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Testing Exponentiality Against UBAC Using Kernel Methods

M. M. Mohie El-Din[1], S. E. Abu-Youssef[1], M. KH. Hassan[2,*]

[1] Department of Mathematics, Faculty of Science,
Al- Azhar University, Cairo, Egypt;
[2] Department of Mathematics, Faculty of Education,
Ain Shams University, Cairo, Egypt

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Abstract

In this paper, the problem of testing exponentiality against used better than aged in convex ordering classes of life distributions is investigated. For this property a nonparametric test is presented based on kernel method. The test is presented for complete and right censored data. Furthermore, Pitman's asymptotic relative efficiency (PARE) is discussed to assess the performance of the test with respect to other tests. Selected critical values are tabulated. Some numerical simulations on the power estimates are presented for proposed test. Finally, examples in medical sciences are used as practical applications for the proposed test.

Keywords: UBAC Classes of Life Distributions; Kernel Method; Survival function; Right censored data; Kaplan-Meier estimator.

1. Introduction

During the past decades, various classes of life distributions have been proposed in order to model different aspects of aging. The best known of these classes are IFR, IFRA, DMRL, NBU, NBUE, and HNBUE. Properties and applications of these aging notions can be found, in Bryson and Siddiqui [6], Barlow and Proschan [4], Rolski [10], Klefsjö [9], and Stoyan [12].

Let X be a random variable describing the life time of a brand new device which begins to work at time $t = 0$. As usual in the reliability, we denoted by X_t the life time of

the device of age t with $t \geq 0$. The probability that the device of age *t* still working till time *x* (the survival function) is,
$\overline{F}_t(x) = P[X > x+t \mid x > t] = \overline{F}(x+t)/\overline{F}(t)$, where $\overline{F}(x)$ is the survival function of *X*. Some properties concerning the asymptotic behavior of X_t as $t \to \infty$ will be used.

Definition (1.1): (Bhattacharjee, [5]), If X is a nonnegative random variable, its distribution function F(x) is said to be finitely and positively smooth if a number γ □ (0, ∞) exists such that $\lim\limits_{t\to\infty} \overline{F}_t(x) = e^{-\gamma x}$ for all $x \geq 0$, where γ will be called the asymptotic decay coefficient of X.
Denoting X_e be an exponentially distributed random variable with mean $1/\gamma$, the following definitions implies that X_t converges to X_e in distribution written as $X_t \xrightarrow{D} X_e$. This property is useful for description of random life times of devices of unknown age.

Definition (1.2): The distribution function F is said to be used better than age (UBA) if for all *x, t* ⩾ 0; $\overline{F}_t(x) \geq e^{-\gamma x}$ or $\overline{F}(x+t) \geq \overline{F}(t)e^{-\gamma x}$. (1.1)

From definition (1.2), we have the following definition:

Definition (1.3): The distribution function F is said to be used better than aged in convex ordering (UBAC) if for all *x, t ⩾ 0*

$$\int_x^\infty \overline{F}(u+t)du \geq \overline{F}(t)\int_x^\infty e^{-\gamma u}du \quad \text{or} \quad \nu(x+t) \geq \frac{1}{\gamma}\overline{F}(t)e^{-\gamma x} \tag{1.2}$$

Where: $\nu(x+t) = \int_{x+t}^\infty \overline{F}(u)du$.

We observe that the inequality of (1.2) is achieved when F(x) has an exponential distribution with mean μ equal to the coefficient of the asymptotic decay γ, where the exponential distribution is the only one which has the lack of memory property. Willmot and Cai [13] showed that the UBA class includes the DMRL class. While Al – Nachawati and Alwasel [2] showed that UBAC class includes the UBA class of life distribution. Thus, we have IFR □ DMRL □ UBA □ UBAC.

Testing exponentiality against the classes of life distribution based on kernel method has seen a good deal of attention. For testing against IFR and DMRL, we refer Ahmed [1]. Our goal in this paper is to propose a new nonparametric test for exponentiality against UBAC based on kernel method in section 2. We use Pitman's asymptotic relative efficiency (PARE) to assess the performance of the test and make a comparison with other available tests, in section 3. A nonparametric test is presented for right censored data in section 4. Finally, examples using data from Attia et al [3] in medical science is given in section 5.

