad (4)$$

Getting recursive formulas: the solution of $t = t_i + \Delta t$ is expressed by the solution of $t = t_i$, the following recursive formulas are got:

$$\begin{cases} (\frac{d^2 y}{dt^2})_{i+1} = [B - A(\frac{dy}{dt})_i]e^{-A\Delta t} \\ (\frac{dy}{dt})_{i+1} = \frac{B}{A} - \frac{1}{A}(\frac{d^2 y}{dt^2})_{i+1} \\ y_{i+1} = y_i + \frac{B}{A}\Delta t + \frac{1}{A}(\frac{dy}{dt})_i - \frac{1}{A}(\frac{dy}{dt})_{i+1} \end{cases} \quad (5)$$

Numerical algorithm: takes a suitable step length Δt , gets A and B of $t = t_i$ by initial values y_i and v_i , and takes A and B of $t = t_i$ as the fixed values of A and B within $[t_i, t_i + 0.5\Delta t]$. Obtains the prediction solution of $t = t_i + 0.5\Delta t$ by substituting the fixed values of A and B within $[t_i, t_i + 0.5\Delta t]$ and step length $0.5\Delta t$ into recursive formulas. According to the prediction solution, gets A and B of $t =$

$t_i + 0.5\Delta t$, and takes A and B of $t = t_i + 0.5\Delta t$ as the fixed values of A and B within $[t_i, t_i + \Delta t]$. Obtains the numerical solution of $t = t_i + \Delta t$ by again substituting the fixed values of A and B within $[t_i, t_i + \Delta t]$ and step length Δt into recursive formulas. Repeats recursive process, until obtains all numerical solution. In the beginning, takes y_0 and v_0 as initial values.

III. COMPUTATION RESULTS AND VALIDATION

In the following examples, numerical solutions are first obtained by SRM, and then numerical solutions are obtained based on fourth-order Runge-Kutta algorithm ODE45 improved by Fehlberg in MATLAB (or RKF for short). For stiff equation, because RKF loses effectiveness, numerical solutions are obtained based on Gear multistep reverse numerical integration algorithm ODE15s in MATLAB (or Gear for short). RKF and Gear are adaptive variable step length algorithm, unless otherwise stated, the relative error of RKF and Gear are set to 10^{-6} in order to ensure computation accuracy. Finally, two kinds of numerical solutions are compared to verify the effectiveness of SRM.

A. Van der Pol Oscillator Equation

Van der Pol self-excitation oscillator equation is as follows:

$$\begin{cases} \frac{d^2 y}{dt^2} + 2(y^2 - 1)\frac{dy}{dt} + y = 0 \\ y(0) = -0.2, \frac{dy}{dt}(0) = -0.7 \end{cases} \quad (6)$$

Lets $A = 2(y^2 - 1)$, $B = -y$, takes $\Delta t = 0.04$, the numerical solutions of SRM and RKF are shown in Fig .1. At this time, the RKF computation points are 1377, and the SRM computation points are only 751, which are about half of the RKF computation points.

B. Duffing Oscillator Equation

Duffing forced oscillator equation is as follows:

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} - y + y^3 = 8\cos 0.7t, y(0) = 1.0, \frac{dy}{dt}(0) = 0.5 \quad (7)$$

Lets $A = 1$, $B = 8\cos 0.7t + y - y^3$, takes $\Delta t = 0.06$, the numerical solutions of SRM and RKF are shown in Fig .2. At this time, the RKF computation points are 2101, and the SRM computation points are only 667, which are less than one-third of the RKF computation points.

Figure 1. The SRM and RKF numerical solutions of Van der Pol self-excited oscillator equation when $\Delta t = 0.04$.

