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# Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves

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**Abstract**

We show that within the framework of linear theory the particle paths in a periodic gravity-capillary or pure capillary deep-water wave are not closed.

## 1 Introduction

The three main types of water waves (capillary, capillary-gravity, and gravity waves) have generally different properties, and in this paper we are interested in the particle trajectories of periodic capillary and capillary-gravity waves propagating over water of infinite depth— a setting suitable for describing these types of waves in deep water. We work within the framework of linear theory. Since the nonlinear governing equations are highly intractable (while there exist some results for the nonlinear governing equations— see [4, 5, 6, 7, 8, 11, 13, 14, 16, 20, 21]— the information available is not sufficiently detailed as to enable a study of the particle paths in the fluid), it appears that the linear framework is appropriate for a first study. In this note we will focus on two-dimensional periodic travelling waves. Previously, formal considerations have suggested that particle trajectories in the fluid are closed (see [12, 17, 19]), and while there are special solutions to the nonlinear governing equations with all particle trajectories closed (see [1, 2]), what we will show in this note is that, over water of infinite depth, within linear capillary and capillary-gravity theory for steady waves this is not the case: if the surface is not flat there are no closed orbits in the fluid– see Theorem 1. It is anticipated that further studies will permit the extension of these results to the nonlinear governing equations (for gravity waves the features observed within the linear theory in [9] were recently proven [3] to hold true for the nonlinear governing equations).

## 2 Preliminaries

In what follows we present the equations of motion for two dimensional capillary-gravity and pure capillary waves in water of infinite depth, together with their standard solutions upon linearisation. We assume the waves are two dimensional, that is, the motion is identical in any direction parallel to the crest line. Thus we need only consider a cross section of the flow, in the direction perpendicular to the crest line, using the Cartesian coordinates $(x, y)$ where the $x-$axis is the direction of wave propagation while the $y-$axis points vertically upwards. Let $(u(t, x, y), v(t, x, y))$ be the velocity field of the flow in the fluid domain $\{(x, y) : x \in \mathbb{R}, y < \eta(t, x)\}$, where $y = \eta(t, x)$ is the water's free surface with the mean water level given by $y = 0$. We adopt the physically reasonable assumption of homogeneity (constant density) in the fluid for capillary and capillary-gravity deepwater waves [17]. This implies the equation of mass conservation

$$u_x + v_y = 0. \tag{2.1}$$

Making the further assumption of inviscid flow, the governing equations for the motion of the waves are given by Euler's equation

$$\begin{cases} u_t + uu_x + vu_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases} \tag{2.2}$$

where $P(t, x, y)$ denotes the pressure and $g$ is the gravitational constant of acceleration. To decouple the motion of the air from that of the free surface particles [15] we introduce the dynamic boundary condition

$$P = P_0 - \frac{\Gamma}{R} \qquad \text{on } y = \eta(t, x), \tag{2.3}$$

where $P_0$ is the constant atmospheric pressure, the parameter $\Gamma(> 0)$ is the coefficient of surface tension, and $1/R$ is the curvature in the $x-$direction given in Cartesian coordinates by

$$\frac{1}{R} = \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}. \tag{2.4}$$

Since the free-surface is always composed of the same particles we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on } y = \eta(t, x). \tag{2.5}$$

The boundary condition at the bottom, expressing the fact that at great depths there is practically no motion, is given by

$$(u, v) \to (0, 0) \text{ as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \tag{2.6}$$

The governing equations for the capillary-gravity deepwater wave problem are encompassed by the nonlinear free boundary problem (2.1)–(2.6), cf. [15]. The equations for pure capillary water-waves follow upon setting $g = 0$. We make the further assumption that

$$u_y = v_x, \tag{2.7}$$

an assumption which characterises irrotational flows. As a result of Kelvin's circulation theorem is reasonable to assume that water which is disturbed from a position of rest will remain irrotational at later times [15, 17].

### 2.1 The linearised problem

The problem (2.1)–(2.7) is nondimensionalised using a typical wavelength $\lambda$ and a typical amplitude $\epsilon$ of the wave. We fix the constant water density to be $\rho = 1$, and define the set of nondimensional variables

$$x \mapsto \lambda x,\ y \mapsto y,\ t \mapsto \frac{\lambda}{\sqrt{g}} t,\ u \mapsto u\sqrt{g},\ v \mapsto v\frac{\sqrt{g}}{\lambda},\ \eta \mapsto \epsilon\eta,$$

where for example we replace $x$ by $\lambda x$, with $x$ now being a nondimensionalised variable, thus avoiding new notation. The pressure in the new nondimensional variables is given by

$$P = P_0 - gy + gp,$$

where the nondimensional pressure variable $p$ measures the deviation from the hydrostatic pressure distribution. If we now apply the scaling

$$p \mapsto \epsilon p, \qquad (u, v) \mapsto \epsilon(u, v),$$

we end up with the following boundary value problem in nondimensional variables

$$\begin{cases} u_t + \epsilon(uu_x + vu_y) = -p_x, \\ \frac{1}{\lambda^2}\{v_t + \epsilon(uv_x + vv_y)\} = -p_y, \\ u_x + v_y = 0, \\ p = \eta - \frac{\Gamma}{g\lambda^2}\frac{\eta_{xx}}{(1+\frac{\epsilon^2}{\lambda^2}\eta_x^2)^{3/2}} \quad \text{and} \quad v = \eta_t + \epsilon u\eta_x \quad \text{on } y = \epsilon\eta, \\ (u, v) \to (0, 0) \quad \text{as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \end{cases} \tag{2.8}$$