2. Testing Exponentiality Versus UBAC Class Based On Kernel Method

Suppose the lifetime X of a component has a distribution function *F* which is unknown to us. Available to us are independent observations on *n* components; i.e. we have at our disposal a random sample $X_1, X_2, ..., X_n$ from the distribution *F*. We have a null hypothesis H_0 and its alternative H_1, where H_0 : F is exponential versus $H_1 : F$ belongs to the class UBAC and F is not exponential.

Ismail and Abu-Youssef [7] introduced testing exponentiality versus UBAC class based on U-statistic but in this paper, we suggest testing exponentiality versus UBAC class of life distribution based on kernel method as follow:

$$\delta_K = \int_0^\infty \int_0^\infty f(x)[\frac{1}{\gamma}\overline{F}(t)e^{-\gamma x} - \nu(x+t)]dF(x)dF(t). \tag{2.1}$$

It is easy to see that if *F* is exponential, then $\delta_K = 0$, while under H_1, we have $\delta_K > (<) 0$. Since the distribution function F is unknown to us, as a consequence we do not know the δ_K. This means we need to construct $\hat{\delta}_K$ a good estimator of δ_K. This can be done in terms of F_n, the empirical distribution function based on the available sample $X_1, X_2, ..., X_n$ from F.

The estimator $\hat{\delta}_K$ of our test statistic is defined by

$$\hat{\delta}_K = \int_0^\infty \int_0^\infty \hat{f}_n(x)[\frac{1}{\gamma}\hat{\overline{F}}_n(t)e^{-\hat{\gamma} x} - \hat{\nu}_n(x+t)]dF_n(x)dF_n(t). \tag{2.2}$$

As usual we denoted by $X_{(1)}, X_{(2)}, ..., X_{(n)}$ the corresponding ordered sample and if $\hat{F}_n$ is the empirical distribution function, then $\hat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} I(X_j > x)$ is the empirical survival function, and $F_n(x) = \frac{i}{n}$ for $x \in [X_{(i)}, X)$, where i=1,2,…,n.

In order to suggest an estimator for δ_K we first construct estimators $\hat{\nu}_n(x)$, $\hat{\gamma}$ and $\hat{f}_n(x)$ for $\nu(x)$, γ and $f(x)$ respectively. $\hat{\nu}_n(x) = \frac{1}{n}\sum_{i=1}^{n}(X_i - x)I(X_i > x)$,

$\hat{\gamma} = \frac{n}{\sum_{i=1}^{n} x_i}$ and $\hat{f}_n(x) = \frac{1}{na_n}\sum_{l=1}^{n} K(\frac{x - X_l}{a_n})$ where $I(X_i > x)$ is the indicator function; $I(X_i > x) = 1$, if $X_i > x$; otherwise $I(X_i > x) = 0$. and *K(.)* be a known probability density function, symmetric and bounded with mean zero and finite variance σ_K^2. Let a_n be a sequence of real such that $a_n \to 0$ and $na_n \to \infty$, $n \to \infty$.

These properties suggest writing the following estimator as

$$\hat{\delta}_K = \frac{1}{n^4 a_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} K\left(\frac{X_i - X_l}{a_n}\right)\left[\frac{1}{\hat{\gamma}} e^{-\hat{\gamma} X_j}(X_i > X_k) - (X_i - X_j - X_k) I(X_i > X_j + X_k)\right] \tag{2.3}$$

To make the test statistic in scale invariant, we take $\hat{\delta}^*_K = \frac{\hat{\delta}_K}{\overline{X}}$. (2.4)

In order to use the U-statistic procedure, we set

$$\varphi(X_1, X_2, X_3, X_4) = K\left(\frac{X_1 - X_4}{a_n}\right)\left[\frac{1}{\hat{\gamma}} e^{-\hat{\gamma} X_2} I(X_1 > X_3) - (X_1 - X_2 - X_3) I(X_1 > X_2 + X_3)\right] \tag{2.5}$$

Here X_1, X_2, X_3 and X_4 are four independent lifetimes each with distribution function F. We define the symmetric kernel as

$$\phi(X_1, X_2, X_3, X_4) = \frac{1}{4!}\sum_R \varphi(X_1, X_2, X_3, X_4)$$

Where the sum is over all permutations of X_1, X_2, X_3 and X_4 .Then $\hat{\delta}_K$ is equivalent to the U-statistic

$$U_n = \frac{1}{\binom{n}{4}} \sum_{i<j<k<l} \varphi(X_1, X_2, X_3, X_4)$$

Our test is rejects for large value of U_n .