C. Van der Pol-Duffing Oscillator Equation

Van der Pol-Duffing forced oscillator equation is as follows:

$$\frac{d^2 y}{dt^2} + (y^2 - 1) \frac{dy}{dt} - y + y^3 = \cos 0.7t, y(0) = 0.1, \frac{dy}{dt}(0) = -0.2 \quad (8)$$

Lets $A = y^2 - 1$, $B = \cos 0.7t + y - y^3$, takes $\Delta t = 0.09$, the numerical solutions of SRM and RKF are shown in Fig .3. At this time, the RKF computation points are 1649, and the SRM computation points are only 556, which are about one-third of the RKF computation points.

Fig .1–Fig .3 show: when step length $\Delta t = 0.04 - 0.09$, the numerical solutions obtained by SRM and RKF have been very close to, almost exactly the same, but the calculating points are greatly reduced, indicates that SRM is very effective to solve Van der Pol-Duffing oscillator equation, not only method is simple, programming is easy, but result is accurate. If step length Δt is further reduced, the numerical solutions obtained by SRM will be more accurate. Meanwhile, their phase diagrams show that (6)–(8) all have stable limit cycle, indicates that their movement all finally are periodic vibration.

IV. STEP LENGTH SELECTION

A. Conventional Computation Step Length

In numerical computation, the absolute error of numerical solution is jointly decided by truncation error and rounding error. According to the principle and method of SRM, the reason caused truncation error by SRM is that A and B are linearized to fixed values within small segment $[t_i, t_i + \Delta t]$, and the recursive formulas are accurate. Therefore, the smaller step length Δt is, the smaller truncation error is, the higher computation accuracy is, but recursive step number will increase correspondingly, thus computation workload will increase, and rounding error will also increase slightly. So a suitable step length should be selected according to accuracy requirement in computation.

For (6), takes $\Delta t = 0.1$, the numerical solutions of SRM and RKF are shown in Fig .4. At this time, the RKF computation points are 1377, and the SRM computation points are only 300, which are less than a quarter of the RKF computation points, because the SRM computation points are too little, the numerical solutions obtained have small error, and error gradually increases as t increases.

Figure 2. The SRM and RKF numerical solutions of Duffing forced oscillator equation when $\Delta t = 0.06$.

Figure 3. The SRM and RKF numerical solutions of Van der Pol-Duffing forced oscillator equation when $\Delta t = 0.09$

Figure 4. The SRM and RKF numerical solutions of Van der Pol self-excited oscillator equation when $\Delta t = 0.1$.

Fig. 4 shows: when step length Δt is too large, because computation points are too little, causes larger linearization truncation error. In most cases, Van der Pol-Duffing oscillator equation is a strong nonlinear equation, namely the nonlinearity of A or B in (2) is strong, so Δt should not be taken too large in order to ensure computation accuracy. In conventional computation, taking step length $\Delta t = 0.01 - 0.05$ can meet accuracy requirement, the stronger nonlinearity of A or B is, the smaller Δt is taken. When the nonlinearity of A and B are weak, linearization error is very little, at this time, even if taking a larger step length Δt , computation accuracy can also meet requirement.

B. Chaotic Motion Step Length

Van der Pol-Duffing forced oscillator equation is as follows:

$$\begin{cases} \frac{d^2 y}{dt^2} + 0.2(y^2 - 1)\frac{dy}{dt} - y + y^3 = 0.53\cos t \\ y(0) = 0.1, \quad \frac{dy}{dt}(0) = -0.2 \end{cases} \quad (9)$$

Lets $A = 0.2(y^2 - 1)$, $B = 0.53\cos t + y - y^3$, when relative error is default value 10^{-3} , the numerical solutions of (13) are obtained by Gear and RKF. Takes $\Delta t = 0.03$ and $\Delta t = 0.01$, the numerical solutions are obtained by SRM. Takes relative error 10^{-6} , the numerical solutions are obtained by RKF. These results are compared in Fig. 5. Fig. 5 shows: when relative error is 10^{-3} , the numerical solutions of Gear and RKF have large error after $t > 10$, especially when $t = 25 - 35$, error is larger, but when relative error is 10^{-6} , the numerical solutions of RKF are very accurate. Meanwhile, takes $\Delta t = 0.03$, the numerical solutions of SRM have also large error after $t > 30$, but takes $\Delta t = 0.01$, the numerical solutions of SRM are also very accurate, almost coincide completely with the

numerical solutions of RKF when relative error is 10^{-6} . Fig. 5 shows that (13) is in chaotic motion, the reaction to truncation error is very sensitive.