The linearised problem is obtained by letting $\epsilon \to 0$ in (2.8), resulting in the equations

$$\begin{cases} u_t = -p_x, \quad \frac{1}{\lambda^2}v_t = -p_y, \\ u_x + v_y = 0, \\ v = \eta_t \quad \text{and} \quad p = \eta - \frac{\Gamma}{g\lambda^2}\eta_{xx} \quad \text{on } y = 0, \\ (u, v) \to (0, 0) \quad \text{as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \end{cases} \tag{2.9}$$

In order to solve the system (2.9) we look for travelling wave solutions, that is waves for which the $(t, x)-$dependence of $u, v, p, \eta$ is in the form of a periodic dependence in $x - c_0 t$, where $c_0 > 0$ represents the nondimensionalised speed of the wave. If we choose the Fourier mode

$$\eta(t, x) = \cos\big(2\pi(x - c_0 t)\big),$$

we obtain the solution

$$\begin{cases} \eta(t, x) = \cos\big(2\pi(x - c_0 t)\big), \\ u(t, x, y) = \frac{F(y)}{\lambda}\cos\big(2\pi(x - c_0 t)\big), \\ v(t, x, y) = F(y)\sin\big(2\pi(x - c_0 t)\big), \\ p(t, x, y) = \frac{c_0 F(y)}{\lambda}\cos\big(2\pi(x - c_0 t)\big), \end{cases} \tag{2.10}$$

so long as

$$c_0^2 = \frac{\lambda}{2\pi} + \frac{2\pi\Gamma}{g\lambda}, \qquad F(y) = 2\pi c_0 \exp\left(\frac{2\pi}{\lambda}y\right).$$

To return to the original physical variables we use the change of variables

$$x \mapsto \frac{x}{\lambda},\ y \mapsto y,\ t \mapsto t\frac{\sqrt{g}}{\lambda},\ u \mapsto \frac{u}{\epsilon\sqrt{g}},\ v \mapsto v\frac{\lambda}{\epsilon\sqrt{g}},\ \eta \mapsto \frac{\eta}{\epsilon},\ p \mapsto \frac{p}{\epsilon}.$$

If we define the wavenumber $k$ and the frequency $\omega$ by

$$k = \frac{2\pi}{\lambda}, \qquad \omega = \sqrt{gk + k^3\Gamma}, \tag{2.11}$$

then the linear wave solution in the physical variables is

$$\begin{cases} \eta(t,x) = \epsilon\cos(kx - \omega t), \\ u(t,x,y) = \epsilon\omega\exp(ky)\cos(kx - \omega t), \\ v(t,x,y) = \epsilon\omega\exp(ky)\sin(kx - \omega t), \\ p(t,x,y) = P_0 - gy + \epsilon\left(g + k^2\Gamma\right)\exp(ky)\cos(kx - \omega t). \end{cases} \tag{2.12}$$

Notice that in the physical variables the wavespeed in (2.12) is given by the dispersion relation

$$c = \frac{\omega}{k} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\Gamma}{\lambda}}. \tag{2.13}$$

If $(x(t), y(t))$ is the path of the particle below the linear wave (2.12), then

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

so that the motion of the particle is described by the system

$$\begin{cases} \frac{dx}{dt} = M\exp(ky)\cos(kx - \omega t) \\ \frac{dy}{dt} = M\exp(ky)\sin(kx - \omega t), \end{cases} \tag{2.14}$$

with initial data $(x_0, y_0)$, where

$$M = \epsilon\omega. \tag{2.15}$$

## 3 Qualitative analysis of solutions

In performing an analysis of the solutions of (2.14) we will not try to solve the system of equations explicitly, since the right hand side of (2.14) is nonlinear. Rather, using phase plane analysis to examine the qualitative features of the solutions c.f. [9], it will be shown that there are no water particles travelling in closed orbits. In fact, we will see that every water particle experiences a forward drift as the wave progresses. We use the following transformation,

$$X(t) = kx(t) - \omega t, \qquad Y(t) = ky(t), \tag{3.1}$$

to give us the new system

$$\begin{cases} \frac{dX}{dt} = kM\exp(Y)\cos(X) - kc, \\ \frac{dY}{dt} = kM\exp(Y)\sin(X), \\ (X(0), Y(0)) = (x_0, y_0). \end{cases} \tag{3.2}$$

Since (3.2) is periodic in $X$, we need only consider the strip

$$\{(X,Y)\in\mathbb{R}^2:\ -\pi\le X\le\pi\}.$$

Furthermore, as (3.2) is a description of our physical model we can restrict our attention to the values $Y>0$. This system corresponds directly to the system considered in [10] for pure-gravity deep-water waves, except with a different for $M$, therefore the approach in [10] applies. Firstly, we present a necessary condition for the wave particles to be periodic c.f. [3].