Theorem (2.1): If $n^4 a_n \to 0$ as n$\to \infty$, $\sqrt{n}(\hat{\delta}^*_K - \delta^*_K)$ is asymptotically normal with mean 0 and variance V^2 where,

$$\begin{aligned} V^2 = Var[2f(X_1)[\frac{1}{\gamma}\int_0^{\infty}\int_0^{u} e^{-\gamma u} dF(y)dF(u) - \int_0^{X_1}\int_0^{X_1-y}(X_1 - u - y)dF(u)dF(y)] \\ + \frac{1}{\gamma}\int_0^{\infty}\int_0^{u} e^{-\gamma X_1} f(u)dF(y)dF(u) - \int_0^{X_1}\int_{X_1-u}^{0}(u - X_1 - y) f(u)dF(y)dF(u) \\ + \frac{1}{\gamma}\int_0^{\infty}\int_0^{X_1} e^{-\gamma y} f(u)dF(y)dF(u) - \int_{X_1}^{0}\int_0^{u-X_1}(u - X_1 - y) f(u)dF(y)dF(u)] \end{aligned}$$

Proof: We use theorem of Serfling [11], which states that if $c_1 < \infty$, then $\sqrt{n}(U_n - \theta) \to N(o, m^2 c_1)$, where $c_1 = Var[\phi(X_1)]$. we need to find $\phi(X_1)$ which is by definition

$$\begin{aligned} \phi(X_1) = E[\varphi(X_1, X_2, X_3, X_4) \mid X_1] + E[\varphi(X_2, X_1, X_3, X_4) \mid X_1] \\ + E[\varphi(X_2, X_3, X_1, X_4) \mid X_1] + E[\varphi(X_2, X_3, X_4, X_1) \mid X_1] \end{aligned}$$

This can be written explicitly as follows:

$$E[\varphi(X_1, X_2, X_3, X_4) \mid X_1] = f(X_1)[\frac{1}{\gamma}\int_0^{\infty}\int_0^{u} e^{-\gamma u} dF(y)dF(u) - \int_0^{X_1}\int_0^{X_1-y}(X_1 u - y)dF(u)dF(y) \tag{2.6}$$

Similarly, we have

$$E[\varphi(X_2, X_1, X_3, X_4) \mid X_1] = \frac{1}{\gamma}\int_0^{\infty}\int_0^{u} e^{-\gamma X_1} f(u)dF(y)dF(u) - \int_0^{X_1}\int_{X_1-u}^{0} (u - X_1 - y) f(u)dF(y)dF(u) \tag{2.7}$$

and

$$E[\varphi(X_2, X_3, X_1, X_4) \mid X_1] = \frac{1}{\gamma}\int_0^{\infty}\int_0^{X_1} e^{-\gamma y} f(u)dF(y)dF(u) - \int_{X_1}^{0}\int_0^{u-X_1} (u - X_1 - y) f(u)dF(y)dF(u) \tag{2.8}$$

Observe that $E[\varphi(X_2, X_3, X_1, X_4) \mid X_1]$ has the same representation as (2.6). By taking the variance of $\phi(X_1)$ we get c_1.

Corollary (2.1): If the null hypothesis H_0 is true, i. e., the lifetime distribution F is exponential, then as we mentioned before, $\delta_K = 0$ and we calculate explicitly the null variance, $V_0^2 = 0.387$.

Proof: Under H_0, $\overline{F}_0 = \overline{F}(x) = e^{-x}$ and by direct calculation we found

$$\phi_0(X_1) = -\frac{1}{12}[6(X_1 - 7e^{-X_1}) + (24X_1 + 37)e^{-2X_1}]$$

Thus, E [$\phi_0(X_1)] = 0$ and Var[$\phi_0(X_1)] = c_1 = 0.387$.

