Journal of Statistical Theory and Applications, Vol. 13, No. 1 (March 2014), 83-85

A characterization of the normal distribution

Hans Volkmer
Department of Mathematical Sciences
University of Wisconsin-Milwaukee
Milwaukee, WI 53201-0413, USA
volkmer@uwm.edu

Received 6 December 2013

Accepted 18 December 2013

It is shown that the normal distribution with mean zero is characterized by the property that the product of its characteristic function and moment generating function is equal to 1.

Keywords: Characterization, normal distribution, characteristic function

2000 Mathematics Subject Classification: 62E10, 60E10

1. The normal distribution

We will obtain a characterization of the normal distribution. Of course, there already exist many characterizations of the normal distribution; see [3, Chapter 4].

The characteristic function of the normal distribution with mean 0 and standard deviation σ is

$$f(t) = \exp(-\tfrac{1}{2}\sigma^2 t^2)$$

while its moment generating function is

$$g(t) = \exp(\tfrac{1}{2}\sigma^2 t^2).$$

Therefore, we obtain

$$f(t)g(t) = 1 \quad \text{for all } t \in \mathbb{R}. \tag{1.1}$$

Weixing Song asked whether there are any other probability distributions for which (1.1) is valid. Using the following result due to Fryntov [1, Theorem 3], we will show that the answer is "no".

Theorem 1.1. *Let*

$$h(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

be a power series with infinite radius of convergence, where $n_1 < n_2 < n_3 < \dots$ *are positive integers such that*

$$\lim_{k\to\infty} \frac{k}{n_k} < \frac{1}{2}.$$

If $\exp(h(z))$ *is a characteristic function, then* h *is a polynomial of degree at most* 2.

Published by Atlantis Press

Theorem 1.2. *Let $F(x)$ be a probability distribution with characteristic function $f(t)$ and moment generating function $g(t)$ (which is assumed to exist for all $t \in \mathbb{R}$.) If (1.1) holds, then $F(x)$ is a normal distribution with mean zero.*

Proof. Since we assume that $g(t)$ exists for all real t, f and g must be entire functions and $f(z) = g(iz)$ for $z \in \mathbb{C}$. This can be proved by a standard argument similar to that which shows analyticity of the Laplace transform; see [4, Chapter 2, Section 5]. By the identity theorem for analytic functions, we obtain that

$$g(z)g(iz) = 1 \quad \text{for all } z \in \mathbb{C}. \tag{1.2}$$

It follows from (1.2) that g is an entire function without zeros, so according to [2, Theorem 2.1, page 360] we can write

$$g(z) = \exp(h(z)) \quad \text{for all } z \in \mathbb{C}, \tag{1.3}$$

where h is an entire function (called the cumulant generating function) with $h(0) = 0$. It follows from (1.2), (1.3) that

$$h(z) + h(iz) = 0 \quad \text{for all } z \in \mathbb{C}.$$

This implies that

$$h(z) = \sum_{k=1}^{\infty} a_k z^{4k-2} = a_1 z^2 + a_2 z^6 + a_3 z^{10} + \dots.$$

Therefore, f is an entire characteristic function of the form

$$f(z) = \exp\left(-\sum_{k=1}^{\infty} a_k z^{4k-2}\right).$$

It follows from Theorem 1 that $f(z) = \exp(-a_1 z^2)$ which completes the proof. □

2. Extension to other distributions

G.G. Hamedani suggested to ask similar questions about other probability distributions. For example, consider the uniform distribution $F(x)$ on the interval $[-1, 1]$. Its characteristic function is

$$f(t) = \frac{\sin t}{t}$$

while its moment generating function is

$$g(t) = \frac{\sinh t}{t}.$$

Therefore, we obtain

$$f(t)g(t) = \frac{\sin t \sinh t}{t^2}. \tag{2.1}$$

If we consider the convolution $\tilde{F}$ of F with a normal distribution with mean zero, its characteristic function is

$$\tilde{f}(t) = \frac{\sin t}{t} \exp(-\tfrac{1}{2}\sigma^2 t^2)$$

and its generating function is

$$\tilde{g}(t) = \frac{\sinh t}{t} \exp(\tfrac{1}{2}\sigma^2 t^2).$$

Therefore, we again have

$$\tilde{f}(t)\tilde{g}(t) = \frac{\sin t \sinh t}{t^2}.$$

The question arises whether there are any other probability distribution for which (2.1) is valid. It appears that Theorem 1 is not sufficient to answer this question.

Acknowledgment: The author thanks colleagues G.G. Hamedani and Weixing Song for communicating these interesting questions.

References

[1] A. E. Fryntov, Characterization of a Gaussian distribution by gaps in its sequence of cumulants, *Theory Probab. Appl.* **33** (1988), 638-644.
[2] S. Lang, Complex Analysis, Third edition, Springer-Verlag, New York, 1993.
[3] J. Patel and C. Read, Handbook of the Normal Distribution, Marcel Dekker, New York, 1996.
[4] D. Widder, The Laplace Transform, Dover reprint, Minealo, New York, 2010.