When there is chaotic motion, the size of truncation error may have a significant effect to chaotic motion, larger error may change the nature of solution, thus chaotic characteristics may be lost. Therefore, when solving the chaotic motion numerical solutions, step length Δt should be taken a smaller value to reduce truncation error and improve computation accuracy, when taking $\Delta t = 0.005 - 0.01$, the results are generally desirable.

C. Adaptive Variable Step Length

When Van der Pol-Duffing oscillator equation is solved by SRM, fixed step length generally can meet computation requirement, adaptive variable step length method also can be used to further improve computation efficiency and reduce computation time.

The relative error is defined as follows:

$$\Delta = \left| \frac{y_{i+1}^{(\Delta t/2)} - y_{i+1}^{(\Delta t)}}{y_{i+1}^{(\Delta t/2)}} \right| \quad (10)$$

where $y_{i+1}^{(\Delta t/2)}$ is the solution after computing two steps by means of half step length $0.5\Delta t$, $y_{i+1}^{(\Delta t)}$ is the solution after computing one step by means of step length Δt .

Given a relative error ε , when $\Delta > \varepsilon$, the numerical solution is computed by means of half step length, until meeting $\Delta < \varepsilon$, takes final $y_{i+1}^{(\Delta t/2)}$ as result. When $\Delta \leq \varepsilon$, the numerical solution is computed by means of double step length, until meeting $\Delta \geq \varepsilon$, then the numerical solution is computed once by means of half step length, takes final $y_{i+1}^{(\Delta t/2)}$ as result.

Figure 5. The SRM and RKF numerical solutions of Van der Pol forced oscillator in chaotic motion.

Van der Pol self-excitation oscillator equation is as follows:

$$\begin{cases} \frac{d^2 y}{dt^2} + 1000(y^2 - 1) \frac{dy}{dt} + y = 0 \\ y(0) = -0.2, \quad \frac{dy}{dt}(0) = -0.7 \end{cases} \quad (11)$$

Lets $A = 1000(y^2 - 1)$, $B = -y$, based on adaptive variable step length method, when simulation time is 3000 and $\varepsilon = 10^{-6}$, the numerical solutions of SRM and Gear are shown in Fig .6. At this time, the Gear computation points are 2036, and the SRM computation points are 2225,

Figure 6. The SRM and Gear numerical solutions of Van der Pol stiff equation with long simulation time.

V. CONCLUSIONS

Segmenting recursion method is very effective to solve Van der Pol-Duffing oscillator equation, not only method is simple, programming is easy, but result is accurate, this method is suitable for strong nonlinear, weak nonlinear and stiff equation, it is a universal new numerical method to solve similar equation. In conventional computation, taking step length 0.01 – 0.05 generally can obtain more accurate results. In chaotic motion computation, in order to avoid losing chaotic characteristics, taking step length can 0.005 – 0.01 generally can obtain desirable results. For stiff equation or long simulation time, adaptive variable step length method can be used to improve computation efficiency.

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which are slightly more than the Gear computation points, the numerical solutions obtained by two methods are also very close to, almost exactly the same.

For adaptive variable step length method, when numerical solutions change quickly, automatically take smaller step length, and when numerical solutions change slowly, automatically take larger step length, not only ensure computation accuracy, but also improve computation efficiency and reduce computation time, of course, programming difficulty will slightly increase. This method is specially suitable for stiff equation or long simulation time.

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