To conduct the test, calculate $\sqrt{n}\hat{\delta}_K / V_0$ and reject H_0 if this value exceeds the standard normal value $Z_{1-\alpha}$. To illustrate the test, we have simulated the upper percentile points for the significance level $\alpha = 0.01$, 0.02, and0.05. The calculation of the test $\hat{\delta}_K$ is based on 10000 simulated samples from the standard exponential distribution. Table (1), gives the critical values of the test statistic $\hat{\delta}_K$. Figure (1) shows the critical values of the test statistic $\hat{\delta}_K$ are decreasing as the sample size increasing as follow:

Table (1): Critical value for $\hat{\delta}_K$

n	95%	98%	99%
4	0.607514	0.722601	0.800363
6	0.50763	0.601598	0.66509
8	0.431089	0.512468	0.567453
10	0.385577	0.458365	0.507546
12	0.351982	0.418428	0.463324
14	0.325872	0.387389	0.428954
16	0.304826	0.362369	0.40125
18	0.290954	0.345206	0.381863
20	0.276294	0.327763	0.362539
22	0.259956	0.30903	0.342187
24	0.247201	0.294185	0.325931
26	0.239125	0.284266	0.314767
28	0.230427	0.273925	0.303317
30	0.222613	0.264637	0.293032
32	0.231136	0.271826	0.299319
34	0.207966	0.24744	0.274112
36	0.203217	0.24158	0.26750
38	0197797	0.235136	0.260365
40	0.194296	0.23069	0.25528
42	0.195178	0.230694	0.254692
44	0.180863	0.215563	0.239009
46	0.179776	0.213714	0.236644
48	0.175991	0.209214	0.231662
50	0.176895	0.209447	0.231441

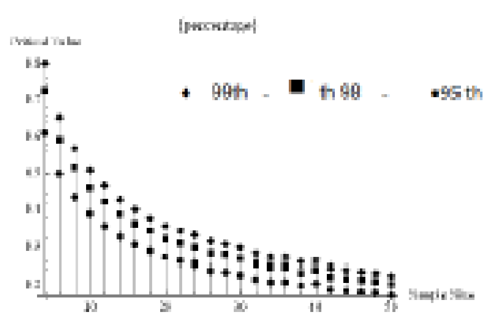

Figure 1: The Relation Between Sample Size and Critical Values

The power estimate of the test statistic $\hat{\delta}_K$ is useful in clarifying how much the test can detect the departure from exponentiality towards the class UBAC. The higher value of the power estimate indicates that the test statistic is more able to detect such a departure. The power of the test statistics $\hat{\delta}_K$ is considered for 5% percentile in Table (2) for three alternatives. These alternatives are:

1. Linear failure rate: $\overline{F}_\theta(x) = e^{-(x+\frac{1}{2}\theta x^2)}$, $x > 0$, $\theta \geq 0$
2. Makeham: $\overline{F}_\theta(x) = e^{-x-\theta(x-1+e^{-x})}$, $x > 0$, $\theta \geq 0$
3. Weibull: $\overline{F}_\theta(x) = e^{-x^\theta}$ $x > 0$, $\theta \geq 0$

For appropriate values of θ, these distributions can be reduced to the exponential distribution. The power estimate of the test statistic $\hat{\delta}_K$, given in Table (2) shows the chance of detecting departure from exponentiality towards the UBAC property as θ increases, or the sample size n increases for the linear failure rate, Makeham, and Weibull distribution.