**Lemma 1.** *If the particle path* $(x(t),y(t))$ *is periodic, with period* $\tau$*, then* $\tau=\frac{2\pi}{\omega}$.

Our main result now follows.

**Theorem 1.** *The system* (2.14) *has no periodic solutions.*

**Proof.** Suppose $(X,Y)$ is a solution of (3.2) with $X(0)=\pi$, and let $\theta_{\frac{\pi}{2}}$ be the time when $X(\theta_{\frac{\pi}{2}})=\frac{\pi}{2}$. Define $Y_{\frac{\pi}{2}}\equiv Y(\theta_{\frac{\pi}{2}})$. Then for the path of any particle below the surface we have

$$\frac{dt}{dX}=\frac{1}{kM\exp(y)\cos X-\omega},$$

since $Y<\log(\omega/kM)$. We note that

$$Y>Y_{\pi/2}\ \text{for}\ X\in(0,\pi/2),\qquad Y<Y_{\pi/2}\ \text{for}\ X\in(\pi/2,\pi),\tag{3.3}$$

and if follows that

$$-\int_0^\pi\frac{dt}{dX}dX=\int_0^\pi\frac{dX}{\omega-kM\exp(Y)\cos(X)}$$

$$\begin{aligned}
&=\int_0^{\pi/2}\frac{dX}{\omega-kM\exp(Y)\cos(X)}+\int_{\pi/2}^{\pi}\frac{dX}{\omega-kM\exp(Y)\cos(X)}\\
&>\int_0^{\pi/2}\frac{dX}{\omega-kM\exp(Y_{\pi/2})\cos(X)}+\int_{\pi/2}^{\pi}\frac{dX}{\omega-kM\exp(Y_{\pi/2})\cos(X)}\\
&=\int_0^{\pi/2}\left(\frac{1}{\omega-kM\exp(Y_{\pi/2})\cos(X)}+\frac{1}{\omega+kM\exp(Y_{\pi/2})\cos(X)}\right)dX\\
&=\int_0^{\pi/2}\frac{2\omega\,dX}{\omega^2-\big(kM\exp(Y_{\pi/2})\cos(X)\big)^2}>\int_0^{\pi/2}\frac{2\omega\,dX}{\omega^2}=\frac{\pi}{\omega}.
\end{aligned}$$

By way of the $X-$symmetry of the system (3.2) we deduce that $X(0)-X(\theta)=2\pi$ implies that $\theta>2\pi/\omega$. It follows from Lemma 1 that there are no periodic solutions of (2.14). ■

We finally note two further results which follow from the qualitative analysis of (3.2) made in the paper [10], which tell us that as we reach greater depths the orbits become nearly closed circles.

**Lemma 2.** *Let $(X, Y)$ be a solution of (3.2) with $(X(0), Y(0)) = (\pi, Y^\pi)$. Then $\Theta \equiv \Theta(Y^\pi)$ defined by $X(\Theta) \equiv -\pi$ is a strictly increasing function of $Y^\pi$, and $\Theta(Y) \to 2\pi/\omega$ as $Y \to -\infty$.*

If $(x, y)$ is the path of a particle we can consider the orbit traced by the particle as it goes from its point of greatest height until it reaches the same height again in the finite time $\theta$, as defined in Lemma 2. Defining the *forward drift* of a fluid particle to be the horizontal distance $x(\theta) - x(0)$, we have the result

**Corollary 1.** *The forward drift of a fluid particle is strictly decreasing with greater depth and vanishes as $y \to -\infty$.*

Analogous to the situation in [10], it is possible to present a qualitative description of the motion of water particles. Namely

$$\begin{array}{lll} \frac{dx}{dt} < 0, \ \frac{dy}{dt} < 0 & \text{for} & X(t) \in (-\pi, -\pi/2), \\ \frac{dx}{dt} > 0, \ \frac{dy}{dt} < 0 & \text{for} & X(t) \in (-\pi/2, 0), \\ \frac{dx}{dt} > 0, \ \frac{dy}{dt} > 0 & \text{for} & X(t) \in (0, \pi/2), \\ \frac{dx}{dt} < 0, \ \frac{dy}{dt} > 0 & \text{for} & X(t) \in (\pi/2, \pi). \end{array}$$

This tells us that the water particles move in an anticlockwise manner, and if starting at the point of greatest depth they first move backward and up, the forward and up, then forward and down, the backward and down, reaching again the point of greatest depth at a time of $\theta > \frac{2\pi}{\omega}$ with $x(\theta) - x(0) = \frac{\theta\omega - 2\pi}{k} > 0$.

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