Table (2): Power estimates for $\hat{\delta}_K$

Distribution	θ	n=10	n=20	n=30
Linear failure rate	2	0.995	0.999	0.999
	3	0.998	0.999	1
	4	0.999	1	1
Makeham	2	0.993	0.999	0.999
	3	0.995	0.999	0.999
	4	0.999	0.999	1
Weibull	2	0.999	1	1

3. Asymptotic relative efficiency:

In this section we compare the power of the test statistic $\hat{\delta}_K$ using the concept of Pitman's asymptotic efficiency (PARE). To do this we need to evaluate the Pitman's asymptotic efficiency (PAE) of our test $\hat{\delta}_K$ and compare the PAR of other test to get PARE. Let F_{θ_n} be a sequence of alternative distributions, where $\theta_n = \theta_0 + k/\sqrt{n}$, k is positive number, and θ_0 corresponds to the exponential distribution. PAE is given by $e_{F_\theta}(T_{n,1}) = \lim_{n\to\infty} \frac{d}{d\theta} E_\theta(T_{n,1})|_{\theta\to\theta_0} (\sigma_0)^{-1}$ where $\sigma_0^2 = V_0^2$ is the null asymptotic variance.

The PARE of $T_{n,1}$ with respect to another test $T_{n,2}$ is then given by $e_{F_\theta}(T_{n,1})/e_{F_\theta}(T_{n,2})$. the efficiencies of $\hat{\delta}_K$ are calculated for the following alternatives:

linear failure rate, Makeham, and weibull distributions. Direct calculation of asymptotic efficiencies of the test U_n and $\hat{\delta}_K$ for NBUE, Kanjo [8] and UBAC classes respectively are summarized in Table (3). In Table (4) we give efficiencies of $\hat{\delta}_K$ with respect to U_n. These calculations clearly indicate that the test proposed in this paper is well comparable with other test widely used in practice, and in some cases is even better.

Table (3): PAR of $\hat{\delta}_K$ and U_n

Distribution	$\hat{\delta}_K$	U_n
Linear failure rate	0.565	0.433
Makeham	0.245	0.144
Weibull	0.424	0.132

Table (4): PARE of $\hat{\delta}_K$ with respect to U_n

Relative efficiency	Linear failure rate	Makeham	Weibull
$e_{F_\theta}(T_{n,1}) / e_{F_\theta}(T_{n,2})$.	1.305	1.701	3.121

4. Testing against UBAC class for right censored data.

In this section, a test statistic proposed to test H_0 versus H_1 with randomly right censored samples. In the censored model, instead of dealing with $X_1, X_2, ..., X_n$, we observe the pair (Z_i, δ_i), i=1,2,…,n, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = 1$ if $Z_i = X_i$, $\delta_i = 0$ if $Z_i = Y_i$, where $X_1, X_2, ..., X_n$ denote their true life time from a distribution F and $Y_1, Y_2, ..., Y_n$ be i.i.d. according to distribution G. Also X's and Y's are independent. Let $Z_{(0)} = 0 \le Z_{(1)} \le Z_{(2)} \le ... \le Z_{(n)}$ denote the order Z's and $\delta_{(i)}$ is the δ_i corresponding to $Z_{(i)}$, respectively. Using the Kaplan Meier estimator in the case of censored data (Z_i, δ_i), i=1… n, the proposed test statistic for right censored data is given by

$$\hat{\delta}_K^c = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{f}(x)[e^{\hat{\gamma} Z_{(j)}} \hat{\bar{F}}_n(t) - \hat{\nu}_n(x+t)][\prod_{p=1}^{i-2} C_p^{\delta_i} - \prod_{p=1}^{i-1} C_i^{\delta_i}][\prod_{q=1}^{j-2} C_j^{\delta_j} - \prod_{q=1}^{j-1} C_j^{\delta_j}] \quad (4.1)$$

Where: $\hat{\nu}_n(x+t) = \int_{x+t}^{\infty} \hat{\bar{F}}_n(z)dz = \hat{\mu} - \int_{0}^{x+t} \hat{\bar{F}}_n(z)dz$

$$= \hat{\mu} - \sum_{k=1}^{l} \prod_{m=1}^{k-1} C_m^{\delta_m} (Z_{(k)} - Z_{(k-1)})$$

Where, $l = i + j$ if $Z_{(i)} + Z_{(j)} < Z_{(n)}$, l= n if $Z_{(i)} + Z_{(j)} > Z_{(n)}$

$$\hat{\mu} = \sum_{j=1}^{l} \prod_{k=1}^{j-1} C_k^{\delta_k} (Z_{(j)} - Z_{(j-1)})$$

$$d\hat{F}_n(Z_j) = [\prod_{q=1}^{j-2} C_q^{\delta_q} - \prod_{q=1}^{j-1} C_q^{\delta_q}]$$

$$\hat{f}(x) = \sum_{m=1}^{n} \delta_m K(x - Z_{(k)})$$

and

$$\hat{\bar{F}}_n(t) = \prod_{m<Z_{(m)}<t} C_m^{\delta_m} \text{ Where } C_m = \frac{n-m}{n-m+1} \text{ and } t \in [0, Z_{(m)}].$$

To illustrate the test, we have simulated the upper percentile points for the significance level $\alpha = 0.01$, 0.02, and0.05. The calculation of the test $\hat{\delta}_K^C$ is based on 10000 simulated samples from the standard exponential distribution. Table (5), gives the critical values of the test statistic $\hat{\delta}_K^C$. Figure (2) shows the critical values of the test statistic $\hat{\delta}_K^C$ are decreasing as the sample size increasing as follow:

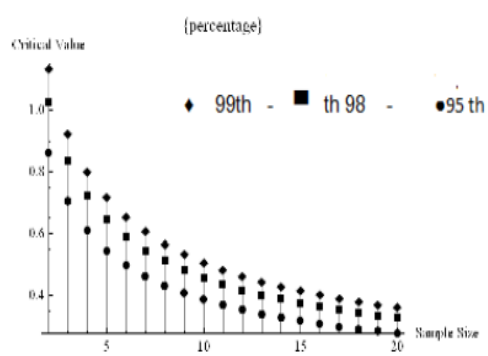

Figure 2: The Relation Between Sample Size and Critical Values (Censored Data)

Table (5): Critical value of $\hat{\delta}_K^C$

n	95%	98%	99%
1	0.862262	1.025020	1.13499
3	0.704056	0.836947	0.926738
4	0.609749	0.724836	0.802598
5	0.545394	0.648331	0.717883
6	0.497891	0.591860	0.655352
7	0.460975	0.547973	0.606755
8	0.431221	0.512600	0.567586
9	0.406579	0.483304	0.535145
10	0.385736	0.458524	0.507704
11	0.367808	0.437209	0.484101
12	0.352176	0.418622	0.463518
13	0.338390	0.402229	0.445363
14	0.326116	0.387633	0.429198
15	0.315099	0.374530	0.414686
16	0.305144	0.362687	0.401568
17	0.296095	0.351921	0.389640
18	0.287833	0.342085	0.378742
19	0.280262	0.33068	0.368747
20	0.273316	0.324785	0.359561
51	0.175919	0.208150	0.229928
61	0.157073	0.186544	0.206457
71	0.145133	0.172450	0.190907
81	0.135724	0.161299	0.178580
91	0.127978	0.152107	0.168411
101	0.121439	0.144342	0.159817

5. Application:

5.1 Application for complete data:

Example (1): The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al [3] the ordered life times (in days) are:

10; 14; 14; 14; 14; 14; 15; 17; 18; 20; 20; 20; 20; 20; 23; 23; 24; 26; 30; 30;
31; 40; 49; 51; 52; 60; 61; 67; 71; 74; 75; 87; 96; 105; 107; 107; 107; 116; 150:

It was found that the test statistic for the data set, $\hat{\delta}_K$ =235.999, which it exceeds the critical value of table 3. Then we reject the null hypothesis of exponentiality.

5.2 Application for censored data:

Example (2): The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al [3] the ordered life times (in days) are:

(i) Non-censored data
10; 14; 14; 14; 14; 14; 15; 17; 18; 20; 20; 20; 20; 20; 23; 23; 24; 26; 30; 30;
31; 40; 49; 51; 52; 60; 61; 67; 71; 74; 75; 87; 96; 105; 107; 107; 107; 116; 150.

(ii) Censored data
30; 30; 30; 30; 30; 60; 150; 150; 150; 150; 150; 185:
It was found that the test statistic for the data set, $\hat{\delta}_K^C$ =0.170761, which it decreases the critical value of table 3. Then we accept the null hypothesis of exponentiality.

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